THE BROWN REPRESENTABILITY THEOREM

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Contents

Fundamental groups and the cohomology groups are defined in remarkably different ways: $\pi_n(X)$ is a satisfyingly natural object, the set $[S^n, X]$ of homotopy classes of based maps from the n-sphere S^n to X. $H^n(X, G)$ is far less geometric object, the homology of the dual of a complex built from formal sums of maps into X. This contrast leads to an amusing dichotomy. Cohomology groups are relatively easy to compute, but many of their formal properties (e.g. h omotopy invariance) are non-trivial to demonstrate. The homotopy groups, on the other hand, are notoriously difficult to compute but relatively easy to study formally. Many of their fundamental properties (for instance, the isomorphism $\pi_{n+1}(X) \simeq \pi_n(\Omega X)$) are encoded in the topological properties of the spheres.

Is conservation of trouble at work here? Is there a geometric definition of singular cohomology? Happily, the answer to at least one of these questions is "yes". $H^n(X, G)$, the n-th cohomology group with coefficients in G, can be realized geometrically, as the set $[X, K(G, n)]$ of homotopy classes of based maps from X into the Eilenberg-MacLane space $K(G, n)$. Even better, something similar is true for every generalized cohomology functor on the category of CW complexes, from K-theory to cobordism. Brown's Representability Theorem shows that for any generalized cohomology theory $h^n(\bullet)$, there exists a sequence of pointed spaces P^n so that the cohomology functor $h^n(\bullet)$ is naturally isomorphic to the functor $[\bullet, P^n]$. Such a sequence is called a spectrum (or more precisely an Ω-prespectrum). This is a strong statement; we are getting much more than an isomorphism of groups. All of the properties of the cohomology groups (for instance, their ring structures) are encoded in the spectrum.

Our goal in this paper is to demonstrate the equivalence of cohomology functors and spectra. We will begin by studying spectra and showing how they give rise to cohomology theories. Then in Section 3 we will prove Brown's theorem, constructing spectra by analogy with Eilenberg-MacLane spaces. Finally, we will sketch the relationship between spectra and generalized homology theories, which is somewhat more subtle than the definition of homotopy groups might lead one to believe.

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Our discussion of Brown's Theorem owes much to [Sel97], but our conventions on spectra are those of May [LMSM86].

1. Ω-Spectra & Generalized Cohomology

We begin by fixing an appropriate category of topological spaces. Henceforth, we will work mainly in CW_*, the categories of based CW complexes. Many of our results can be proved in more general contexts, but the extra effort required doesn't lead to any deeper insights.

Recall that CW_* comes equipped with smash and wedge products, based analogues of the cartesian product and the disjoint union. We can use the smash product to define a functor $\Sigma = S^1 \wedge \bullet : CW_* \to CW_*$. We have also the loop space functor $\Omega = \mathrm{Hom}_{CW_*}(S^1, \bullet)$, which is left adjoint to suspension: $\mathrm{Hom}_{CW_*}(\Sigma X, Y) \simeq \mathrm{Hom}_{CW_*}(X, \Omega Y)$. We will make frequent use of these functors.

We want to show that any spectrum gives rise to a generalized cohomology theory. Since the definition of spectra is not immediately transparent, we will begin by recalling the axioms for generalized cohomology and motivating the introduction of the so-called Ω-prespectra. Since we are working on based spaces, we will actually work with the following technically convenient variation.

Definition 1 (Generalized Cohomology). *A* **reduced generalized cohomology theory** *is a sequence of functors $h^n : CW_* \to Ab$ satsifying the following axioms:*

- *: Homotopy Invariance: $f \sim g$ implies $h^n(f) = h^n(g)$.*
- *: Suspension: $\forall X \in CW_*$, there is a natural isomorphism $\Sigma_X : h^n(X) \simeq h^{n+1}(\Sigma X)$*
- *: Exactness: For every cofibration $A \xrightarrow{i} X \xrightarrow{p} X/A$, we get an exact sequence*

$$\xrightarrow{p^*} h^n(X) \xrightarrow{i^*} h^n(A)$$

- *: Wedge Sum: For any wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ induce an isomorphism $\Pi_\alpha i_\alpha^* : h^n(\bigvee_\alpha X_\alpha) \to \Pi_\alpha h^n(X_\alpha)$.*

The reader should think of the n in h^n as ranging over $\mathbb{N}$ for now. We will show later how to extend its range to $\mathbb{Z}$.

