Math 113 Homework # 1, due 9/9/9 at 2:10 PM

0. (optional, not for credit) If you want practice with proof by induction, read chapter 4 of the notes on proofs and do the exercises at the end.

1. Fraleigh, section 0, exercises 29–34.

2. In this exercise you will construct $\mathbb{Q}$ starting from $\mathbb{Z}$. Let $S = \{(a,b) \mid a,b \in \mathbb{Z},\ b \neq 0\}$. Define a relation $\sim$ on S by
$$(a,b) \sim (c,d) \iff ad = bc.$$
   (a) Show that $\sim$ is an equivalence relation.
   (b) Let $\mathbb{Q}$ denote the set of equivalence classes. Denote the equivalence class of (a,b) by $[a,b]$. (Ordinarily we denote this by a/b.) Show that the following operations of "addition" and "multiplication" on $\mathbb{Q}$ are well defined:
$$[a,b] + [c,d] = [ad + bc, bd],$$
$$[a,b][c,d] = [ac, bd].$$

3. Recall the Division Theorem: if a and b are integers with $b > 0$, then there are unique integers q, r such that $a = qb + r$ and $0 \leq r < b$.
   (a) Show that $\gcd(a,b) = \gcd(b,r)$. (This is the key step in proving that the euclidean algorithm works.)
   (b) Prove that there exist integers x, y such that $ax + by = \gcd(a,b)$. *Hint:* use induction on $\max(a,b)$ and part (a).

4. Find an integer solution x, or explain why no solution exists:
   (a) $83x \equiv 4 \pmod{157}$.
   (b) $1001x \equiv 131 \pmod{611}$.

5. (a) Show that every positive integer n has a **binary expansion**, i.e. can be expressed as a sum of *distinct* powers of 2. For example, $2009 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^0$. *Hint:* use the division theorem to write $n = 2q + r$ with $r \in \{0, 1\}$, and use induction.
   (b) *Extra credit:* Show that the binary expansion of a given positive integer is unique.

6. How challenging did you find this assignment? How long did it take?