Notes for Math 185, Part 2: Integration

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1 Review of some multivariable calculus

1.1 Curves

We begin by reviewing some definitions involving curves in the plane. Intuitively one thinks of a curve as a "one-dimensional" set of points in the plane. However to discuss integrals over curves, it is more convenient to think of curves as "parametrized", as follows.

Definition 1.1. A C^1 *parametrized curve* in $\mathbb{R}^2$ is a continuously differentiable map $\gamma : [a, b] \to \mathbb{R}^2$, for some a, b, such that $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

The image $\gamma([a, b])$ is the curve in the plane that we care about. One can think of t as a "time" variable, and the map γ as a "schedule" which describes how to move along the curve, describing where we should be at each time. Here $\gamma'(t)$ is the velocity vector of γ at time t. If we write $\gamma(t) = (x(t), y(t))$, then $\gamma'(t) = (x'(t), y'(t))$. We impose the condition $\gamma'(t) \neq 0$ in order to avoid strange parametrizations which stop for some amount of time, and to make the following definitions cleaner.

Definition 1.2. An *oriented* C^1 *curve* in $\mathbb{R}^2$ is an equivalence of C^1 parametrized curves in $\mathbb{R}^2$, under the following equivalence relation. We declare $\gamma : [a, b] \to \mathbb{R}^2$ to be equivalent to $\rho : [c, d] \to \mathbb{R}^2$ if there is a C^1 bijection $\phi : [a, b] \to [c, d]$ with $\phi'(t) > 0$ for all t, such that $\gamma = \rho \circ \phi$. (Exercise: check that this is an equivalence relation.)

When γ and ρ are equivalent as above, they traverse the same set of points in the plane, but on different schedules. The map ϕ is a "reparametrization" which specifies how to convert from one schedule to the other.

The word "oriented" means that the curve has a distinguished "direction" along which we traverse it. If we modified the above definition to only require $\phi'(t) \neq 0$ and not $\phi'(t) > 0$, then the resulting equivalence classes would describe "unoriented" curves.

We typically denote an equivalence class by C, and we refer to an element γ of this equivalence class as a *parametrization* of the curve C. Since we assume that all curves and parametrizations are C^1, we will not always write "C^1" below.

1.2 Line integrals with respect to arc length

Let C be an oriented curve in $\mathbb{R}^2$. We now review how to define integrals over C.

Suppose $f : U \to \mathbb{C}$ is a continuous function[1], where $U \subset \mathbb{R}^2$ is a set containing C (i.e. the image of any parametrization). Can we integrate f over C? The most naive thing to try would be to choose a parametrization $\gamma : [a, b] \to \mathbb{C}$ and define

$$\int_C f = \int_a^b f(\gamma(t))dt. \tag{1.1}$$

To justify this definition, we would need to show that this definition does not depend on the choice of a parametrization. Let $\rho : [c, d] \to \mathbb{R}^2$ be another parametrization such that $\gamma = \rho \circ \phi$ where $\phi : [a, b] \to [c, d]$ is a bijection with $\phi' > 0$. Let $\tau \in [c, d]$ denote the parameter of ρ. By the change of variables formula for one-variable integrals, we have

$$\int_c^d f(\rho(\tau))d\tau = \int_a^b f(\gamma(t))\phi'(t)dt. \tag{1.2}$$

The right hand side will usually not agree with (1.1) (unless $\phi'(t) \equiv 1$). Thus the definition (1.1) does not work.

Summary: the factor $\phi'(t)$ in the change of variables formula (1.2) makes the naive definition (1.1) not invariant under reparametrization.

There are various ways to fix this. One way is to define the *integral with respect to arc length*

$$\int_C f\,ds = \int_a^b f(\gamma(t))|\gamma'(t)|dt \tag{1.3}$$

where γ is a parametrization of t. To check that this is well-defined, let ρ be another parametrization as above. By the change of variables formula, we have

$$\int_c^d f(\rho(\tau))|\rho'(\tau)|d\tau = \int_a^b f(\gamma(t))|\rho'(\phi(t))|\phi'(t)dt.$$

[1] In your multivariable calculus class you probably just integrated real-valued functions. However it makes little difference in this discussion if we allow the functions that we are integrating to be complex-valued.

