**Math 214 HW#9, due 4/7/15 at 12:40 PM**

1. Lee 11.11.

2. Lee 11.17.

3. Check in local coordinates that if $\alpha$ is a 1-form and $V$ and $W$ are vector fields on $M$, then

$$d\alpha(V, W) = V\alpha(W) - W\alpha(V) - \alpha([V, W]).$$

4. Let $\omega : \mathbb{R}^4 \otimes \mathbb{R}^4 \to \mathbb{R}$ be an alternating bilinear form. Show that there exist linear maps $\alpha, \beta : \mathbb{R}^4 \to \mathbb{R}$ with $\omega = \alpha \wedge \beta$ if and only if $\omega \wedge \omega = 0$. *Hint:* Choose a basis in which $\omega$ looks simple.

5. Lee 13.10.

6. Lee 13.24.

7. Let $M$ be a smooth manifold with a Riemannian metric $g : TM \otimes TM \to \mathbb{R}$. If $f : M \to \mathbb{R}$ is a smooth function, the *gradient* of $f$ with respect to $g$ is the vector field $\nabla f$ defined by

$$df = g(\nabla f, \cdot).$$

   (a) In local coordinates $\{x^i\}$, if $g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij}$, explain how to compute $\nabla f$ in terms of $g_{ij}$ and $\partial f/\partial x^i$. *Hint:* See HW#1.

   (b) Let $f : M \to \mathbb{R}$ and let $p \in M$. Show that if $V \in T_pM$ satisfies $df_p(V) > 0$, then there exists a Riemannian metric $g$ on $M$ with $\nabla f(p) = V$.

8. Consider the one-form $\alpha$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\alpha = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

(which we can think of as "$d\theta$"). Let $A \subset \mathbb{R}^2 \setminus \{(0,0)\}$ denote the positive $x$-axis, and let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a loop which is transverse to $A$. Show that the intersection number $A \cdot \gamma \in \mathbb{Z}$ satisfies

$$\frac{1}{2\pi} \int_\gamma \alpha = A \cdot \gamma.$$

9. Lee 14.6.

10. For $k > 0$, define a map $K$ from $k$-forms on $\mathbb{R}^n$ to $(k-1)$-forms on $\mathbb{R}^n$ as follows. If $\alpha$ is a $k$-form, write $\alpha = dx^1 \wedge \beta_I dx^I + \gamma_J dx^J$ where the multi-indices $I$ and $J$ do not include 1. Define

$$K\alpha(x^1, \ldots, x^n) = \left( \int_0^{x^1} \beta_I(t, x^2, \ldots, x^n) dt \right) dx^I.$$

Let $V$ denote the subspace $(x^1 = 0) \subset \mathbb{R}^n$, let $\imath : V \to \mathbb{R}^n$ denote the inclusion map, and let $\pi : \mathbb{R}^n \to V$ denote the projection $(x^1, \ldots, x^n) \mapsto (0, x^2, \ldots, x^n)$.

   (a) Show that
   $$d \circ K + K \circ d = 1 - \pi^* \circ \imath^*.$$

   (b) Use the above result and induction on $n$ to show that if $k > 0$ then every closed $k$-form on $\mathbb{R}^n$ is exact.

11. How difficult was this assignment?