SIMPLE SINGULARITIES AND INTEGRABLE HIERARCHIES

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To Alan Weinstein on his 60-th birthday

ABSTRACT. The paper [11] gives a construction of the *total descendent potential* corresponding to a semisimple Frobenius manifold. In [12], it is proved that the total descendent potential corresponding to K. Saito's Frobenius structure on the parameter space of the miniversal deformation of the A_{n-1}-singularity satisfies the modulo-n reduction of the KP-hierarchy. In this paper, we identify the hierarchy satisfied by the total descendent potential of a simple singularity of the A, D, E-type. Our description of the hierarchy is parallel to the vertex operator construction of Kac – Wakimoto [17] except that we give both some general integral formulas and explicit numerical values for certain coefficients which in the Kac – Wakimoto theory are studied on a case-by-case basis and remain, generally speaking, unknown.

1. The ADE-hierarchies. According to Date–Jimbo–Kashiwara–Miwa [6] and I. Frenkel [10], the KdV-hierarchy of integrable systems can be placed under the name A_1 into the list of more general integrable hierarchies corresponding to the ADE Dynkin diagrams. These hierarchies are usually constructed (see [16]) using representation theory of the corresponding loop groups. V. Kac and M. Wakimoto [17] describe the hierarchies even more explicitly in the form of the so called *Hirota quadratic equations* expressed in terms of suitable *vertex operators*.

One of the goals of the present paper is to show how the vertex operator description of the Hirota quadratic equations (certainly the same ones, even though we don't quite prove this) emerges from the theory of vanishing cycles associated with the ADE singularities.

Let f be a weighted-homogeneous polynomial in $\mathbb{C}^3$ with a simple critical point at the origin. According to V. Arnold [1] simple singularities of holomorphic functions are classified by simply-laced Dynkin diagrams:

$$A_N,\ N\geq 1:\quad f=\frac{x_1^{N+1}}{N+1}+\frac{x_2^2}{2}+\frac{x_3^2}{2},\quad D_N,\ N\geq 4:\quad f=x_1^2x_2-x_2^{N-1}+x_3^2,$$

$$E_6:\quad f=x_1^3+x_2^4+x_3^2,\quad E_7:\quad f=x_1^3+x_1x_2^3+x_3^2,\quad E_8:\quad f=x_1^3+x_2^5+x_3^2.$$

Let $H=\mathbb{C}[x_1,x_2,x_3]/(f_{x_1},f_{x_2},f_{x_3})$ denote the local algebra of the critical point. We equip H with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ by picking a weighted - homogeneous holomorphic volume $\omega=dx_1\wedge dx_2\wedge dx_3$ and using the residue pairing:

$$(\varphi,\psi):=\mathrm{Res}_0\,\frac{\varphi(x)\ \psi(x)\ \omega}{f_{x_1}f_{x_2}f_{x_3}}.$$

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Let $\mathcal{H} = H((z^{-1}))$ be the space of Laurent series $\mathbf{f}(z) = \sum_{k\in\mathbb{Z}} \mathbf{f}_k z^k$ in one indeterminate z^{-1} (*i.e.* finite in the direction of positive k) with vector coefficients $\mathbf{f}_k \in H$. We endow $\mathcal{H}$ with the symplectic form

$$\Omega(\mathbf{f},\mathbf{g}) := \frac{1}{2\pi i}\oint (\mathbf{f}(-z),\mathbf{g}(z))\, dz.$$

The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$ is Lagrangian and identifies $\mathcal{H}$ with the cotangent bundle space $T^*\mathcal{H}_+$. The Hirota quadratic equations are imposed on *asymptotical functions* of $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 z + \mathbf{q}_2 z^2 + ... \in \mathcal{H}_+$. By an asymptotical function we mean an expression of the form

$$\Phi = \exp \sum_{g=0}^{\infty} \hbar^{g-1}\mathcal{F}^{(g)}(\mathbf{q})$$

where usually $\mathcal{F}^{(g)}$ will be formal functions on $\mathcal{H}_+$. By definition, vertex operators are elements of the Heisenberg group acting on such functions. Given a sum $\mathbf{f} = \sum \mathbf{f}_k z^k$ (possibly infinite in both directions) one defines the corresponding vertex operator of the form

$$e^{\Omega(\mathbf{f}_-,\mathbf{q})/\sqrt{\hbar}} e^{\sqrt{\hbar}\partial_{\mathbf{f}_+}} = \exp\{\sum_{k\geq 0}(-1)^{k+1}\sum_a f^a_{-1-k} q^a_k/\sqrt{\hbar}\}\ \exp\{\sqrt{\hbar}\sum_{k\geq 0}\sum_a f^a_k \partial/\partial q^a_k\}.$$