Given a reduced generalized cohomology theory h, we can recover an unreduced theory $\hat{h}$ by setting $\hat{h}(X) = h(X^+)$, where X^+ is the CW complex with underlying space $X \sqcup \{*\}$ and basepoint $*$. And likewise, given a generalized cohomology theory h, we get a reduced theory $\tilde{h}$ by taking the relative cohomology with respect to the basepoint: $\tilde{h}(X) = h(X, *)$. On CW_*, reduced and unreduced theories are equivalent.

The term "reduced generalized cohomology theory" is somewhat cumbersome. Henceforth, we will adhere to modern terminology, ignoring whenever possible the distinction between reduced and unreduced theories, using the word "cohomology" to refer to the broad class of theories, and reserving the term "ordinary cohomology" for $H^n(\bullet, G)$.

Now, given any CW-complex P, we can define a homotopy invariant cofunctor from CW_* to Set by sending X to $[X, P]$. (A map $f : Y \to X$ induces a map $f^* : [X, P] \to [Y, P]$ via composition.) This is a promising approach to cohomology: The functor $\mathrm{Hom}_{CW_*}(\bullet, P) : CW_* \to Set$ respects the wedge sums and short exact sequences, and consequently so does the functor $[\bullet, P]$. However, these randomly chosen homotopy invariants are not likely to be terribly interesting, unless we equip the space P with extra structure. We want group structures, coboundary maps, and the like. It turns out that essentially all of these structures follow from the suspension axiom, which is the only axiom relating cohomology groups of different degree.

Suppose that we have a sequences of CW-complexes $\{P^n\}$. What do we need to make the suspension axiom hold for the functors $[\bullet, P^n]$? A short computation points towards the answer. Demanding suspension forces $[X, P^n] = [\Sigma X, P^{n+1}] = [X, \Omega P^{n+1}]$. This suggests that P^n should be homotopy equivalent to the loop space of P^{n+1}. It turns out that this is enough for our purposes.

Definition 2 (Ω-Prespectra)**.** *An* Ω-**prespectrum** P *is a collection of pointed spaces* $\{P^n\}$ *equipped with homotopy equivalences* $e_n : P^n \to \Omega P^{n+1}$.

Given an Ω-prespectrum, we can define a cohomology theory by setting $h^n(\bullet) = [\bullet, P^n]$. This theory satisfies in more or less trivial fashion all of the above axioms, except one. We must do a little bit of work to show that $[X, P^n]$ actually is an abelian group.

Proposition 1. $[X, P^n]$ *has a natural abelian group structure.*

Proof. The proof is pretty much the same as for higher homotopy groups. The reader might want to just draw pictures.

Observe that $[X, P^n] = [X, \Omega^2 P^{n+2}] = [\Sigma X, \Omega P^{n+2}]$. ΩP^{n+2} has (up to homotopy) a natural group structure, given by concatenation of paths. Thus, $[X, P^n]$ is a group. $[X, P^n]$ also gets a group structure from an up-to-homotopy comultiplication on ΣX. These two multiplicative structures on $[X, P^n]$ (denoted by $\bullet$ and $\star$) are compatible in the sense that they share a unit, and distribute across each other. The latter condition means that, for any $a, b, c, d \in [X, P^n]$, $(a \bullet b) \star (c \bullet d) = (a \star c) \bullet (b \star d)$. Setting $b = c = id$, we see that $a \star d = a \bullet d$, so the two group structures agree. Using this fact and setting $a = d = id$, we see that $b \star c = c \bullet b = c \star b$, so the group structure is abelian

□

It's natural to wonder what a spectrum is, now that we know what an "Ω-prespectrum" is. This topic is actually somewhat outside of our main line of inquiry; spectra are not really necessary for discussing cohomology. The objects we call Ω-prespectra were originally called spectra, when they were discovered in the 1960s. They eventually proved to be unsuitable for the study of stable phenomena, and a new class of spectra were introduced.

To define spectra, we first need to define prespectra.

Definition 3 (Prespectra)**.** *A* **prespectrum** P *is a collection of pointed spaces* $\{P^n\}$ *equipped with maps* $\sigma_n : \Sigma P^n \to P^{n+1}$.