To check that this agrees with (1.3), we know that

$$\gamma(t) = \rho(\phi(t)).$$

By the chain rule, it follows that

$$\gamma'(t) = \rho'(\phi(t))\phi'(t).$$

Here $\gamma'(t)$ and $\rho'(\phi(t))$ are vectors, while $\phi'(t)$ is a positive scalar. Since multiplying a vector by a scalar multiplies its absolute value by the absolute value of the scalar, we have

$$|\gamma'(t)| = |\rho'(\phi(t))|\phi'(t).$$

Plugging this into (1.3) gives us what we want.

Summary: the factor $|\gamma'(t)|$ in (1.3) transforms under a change of variables in such a way that it cancels the error arising from the change of variables formula (1.2), so that the definition (1.3) is invariant under reparametrization.

In fact, integration with respect to arc length is also well-defined for unoriented curves, although we will not need this.

There is another, equivalent way to define $\int_C f\,ds$ which is more evidently invariant under reparametrization. Namely, let N be a large positive integer and choose times

$$a = t_0 < t_1 < \cdots < t_N = b.$$

Let $\Delta t = \max(t_i - t_{i-1})$. Define

$$\int_C f\,ds = \lim_{N\to\infty,\Delta t\to 0} \sum_{i=1}^{N} f(t_i^*)|\gamma(t_i) - \gamma(t_{i-1})| \tag{1.4}$$

where $t_i^* \in [t_{i-1}, t_i]$. Intuitively, this means that we approximate the curve by N line segments, and sum up the lengths of the line segments times sample values of f. One can show, similarly to the definition of the Riemann integral, that the above limit is well-defined and agrees with (1.3).

We define the length of C to be $\int_C 1\,ds$. The above paragraph should make this seem reasonable.

1.3 Integration of 1-forms over oriented curves

Let U be a subset of $\mathbb{R}^2$.

Definition 1.3. A 1-*form* on U is an expression $P\,dx + Q\,dy$ where $P, Q : U \to \mathbb{C}$ are functions. We will always assume that P and Q are at least continuous.

If C is an oriented curve in U, we define the integral of a 1-form over C by

$$\int_C (P\,dx + Q\,dy) = \int_a^b (P(\gamma(t))x'(t) + Q(\gamma(t))y'(t))dt \tag{1.5}$$

where $\gamma : [a,b] \to \mathbb{R}^2$ is a parametrization of C, and we write $\gamma(t) = (x(t), y(t))$.

Let us check that this does not depend on the choice of parametrization. We will show that $\int_C P\,dx$ does not depend on the choice of parametrization; the proof for $\int_C Q\,dy$ is similar. Let $\rho : [c,d] \to \mathbb{R}^2$ be another parametrization as before. Using slightly informal notation, we need to check that

$$\int_a^b P(\gamma(t))\frac{dx}{dt}dt = \int_c^d P(\rho(\tau))\frac{dx}{d\tau}d\tau.$$

By the change of variables formula, we have

$$\int_c^d P(\rho(\tau))\frac{dx}{d\tau}d\tau = \int_a^b P(\gamma(t))\frac{dx}{d\tau}\frac{d\tau}{dt}dt.$$

By the chain rule,

$$\frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{dt}.$$

Putting this into the above equation gives what we want.

For example, $\int_C 1\,dx = x(b) - x(a)$ and $\int_C 1\,dy = y(b) - y(a)$.

Once again, the factors of $x'(t)$ and $y'(t)$ in (1.5) ensure invariance under reparametrization. Note that it is crucial here that C is an *oriented* curve. If we allowed the reparametrization ϕ to have $\phi' < 0$, then in the change of variables formula, the limits of integration would get switched. Thus, switching the orientation of a curve will multiply the integral of a 1-form by -1.