Here f^a_k, q^a_k are components of the vectors $\mathbf{f}_k, \mathbf{q}_k$ in an *orthonormal* basis. We will make use of the vertex operators $\Gamma^\phi(\lambda)$ corresponding to 2-dimensional homology classes $\phi \in H_2(f^{-1}(1)) \simeq \mathbb{Z}^N$ and defined as follows. Take

$$\mathbf{f} := \sum_{k\in\mathbb{Z}} I^{(k)}_\phi(\lambda)(-z)^k \quad \text{where} \quad dI^{(k)}_\phi/d\lambda = I^{(k+1)}_\phi,$$

and $I^{(-1)}_\phi(\lambda) \in H$ is the following period vector:

$$(I^{(-1)}_\phi(\lambda), [\psi_a]) := \frac{1}{2\pi}\int_{\phi\subset f^{-1}(\lambda)} \psi_a(x)\frac{\omega}{df}.$$

The cycle ϕ is transported from the level surface $f^{-1}(1)$ to $f^{-1}(\lambda)$, and ψ_a are *weighted-homogeneous* functions representing a basis in the local algebra H. [1] The functions $(I^{(k)}_\phi, [\psi_a])$ are proportional to the fractional powers $\lambda^{m_a/h-k-1}$ where h is the Coxeter number and $m_a = 1 + h\deg\psi_a$ are the exponents of the appropriate reflection group A_N, D_N or E_N.

The lattice $H_2(f^{-1}(1))$ carries the action of the monodromy group (defined via morsification of the function f) which is the reflection group with respect to the intersection form of cycles. The form is negative definite, and we will denote $\langle\cdot,\cdot\rangle$ the positive definite form opposite to it. Let A denote the set of *vanishing cycles*, *i.e.* the set of classes $\alpha \in H_2(V_1)$ with $\langle\alpha,\alpha\rangle = 2$ such that the reflections $\phi \mapsto \phi - \langle\alpha,\phi\rangle\alpha$ belong to the monodromy group. The *Hirota quadratic equation* of the

[1] As it follows, for instance, from [13], the integral on the R.H.S. depends only on the class $[\psi_a] \in H$.

ADE-type takes on the form

$$\text{(1)} \qquad \mathrm{Res}_{\lambda=\infty}\ \frac{d\lambda}{\lambda}\ \left[\sum_{\alpha\in A} a_\alpha\ \Gamma^{\alpha}(\lambda)\otimes\Gamma^{-\alpha}(\lambda)\right]\ (\Phi\otimes\Phi) = \frac{N(h+1)}{12h}\ (\Phi\otimes\Phi) +$$

$$\text{(2)} \qquad \sum_{k\geq 0}\sum_a \left(\frac{m_a}{h}+k\right)(q_k^a\otimes 1 - 1\otimes q_k^a)\left(\frac{\partial}{\partial q_k^a}\otimes 1 - 1\otimes\frac{\partial}{\partial q_k^a}\right)\ (\Phi\otimes\Phi).$$

The tensor product sign means that the functions depend on two copies $\mathbf{q}'$ and $\mathbf{q}''$ of the variable $\mathbf{q}$, and the objects on the left of $\otimes$ refer to $\mathbf{q}=\mathbf{q}'$ while those on the right to $\mathbf{q}=\mathbf{q}''$. The equation can be interpreted as follows. Set $\mathbf{q}'=\mathbf{x}+\mathbf{y}, \mathbf{q}''=\mathbf{x}-\mathbf{y}$. and expand (1,2) as a power series in $\mathbf{y}$. Namely, rewrite the vertex operators:

$$\Gamma^{\alpha}(\lambda)\otimes\Gamma^{-\alpha}(\lambda) = \exp\{\sum 2(-1)^{k+1} f_{-1-k}^a \hbar^{-1/2} y_k^a\}\ \exp\{\sum f_k^a \hbar^{1/2}\partial_{y_k^a}\}$$

where the coefficients f_k^a (respectively f_{-1-k}^a) are proportional to negative (respectively positive) fractional powers of λ. The residue sum (which should be understood here as the coefficient at λ^0) can therefore be written as a power series $\sum \mathbf{y}^{\mathbf{m}} P_{\mathbf{m}}(\partial_{\mathbf{y}})$ in $\mathbf{y}$ with coefficients $P_{\mathbf{m}}$ which are differential polynomials. Also, $\Phi\otimes\Phi = \Phi(\mathbf{x}+\mathbf{y})\Phi(\mathbf{x}-\mathbf{y})$ can be expanded into the Taylor power series in $\mathbf{y}$ with coefficients which are quadratic expressions in partial derivatives of $\Phi(\mathbf{x})$. Finally the operator in (2) assumes the form $2\sum_{a,k}\ (m_a/h\ +\ k) y_k^a \partial_{y_k^a}$. Equating coefficients in (1,2) at the same monomials $\mathbf{y}^{\mathbf{m}}$ we obtain a hierarchy of quadratic relations between partial derivatives of $\Phi(\mathbf{x})$.