Note that the σ_n give rise (via the isomorphism $\mathrm{Hom}_{CW_*}(\Sigma X, Y) \simeq \mathrm{Hom}_{CW_*}(X, \Omega Y)$) to adjoint maps $\hat{\sigma}_n : P^n \to \Omega P^{n+1}$ (and vice versa). This tells us that Ω-prespectra are precisely those prespectra for which the $\hat{\sigma}_n$ are homotopy equivalences.

One example of a prespectrum which is not an Ω-spectrum is the set of n-spheres S^n, together with the natural maps $S^1 \wedge S^n \simeq S^{n+1}$.

Spectra are prespectra which are even more tightly constrained than the Ω-prespectra.

Definition 4 (Spectra)**.** *A* **spectrum** *is a prespectrum for which the adjoint maps* $\hat{\sigma}_n$ *are homeomorphisms.*

The forgetful functor from spectra to prespectra has a left adjoint. Given a prespectrum P, we can define a spectrum $\tilde{P}$ so that any maps from P into a spectrum S factor through a map $\tilde{P} \to S$. Defining this functor in general takes a bit of work (see [Ada74]) , so we will content ourselves with the case where the adjoint maps $\hat{\sigma}_n : P^n \to \Omega P^{n+1}$ are inclusions. In this case, we can set $\tilde{P}^n = \operatorname{colim}_k \Omega^k P^{n+k}$.

We may associate a cohomology theory h to any prespectrum P by setting $h^n(X) = [X, \operatorname{colim}_k \Omega^k P^{n+k}]$. This definition lets us extend our definition of h to negative n, by starting the colimit at $k = -n$. If P is an Ω-prespectrum, the homotopy equivalences $\Omega^k P^{n+k} \sim P^n$ descend to isomorphisms $[X, \operatorname{colim}_k \Omega^k P^{n+k}] \simeq [X, P^n]$. Ω-prespectra strike a nice balance between being encoding the properties of cohomology functors and being easy to think about. Spectra take more effort to define and study – for instance, the spectrum associated to the sphere prespectrum consists of infinite loop spaces – but they have very good formal properties. The isomorphism $P^n \simeq \Omega^k P^{n+k}$ actually holds at the point set level!

In what follows, we will continue to work explicitly with prespectra. The reader should imagine that we have used the spectrification functor where appropriate.

2. Examples

We just mention here a few examples of spectra. We omit proofs, as proving that a given collection of spaces constitutes a spectrum tends to be extremely non-trivial. For example, the essential part of the proof that the K-theory spectrum is in fact a spectrum is the proof of Bott Periodicity.

Example 1 (The sphere spectrum). The n-spheres are naturally a prespectrum, since $\Sigma S^n \simeq S^{n+1}$. We obtain the sphere spectrum $\mathbb{S}$ by setting $\mathbb{S}^n = \operatorname{colim}_k \Omega^k S^{n+k}$. The associated cohomology theory is known as stable cohomotopy.

Example 2. The sphere spectrum above arise by applying the suspension functor to the two-pointed space S^0. If we start with some other space X, we obtain a spectrum $\mathbb{X}^n = \operatorname{colim}_i \Omega^i \Sigma^n X$.

Example 3. Denote by U the infinite dimensional unitary group and by BU the classifying space for principal $\mathrm{U}(n)$ bundles. There is a canonical isomorphism $U \simeq \Omega BU \simeq \Omega(BU \times \mathbb{Z})$ and Bott Periodicity tells us that $BU \times \mathbb{Z} \simeq \Omega U$. Thus, we can define an Ω-prespectrum by setting $P^{2i} = BU \times \mathbb{Z}$ and $P^{2i+1} = U$. The zeroth degree part of this cohomology theory is actually complex K-theory, the Grothendieck group of complex vector bundles. For more details, see chapter 24 of [May99]. Observe that this cohomology theory is periodic, and can have infinitely many non-zero groups.

A nearly identical construction applies to the classifying BSO and real K theory, except that the periodicity is eightfold instead of twofold.

Example 4. The Eilenberg-MacLane spaces $K(G, n)$ form an Ω-prespectrum, and their associated cohomology theory is ordinary cohomology with coefficients in G.