There is also an alternate definition along the lines of (1.4), namely

$$\int_C P\,dx = \lim_{N\to\infty,\Delta t\to 0} \sum_{i=1}^N P(t_i^*)(x(t_i) - x(t_{i-1})),$$

$$\int_C Q\,dy = \lim_{N\to\infty,\Delta t\to 0} \sum_{i=1}^N Q(t_i^*)(y(t_i) - y(t_{i-1})).$$

That is, to integrate $P\,dx$, we approximate C by N intervals, and sum up the x displacement of each interval times a sample value of P.

1.4 0-forms

Let U be a subset of $\mathbb{R}^2$.

Definition 1.4. A 0-*form* on U is a function $f : U \to \mathbb{C}$. If f is continuously differentiable, we define a 1-form df on U by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Let C be an oriented curve in U from A to B. That is, if $\gamma : [a, b] \to \mathbb{R}^2$ is a parametrization of C, then $\gamma(a) = A$ and $\gamma(b) = B$.

Theorem 1.5. *("fundamental theorem of line integrals") If C is an oriented curve in U from A to B, and if $f : U \to \mathbb{C}$ is continuously differentiable, then*

$$\int_C df = f(B) - f(A).$$

Proof. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a parametrization of C. Write $\gamma(t) = (x(t), y(t))$. By the chain rule,

$$\frac{d}{dt} f(\gamma(t)) = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t).$$

Integrating the above equation from $t = a$ to $t = b$ and using the fundamental theorem of calculus, we obtain

$$f(B) - f(A) = \int_a^b \left(\frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) \right) dt.$$

The right hand side of this equation is, by definition, equal to $\int_C df$. $\square$

1.5 2-forms and Green's theorem

Let U be a subset of $\mathbb{R}^2$.

Definition 1.6. A 2-*form* on U is an expression $f\, dxdy$ where $f : U \to \mathbb{C}$. We will always assume that f is at least continuous.

If U is compact, then the double integral $\iint_U f\, dxdy$ is defined as in the definition of the Riemann integral. We will not review this here; we just recall that if U is a rectangle $[a, b] \times [c, d]$, then

$$\iint_U f\, dxdy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$$

Definition 1.7. If $\alpha = P\,dx + Q\,dy$ is a continuously differentiable 1-form on U, define a 2-form $d\alpha$ on U by

$$d\alpha = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

The significance of the above definition is as follows. Let U be a compact domain in $\mathbb{R}^2$ with piecewise C^1 boundary. Here "piecewise smooth" means that the boundary of U is the union of the images of finitely many C^1 curves, intersecting only at the endpoints. There is then a preferred way to orient the boundary curves. Without giving a rigorous definition here, let us just informally say that each boundary curve is oriented so that U is "on the left" as one traverses the boundary curve.

Green's theorem asserts that in the above situation, if α is a continuously differentiable 1-form on U, then

$$\int_{\partial U} \alpha = \iint_U d\alpha.$$

Here $\int_{\partial U}$ denotes the sum of the line integrals over all of the boundary curves with their preferred orientations. More explicitly, writing $\alpha = P\,dx + Q\,dy$, we have

$$\int_{\partial U} (P\,dx + Q\,dy) = \iint_U (Q_x - P_y) dxdy.$$

Let us recall the proof in the special case when U is a rectangle $[a, b] \times [c, d]$. It is enough to separately consider the cases $\alpha = P\,dx$ and $\alpha = Q\,dy$. We will just check the first case, leaving the second case as an exercise. We can divide the boundary of U into four curves $C_1, \ldots, C_4$, where C_1 is the bottom edge (oriented to the right), C_2 is the right edge (oriented upward), C_3 is the top edge (oriented to the left), and C_4 is the left edge (oriented downward). We have