In particular, the equation corresponding to $\mathbf{y}^{\mathbf{0}}$ shows that

$$\text{(3)} \qquad \sum_{\alpha\in A} a_\alpha = \frac{N(h+1)}{12h}$$

is a necessary condition for consistency of the hierarchy (*i.e.* for existence of a non-zero solution Φ).

According to C. Hertling (see the last chapter in [15]) for any weighted - homogeneous singularity the expressions $N(h+1)/(12h)$ and $h^{-2}\sum_a m_a(h-m_a)/2$ coincide. Therefore the operator on the R.H.S. of the Hirota equation is twice the *Virasoro operator*[2]

$$\sum_{a,k}\ \left(\frac{m_a}{h}+k\right)\ y_k^a\ \partial_{y_k^a}\ +\ \sum_a\ \frac{m_a(h-m_a)}{4h^2}.$$

The coefficients a_α actually depend only on the orbit of the vanishing cycle α under the action of the *classical monodromy operator* defined by transporting the cycles in $f^{-1}(\lambda)$ around $\lambda=0$ and acting as one of the Coxeter elements in the reflection group. In fact the root system A consists of N such orbits with h elements each. Summing the vertex operators within the same orbit acts as taking the average over all h branches of the function $\lambda^{1/h}$. Thus the total sum does not contain fractional powers of λ when expanded near $\lambda=\infty$.

[2]In a sense it corresponds to the vector field $\lambda\partial_\lambda$ in the Lie algebra of vector fields on the line — see Section 7 for further information about this.

The exact values of the coefficients a_α can be described as follows. To a vector $\beta \in H_2(f^{-1}(1), \mathbb{C}) \simeq \mathbb{C}^N$, associate the meromorphic 1-form on $\mathbb{C}^N$

$$\mathcal{W}_\beta := -\frac{1}{2} \sum_{\gamma \in A} \langle \beta, \gamma \rangle^2 \frac{d\langle \gamma, x\rangle}{\langle \gamma, x\rangle}. \tag{4}$$

Let w be an element of the reflection group and α and $\beta = w\alpha$ be two roots. Then

$$a_\beta / a_\alpha = \exp \int_\kappa^{w^{-1}\kappa} \mathcal{W}_\alpha = \prod_{\gamma \in A} \langle \kappa, \gamma \rangle^{\langle \alpha, \gamma\rangle^2/2 - \langle \beta, \gamma \rangle^2/2}, \tag{5}$$

where $\kappa \in \mathbb{C}^N$ denotes an eigenvector of the classical monodromy operator M with the eigenvalue $\exp(2\pi i/h)$. The R.H.S. does not depend on the path connecting κ with $w^{-1}\kappa$ since $\mathcal{W}_\alpha$ is closed with logarithmic poles on some mirrors and with periods which are integer multiples of $2\pi i$. It does not depend on the normalization of κ since $\mathcal{W}_\alpha$ is homogeneous of degree 0. Also, the identity (see *i.g.* [5], Section *V*.6.2)

$$\sum_{\gamma \in A} \langle \gamma, x \rangle^2 = 2h \ \langle x, x \rangle$$

implies that $i_{\sum x_a \partial/\partial x_a} \mathcal{W}_\alpha = -2h$ and shows that $\int_\kappa^{M^{-1}\kappa} \mathcal{W}_\alpha = h^{-1} \int_\kappa^{M^{-h}\kappa} \mathcal{W}_\alpha = -4\pi i$ so that $a_{M\alpha} = a_\alpha$ as expected. While the ratios of a_α are determined by (5), the normalization of a_α is found from (3) which says that the average value of a_α is $(h+1)/12h^2$. Later we give two other description of the coefficients a_α — as certain limits and as explicit case-by-case values.

Conjecture. *The Hirota quadratic equation (1—5) coincides (up to certain rescaling of the variables q_k^a) with the corresponding ADE-hierarchy of Kac – Wakimoto* [17].