3. The Brown Representability Theorem

In the previous section, we saw that spectra give rise to cohomology theories. In this section, we will prove a converse to this result: all cohomology theories come from spectra. More precisely, we have

Theorem 1 (Brown's Representability Theorem)**.** *If $h : CW_* \to Ab$ is a reduced cohomology functor, then there exists an Ω-prespectrum P (unique up to homotopy equivalence) such that $h^n(\bullet)$ is naturally isomorphic to $[\bullet, P^n]$.*

The proof we will give is a fairly explicit construction, done by analogy with the construction of Eilenberg-Maclane spaces. This construction depends on some properties of cohomology theories on CW-complexes.

Theorem 2 (Milnor)**.** *Suppose $\{i_n : X_n \hookrightarrow X_{n+1}\}$ is a sequence of nested subspaces, and let $X = \cup_n X_n$. If $h : CW_* \to Ab$ is a reduced cohomology theory, then for all $i \geq 1$*

$$0 \longrightarrow \lim_n{}^1 h^{i-1}(X_n) \longrightarrow h^i(X) \longrightarrow \lim_n h^i(X_n) \longrightarrow 0$$

Proof. This result is really just a generality from homological algebra (see Chapter 3 of [Wei94]), but for CW-complexes we can give a direct construction.

Let the mapping telescope of X be given by

$$T = \frac{\coprod_n X_n \times [n-1, n]}{X_n \times \{n\} \sim j_n(X_n) \times \{n\}}$$

We can divide T up into two parts. If π is the projection corresponding to the equivalence relation in the definitition above, then write

$$T_1 = \pi(\cup_n X_{2n-1} \times [2n-2, 2n-1] \bigcup \{\star\} \times \mathbb{R}_{\geq 0})$$

and

$$T_2 = \pi(\cup_n X_{2n} \times [2n-1, 2n] \bigcup \{\star\} \times \mathbb{R}_{\geq 0}).$$

The wedge axiom tells us that $h^i(T_1) = \Pi_n h^i(T_{2n-1})$, $h^i(T_2) = \Pi_n h^i(T_{2n})$, and $h^i(T_1 \cap T_2) = \Pi_n h^i(T_n)$. Since $T = T_2 \cup T_2$, we get a Mayer-Vietoris sequence, and in particular a map $h^i(T_1 \cap T_2) \to h^i(T_1) \oplus h^i(T_2)$. This map can be written as

$$I_i : \Pi_n h^i(X_n) \to \Pi_n h^i(X_n)$$

which is the map whose projection to $h^i(X_n)$ is $1 + (i_n)^*$. This gives us an short exact sequence

$$0 \longrightarrow \operatorname{coker}(I_{i-1}) \longrightarrow h^i(T) \longrightarrow \ker(I_i) \longrightarrow 0$$

In the category of abelian groups we have $\operatorname{coker}(I_{i-1}) \simeq \lim_n^1 h^{i-1}(X_n)$ and $\ker(I_i) \simeq \lim_n h^i(X_n)$, more or less by definition, so the result follows. $\square$

With this fact in hand, we are ready to tackle Brown's Theorem.

Proof. (Brown Representability)

We first prove the theorem on spheres, explicitly constructing the prespectrum, and then extend to CW complexes by gluing.

Observe that any element $c_n \in h^n(P)$ gives rise to maps $(c_n)^*_X : [X, P] \to h^n(X)$ via $(c_n)^*_X([f]) = f^*(c_n)$. We will construct by induction for each k a CW-complex $P^n(k)$ and an element $c_n(k) \in h^n(P^n(k))$ so that $(c_n(k))^*_{S^m}$ is an isomorphism for $m < k$ and an epimorphism for $m = k$.

The opening step of the induction is easy: set $P^n(-1) = \{*\}$ and $c_n(-1) = 0$. Now suppose that $P_n(k)$ and $c_n(k)$ exist. Let $K \lhd [S^k, P_n(k)]$ be the kernel of $(c_n(k))^*_{S^k}$. Every point of K is a homotopy class of maps from S^k to $P^n(k)$. If we wedge all of these maps together and then compose with the inclusion $i_n(k) : P^n(k) \hookrightarrow P^n(k) \wedge \bigwedge_{f \in h^n(S^{k+1})} S^{k+1}$, we get a cofibration

$$\phi_n(k) : \bigwedge_{x \in K} S^k \to P^n(k) \wedge \bigwedge_{y \in h^n(S^{k+1})} S^{k+1}$$

