Math 185 Review problems

The following questions are from the complex analysis part of recent prelim exams for first-year grad students. If you can get more than about half of the prelim exam right (some of which consists of easier topics than this), then you are considered ready for graduate study in mathematics.

1. Find the Laurent expansion of
$$f(z) = (1+z)^{-1} + (z^2-9)^{-1}$$
in the annulus $1 < |z| < 3$.

2. Compute
$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

3. (a) Find the poles and residues of $1/(z^3\cos(z))$.
   (b) Show that the integral of the function above over a square contour centered at the origin with side $2\pi N$ tends to zero as the integer N tends to infinity.
   (c) Find the sum $1/1^3 - 1/3^3 + 1/5^3 - 1/7^3 + \cdots$.

4. Let $D \subset \mathbb{C}$ denote the open unit disc. If $a, b \in D$, show that there exists a holomorphic bijection $f : D \to D$ with $f(a) = b$.

5. Calculate
$$\int_0^{2\pi} \frac{1}{1+\frac{1}{2}\sin\theta} d\theta.$$

6. If $0 < r < 1$, find
$$\sum_{k=0}^{\infty} r^k \cos(k\theta).$$
Your final answer should not involve any complex numbers.

7. Find a conformal bijection from the unit disk $|z| < 1$ to the sector $0 < \arg(z) < \pi/4$.

8. Compute

$$\int_{|z|=2} \frac{z^4}{z^5 - z - 1} dz.$$

9. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that if $\mathrm{Re}(f(z)) > \mathrm{Re}(g(z))$ for all z with $|z| = 1$, then $\mathrm{Re}(f(z)) > \mathrm{Re}(g(z))$ for all z with $|z| < 1$.

10. Let $U \subset \mathbb{C}$ be simply connected, and suppose $f : U \to \mathbb{C}$ is holomorphic and never zero. Show that there is a holomorphic function $g : U \to \mathbb{C}$ with $e^g = f$. *Hint:* Use the fact that every closed 1-form on a simply connected domain is exact.