In Section 8 we confirm this conjecture in the cases A_N, D_4 and E_6. [3]

2. The total descendent potential. The second goal of this paper is to generalize to the ADE-singularities the result of [12] that the *total descendent potential* associated to the A_{n-1}-singularity in the axiomatic theory of topological gravity is a tau-function of the *nKdV* (or *Gelfand-Dickey*) hierarchy.

According to E. Witten's conjecture [22] proved by M. Kontsevich [18], the following generating function for intersection indices on the Deligne – Mumford spaces satisfies the equation of the KdV-hierarchy: [4]

$$\mathcal{D}_{A_1} = \exp \sum_{g,m} \frac{\hbar^{g-1}}{m!} \int_{\overline{\mathcal{M}}_{g,m}} \prod_{i=1}^m (\psi_i + \sum_{k=0}^\infty q_k \psi_i^k). \tag{6}$$

In the axiomatic theory, the total descendent potential is, by definition, an asymptotical function of the form

$$\mathcal{D} = \exp \sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^{(g)}(\mathbf{q})$$

[3]There has been a new development in the subject which leads, in particular, to the proof of the conjecture: motivated partly by this paper, E. Frenkel found a simple formula for the analogues in the Kac–Wakimoto theory of the coefficients a_α.

[4]Here ψ_i is the 1-st Chern class of the line bundle over $\overline{\mathcal{M}}_{g,m}$ formed by the cotangent lines to the curves at the i-th marked points.

where $\mathcal{F}^{(g)}$ are formal functions on $\mathcal{H}_+$ *near the point* $\mathbf{q} = -1z$. (Here 1 is the unit element in the local algebra H.) This convention called the *dilaton shift* is already explicitly present in (6). The formal functions $\mathcal{F}^{(g)}$ called the *genus g descendent potentials* are supposed to satisfy certain axioms dictated by Gromov–Witten theory. The axioms (while not entirely known) are to include the so called *string equation* (SE), *dilaton equation* (DE), *topological recursion relations* (TRR or $3g-2$-jet property) and *Virasoro constraints*.

According to [14], the genus-0 axioms SE+DE+TRR for $\mathcal{F}^{(0)}$ are equivalent to the following geometrical property $(*)$ of the Lagrangian submanifold $\mathcal{L} \subset \mathcal{H} = T^*\mathcal{H}_+$ defined as the graph of $d\mathcal{F}^{(0)}$:

$(*)$ $\mathcal{L}$ is a Lagrangian cone with the vertex at the origin and such that tangent spaces L to $\mathcal{L}$ are tangent to $\mathcal{L}$ exactly along zL.

In other words, the cone $\mathcal{L}$ is swept by the family $\tau \in H \mapsto zL_\tau$ of isotropic subspaces which form a variation of semi-infinite Hodge structures in the sense of S. Barannikov [3]. According to his results, this defines a Frobenius structure on the space of parameters τ.

In the case of ADE-singularities (and, more generally, finite reflection groups) the Frobenius structures have been constructed by K. Saito [20]. Consider the miniversal deformation

$$f_\tau(x) = f(x) + \tau^1\psi_1(x) + ... + \tau^N\psi_N(x),$$

where $\{\psi_a\}$ form a weighted-homogeneous basis in the local algebra H, and $\psi_N = 1$. The tangent spaces $T_\tau\mathcal{T}$ to the parameter space $\mathcal{T} \simeq \mathbb{C}^N$ are canonically identified with the algebras of functions on the critical schemes $crit(f_\tau)$: $\partial_{\tau^a} \mapsto \partial f_\tau/\partial\tau^a$ mod $(\partial f_\tau/\partial x)$. The multiplication $\bullet$ on the tangent spaces is Frobenius with respect to the following *residue metric*:

$$(\partial_{\tau_a}, \partial_{\tau_b})_\tau := (\frac{1}{2\pi i})^3 \oint\oint\oint \frac{\psi_a(x)\psi_b(x)\ \omega}{\frac{\partial f_\tau}{\partial x_1}\frac{\partial f_\tau}{\partial x_2}\frac{\partial f_\tau}{\partial x_3}}.$$

The residue metric is known to be flat and together with the Frobenius multiplication, the unit vectors ∂_{τ^N} and the *Euler vector field*

$$E := \sum_{a=1}^{N} (\deg\tau^a)\ \tau^a\ \partial_{\tau^a}, \quad \deg\tau^a = 1 - (m_a - 1)/h,$$

forms a conformal Frobenius structure on $\mathcal{T}$ (see [7]).