$$\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$$

because the curves C_2 and C_4 do not move in the x direction, so the $x'(t)$ factor in (1.5) vanishes. We also have

$$\int_{C_1} P\,dx = \int_a^b P(x, c)dx,$$

$$\int_{C_3} P\,dx = -\int_a^b P(x, d)dx$$

where the second equation has a minus sign because of the orientation. Adding up the above gives

$$\int_{\partial U} P\,dx = \int_a^b (P(x, c) - P(x, d))\,dx.$$

By the fundamental theorem of calculus, for a fixed $x \in [a,b]$ we have

$$P(x,d) - P(x,c) = \int_c^d \frac{\partial P}{\partial y}(x,y)dy.$$

Plugging this into the previous equation, we obtain

$$\begin{aligned}\int_{\partial U} P\,dx &= -\int_a^b \left(\int_c^d P_y(x,y)dy \right) dx \\ &= -\iint_U P_y\,dxdy\end{aligned}$$

which is what we wanted.

1.6 The mean value property of harmonic functions

[this is an important fact which we will need, and a nice application of Green's theorem...]

1.7 Closed and exact 1-forms

To review, we have a d operator from (continuously differentiable) 0-forms to 1-forms defined by

$$df = f_x dx + f_y dy,$$

and we also have a d operator from (continuously differentiable) 1-forms to 2-forms defined by

$$d(Pdx + Qdy) = (Q_x - P_y)dxdy.$$

If f is continuously twice differentiable, then

$$d(df) = d(f_x dx + f_y dy) = (f_{yx} - f_{xy})dxdy = 0$$

by Clairaut's theorem.

A (continuously differentiable) 1-form α is called *closed* if $d\alpha = 0$, and *exact* if $\alpha = df$ for some f. Since $d(df) = 0$, it follows that every exact form is closed.

Conversely, is every closed 1-form exact? The answer depends on the domain on which the 1-form is defined. To start, we have

Theorem 1.8. *Let α be a closed 1-form on $\mathbb{R}^2$. Then α is exact. That is, there exists $f : \mathbb{R}^2 \to \mathbb{C}$ such that $df = \alpha$. Moreover, f is unique up to addition of a constant.*

Proof. Uniqueness: Write $\alpha = Pdx + Qdy$. Suppose that $df = \alpha$. Write $c = f(0,0)$. Applying the "fundamental theorem of line integrals" to a horizontal curve from $(0,0)$ to $(x,0)$ shows that

$$f(x,0) = c + \int_0^x P(t,0)dt.$$

Then applying the "fundamental theorem of line integrals" to a vertical curve from $(x,0)$ to (x,y) shows that

$$f(x,y) = c + \int_0^x P(t,0)dt + \int_0^y Q(x,t)dt. \tag{1.6}$$

Thus f is uniquely determined by P and Q and the additive constant c.

Existence: Let c be an arbitrary constant and define f by (1.6). We will show that $df = \alpha$, that is, $f_x = P$ and $f_y = Q$.

If we differentiate (1.6) with respect to y, then only the last term depends on y, and its derivative is $Q(x,y)$ by the fundamental theorem of calculus.

To differentate the right hand side of (1.6) with respect to x, we differentiate the second term using the fundamental theorem of calculus as above, and the third term by differentiating under the integral sign, to obtain

$$f_x(x,y) = P(x,0) + \int_0^y Q_x(x,t)dt.$$

We now use the assumption that α is exact to write $Q_x = P_y$. Thus

$$f_x(x,y) = P(x,0) + \int_0^y P_y(x,t)dt.$$

By the fundamental theorem of calculus,

$$\int_0^y P_y(x,t)dt = P(x,y) - P(x,0).$$

Combining the above two equations gives $f_x(x,y) = P(x,y)$ as desired[2]. $\square$

The above argument also works when the domain is a rectangle or a disk, and also for some other domains. In general, if the domain is *simply connected*, which roughly speaking means that it "has no holes", then every

[2] Another way to show that $f_x = P$ is to use Green's theorem (and the assumption that α is exact) to rewrite

$$f(x,y) = c + \int_0^y Q(0,t)dt + \int_0^x P(t,y)dt$$

and then argue similarly to the proof that $f_y = Q$.

closed 1-form is exact. However, for non-simply connected domains, closed 1-forms need not be exact.