$P^n(k+1)$ is defined to be the cofiber of this map (with quotient map $p_n(k)$. Now we use the axioms for cohomology to produce $c_n(k+1)$. The wedge axiom gives natural isomorphisms $h^n(P^n(k) \wedge \bigwedge_{y \in h^n(S^{k+1})} S^{k+1}) \simeq h^n(P^n(k) \times \Pi_{y \in h^n(S^{k+1})} h^n(S^{k+1})$, so we can think of $c_n(k)$ as living in the cohomology of the bigger space. Now, the exactness axiom gives us an exact sequence

$$0 \longrightarrow h^n(P^n(k+1)) \xrightarrow{p_n(k)^*} h^n(P^n(k) \wedge \bigwedge_{y \in h^n(S^{k+1})} S^{k+1}) \xrightarrow{\phi_n(k)^*} h^n(\bigwedge_{x \in K} S^k) \longrightarrow 0$$

and $\phi_n(k)^*(c_n(k)) = 0$ by construction. It follows that there is an element $c_n(k+1) \in h^n(P^n(k+1))$ such that $p_n(k)^*(c_n(k+1)) = (c_n(k), (y)_{y \in h^n(P^{n+1})})$ which demonstrates that $(c_n(k+1))^*_{S^{k+1}}$ is a surjection. This completes the induction. The next step is to glue all of the $P^n(k)$ together to get a single space P^n.

To achieve this, let $j^n(k) : P^n(k) \to P^n(k+1)$ be the composite of $P^n(k) \hookrightarrow P^n(k) \wedge \bigwedge_{f \in h^n(S^{k+1})} S^{k+1}$ and $p_n(k)$, and note that $j^n(k)^*(c_n(k+1)) = c_n(k)$. Milnor's Theorem above guarantees that if we set $P^n = \operatorname{colim}_k P^n(k)$, there will exist an element $c_n \in h^n(P^n)$ which pulls back to $c_n(k)$ on each $P^n(k)$.

$(c_n)^*_{S^m}$ is by construction an isomorphism for any m. We can use the 5-lemma to extend this isomorphism to any finite CW-complex X. By the argument from section 1, there exist natural isomorphisms $[X, P^n] \simeq [X, \Omega P^{n+1}]$. Substituting finite subcomplexes of P^n for X, we obtain maps to ΩP^{n+1} which are compatible with the colimit above, and so induce a map $e_n : P^n \to \Omega P^{n+1}$. e_n induces isomorphisms on the homotopy groups of these spaces, so by Whitehead's Theorem, e_n is a homotopy equivalence. Thus, $\{P^n, e_n\}$ is an Ω-prespectrum.

So now, we have a prespectrum P^n and natural transformations $(c_n)^* : [\bullet, P^n] \to h^n(\bullet)$ which induce isomorphisms for any finite CW-complex X. The theory of CW-complexes tells us that this is good enough: $(c_n)^*$ is a natural isomorphism.

Uniqueness follows by categorical nonsense.

□

Brown's theorem gives us a prespectrum. We can sharpen his result slightly be applying the spectrification functor, thus obtaining a bijection between naturally isomorphic cohomology functors and homeomorphism classes of spectra.

4. Spectra & Homology

We have only discussed cohomology theories up to this point. There are generalized homology theories also. Do these theories also come from spectra? The answer is, more or less, yes. Any prespectrum P determines a homology theory, via the assignment

$$h_n(X) = \operatorname{colim}_i \pi_{i+n}(P^i \wedge X),$$

where the colimit is taken relative to the maps $\sigma_i \wedge 1_x : \Sigma P^i \wedge X \to P^{i+1}$. Examples include regular homology with coefficients in G (given by $P^n = K(G,n)$) and stable homotopy theory (given by the sphere spectrum $P^n = \mathbb{S}^n = \operatorname{colim}_i \Omega^i S^{i+n}$).

The converse, however, is more difficult. We made crucial use of the wedge axiom in our proof of Brown's Representability Theorem. But this axiom is not strong enough to give us a bijection between spectra and homology functors. We can only get a result like this if we impose the stronger condition $h_i(\operatorname{colim}_n X_n) = \operatorname{colim}_n h_i(X_n)$. The reader wishing to see more details should start with [Hat02]

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