Math 113 Homework # 2, due 2/6/03 at 8:10 AM

1. In this exercise you will construct $\mathbb{Q}$ starting from $\mathbb{Z}$. Let $S = \{(a,b) \mid a,b \in \mathbb{Z},\ b \neq 0\}$. Define a relation $\sim$ on S by
$$(a,b) \sim (c,d) \Longleftrightarrow ad = bc.$$
   (a) Show that $\sim$ is an equivalence relation.

   (b) Let $\mathbb{Q}$ denote the set of equivalence classes. Denote the equivalence class of (a,b) by $[a,b]$. (Ordinarily we denote this by a/b.) Show that the following operations of "addition" and "multiplication" on $\mathbb{Q}$ are well defined:
$$[a,b] + [c,d] = [ad + bc, bd],$$
$$[a,b][c,d] = [ac, bd].$$

2. (a) Show by induction that if k is a positive integer then $2^k > k$.

   (b) Show that every positive integer has a **binary expansion**, i.e. can be expressed as a sum of *distinct* powers of 2. For example, $2003 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^1 + 2^0$.

   (c) (Extra credit) Show that the binary expansion of a given positive integer is unique.

3. (a) Show that if a and b are integers with $\gcd(a,b) = 1$ and $b > 1$ then $a/b \notin \mathbb{Z}$.

   (b) Show that if $\gcd(a,b) = 1$ then $\gcd(a^2, b^2) = 1$.

   (c) Show that if n is an integer then $\sqrt{n} \in \mathbb{Q} \Longleftrightarrow \sqrt{n} \in \mathbb{Z}$.

4. For each of the following congruences, either find an integer solution x or explain why no solution exists:

   (a) $83x \equiv 4 \pmod{157}$.

   (b) $1001x \equiv 131 \pmod{611}$.

5. Fraleigh Section 2, exercises 26, 30, 32, 34.

6. How challenging did you find this assignment? How long did it take? (Note that the homework may get a little more challenging [and interesting] next week once we start playing with groups.)