Math 185 HW#2, due 9/11/12 at 12:40 PM

1. For $z \in \mathbb{C}$ define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

   Prove that $\cos(z)^2 + \sin(z)^2 = 1$ and that the usual angle addition formulas still hold.

2. Gamelin, page 27, exercise 1.

3. Prove that there is no continuous logarithm function defined for all nonzero complex numbers. That is, there is no continuous function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $e^{f(z)} = z$ for all $z \in \mathbb{C} \setminus \{0\}$. *Hint:* Modify the proof in the notes that there is no continuous square root function.

4. Prove that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \overline{f(\overline{z})}$ is holomorphic.

5. Let U be a connected open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be holomorphic. Show that if $|f|$ is constant, then f is constant.

6. Gamelin, page 50, exercises 6, 8.

7. Gamelin, page 53, exercise 6.

8. For which quadruples of real numbers is the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ harmonic? For quadruples (a, b, c, d) such that u is harmonic, find all conjugate harmonic functions v by integration, and describe the corresponding holomorphic functions $u + iv$.

9. *Extra credit:* Let $a \in \mathbb{C}$ and assume that $|a| \neq 0, 1$. Show that all circles that pass through a and $1/\overline{a}$ intersect the circle $|z| = 1$ at right angles.