An important example (for reasons to be discussed later) is the 1-form

$$\alpha = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

defined on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$. Here the removed origin counts as a "hole", so this domain is not simply connected. A short calculation shows that $d\alpha = 0$ (do it). However α is not exact. To see why, let C denote the unit circle, oriented counterclockwise. Then another short calculation shows that $\int_C \alpha = 2\pi$ (do this one too). Since C is a closed curve (i.e. any parametrization will start and end at the same point on the curve), the "fundamental theorem of line integrals" implies that the integral of any exact 1-form on C is zero. Hence α is not exact.

2 Integration of holomorphic functions

2.1 Complex notation

We now identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way. The identity map $z \mapsto z$ is a perfectly good 0-form on $\mathbb{C}$, and so we can take d of it to obtain a 1-form dz. Since $z = x + iy$, it follows that

$$dz = dx + idy.$$

Likewise, $\overline{z} = x - iy$ is a 0-form on $\mathbb{C}$, and d of it is

$$d\overline{z} = dx - idy.$$

If f is a continuously differentiable (in the real sense) complex-valued function on a domain U in $\mathbb{C}$, define

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \\ \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \end{aligned}$$

Observe that f is holomorphic if and only if $\partial f/\partial\overline{z} = 0$ (this is just a rewrite of the Cauchy-Riemann equations). In this case, the complex derivative of f is given by $f'(z) = \partial f/\partial z$.

There is a nice formula for the d operator on 0-forms in terms of the strange-looking operators $\partial/\partial z$ and $\partial/\partial\overline{z}$ (which is the reason they are defined the way they are). Namely, if f is a continuously differentiable (not necessarily holomorphic) function, then

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}. \tag{2.1}$$

(Do the calculation.)

As a corollary, we have:

Proposition 2.1. *Let C be an oriented curve in $\mathbb{C}$ from A to B, and let f be a holomorphic function on a domain containing C. Then*

$$\int_C f'(z)dz = f(B) - f(A).$$

Proof. Since f is holomorphic, $\partial f/\partial \overline{z} = 0$. Then the formula (2.1) implies that $df = f'(z)dz$. Now apply the "fundamental theorem of line integrals". □

2.2 Cauchy's theorem and Cauchy's integral formula

Let $U \subset \mathbb{C}$ be a compact domain with piecewise smooth boundary, as in the statement of Green's theorem. Let ∂U be the union of the boundary curves of U, with their preferred orientations.

Theorem 2.2. *(Cauchy's theorem) If $f : U \to \mathbb{C}$ is continuously differentiable (in the real sense) and holomorphic, then*

$$\int_{\partial U} f\,dz = 0.$$

Proof. By Green's theorem, it is enough to show that $d(fdz) = 0$. We compute

$$d(fdz) = d(fdx + ifdy) = (if_x - f_y)dxdy.$$

By the Cauchy-Riemann equations, $f_y = if_x$, so the right hand side of the above equation vanishes. □

We remark that more generally, by a bit of linear algebra, any 1-form α can be uniquely written as $\alpha = fdz + gd\overline{z}$. A short calculation then shows that

$$d(fdz + gd\overline{z}) = 2i\left(\frac{\partial f}{\partial \overline{z}} - \frac{\partial g}{\partial z}\right)dxdy.$$

We also remark that by a more clever argument, in Cauchy's theorem one can drop the assumption that the derivatives of f are continuous. This fact is known as Goursat's theorem.

2.3 Cauchy's integral formula

[This is all that I am writing for now...]