On the other hand, the condition $(*)$ involves only the symplectic structure Ω on $\mathcal{H}$ and the operator of multiplication by z and thus admits the following *twisted loop group* of symmetries:

$$L^{(2)}GL(H) = \{M \in End(H)\ ((1/z))\ |\ \ M(-z)^*M(z) = 1\}.$$

According to a result from [14], when the Frobenius structure associated to the cone $\mathcal{L}$ is semisimple, one can identify $\mathcal{L}$ with the Cartesian product $\mathcal{L}_{A_1} \times ... \times \mathcal{L}_{A_1}$ of $N = \dim H$ copies of the cone $\mathcal{L}_{A_1}$ defined by the genus 0 descendent potential $\mathcal{F}^{(0)}_{A_1} = \lim_{\hbar\to 0}\hbar\ln\mathcal{D}_{A_1}$. The identification is provided by a certain transformation M_τ from (a completed version of) $L^{(2)}GL(H)$ whose construction depends on the choice of a semisimple point τ.

A number of results in Gromov – Witten theory suggests that the higher genus theory inherits the symmetry group $L^{(2)}GL(H)$ (see [11, 14]). This motivates the following construction of the total descendent potential of a *semisimple* Frobenius manifold.

Adopt the following rules of quantization $\hat{}$ of quadratic hamiltonians. Let $\{..., p_a, ..., q_b, ...\}$ be a Darboux coordinate system on the symplectic space $(\mathcal{H}, \Omega)$ compatible with the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Then

$$(q_a q_b)\hat{} = q_a q_b/\hbar, \quad (q_a p_b)\hat{} = q_a \partial/\partial q_b, \quad (p_a p_b)\hat{} = \hbar \partial^2/\partial q_a \partial q_b.$$

This gives a *projective* representation of the Lie algebra $L^{(2)}gl(H)$ in the Fock space. The central extension is due to

$$[\hat{F}, \hat{G}] = \{F, G\}\hat{} + \mathcal{C}(F, G)$$

where $\mathcal{C}$ is the cocycle satisfying

$$\mathcal{C}(p_\alpha p_\beta, q_\alpha q_\beta) = \begin{cases} 1 \text{ if } \alpha \neq \beta, \\ 2 \text{ if } \alpha = \beta \end{cases}$$

and equal 0 for any other pair of quadratic Darboux monomials.

Introduce the *total descendent potential* as an asymptotical function:

$$\mathcal{D} := C(\tau)\ \hat{M}_\tau\ [\mathcal{D}_{A_1} \otimes ... \otimes \mathcal{D}_{A_1}],$$

where $\hat{M} := \exp(\ln M)\hat{}$, and $C(\tau)$ is a normalizing constant possibly needed to keep the R.H.S. independent of the choice of a semisimple point τ. This definition has been tested in [11, 12] and is known to agree with the TRR, SE, DE and the Virasoro constraints. Here is a more explicit description of M_τ and $C(\tau)$ in the form applicable to Frobenius manifolds of simple singularities.

Consider the complex oscillating integral

$$\mathcal{J}_{\mathfrak{B}}(\tau) = (-2\pi z)^{-3/2} \int_{\mathfrak{B}} e^{f_\tau(x)/z} \omega.$$

Here $\mathfrak{B}$ is a non-compact cycle from the relative homology group

$$\lim_{C \to \infty} H_m(\mathbb{C}^m, \{x : \mathrm{Re}(f_\tau/z) \leq -C\}) \simeq \mathbb{Z}^N.$$

We will use the notation $\partial_1, ..., \partial_N$ for partial derivative with respect to a flat (and weighted - homogeneous) coordinate system $(t^1, ..., t^N)$ of the residue metric. We treat the derivatives $z\partial_a \mathcal{J}_{\mathfrak{B}}$ as components of a covector field $z \sum \partial_a \mathcal{J}_{\mathfrak{B}} dt^a \in T^*\mathcal{T}$ which can be identified with a vector field via the residue metric and — via its Levi-Civita connection — with an H-valued function $J_{\mathfrak{B}}(z, \tau)$. According to K. Saito's theory these functions satisfy in flat coordinates the differential equations

$$z\partial_a J = (\partial_a \bullet) J \tag{7}$$

together with the homogeneity condition:

$$(z\partial_z - \mu + z^{-1} E \bullet) J = 0 \tag{8}$$

where $\mu = -\mu^*$ is the diagonal operator with the eigenvalues $1/2 - m_a/h$. The latter equation yields an *isomonodromic* family of connection operators $\nabla_\tau = \partial_z - \mu/z + (E\bullet)/z^2$ regular at $z = \infty$ and turning into $\partial_z - \mu/z$ at $\tau = 0$.

According to [8], there exists a (unique in the ADE-case) gauge transformation of the form $S_\tau(z) = \mathbf{1} + S_1(\tau) z^{-1} + S_2(\tau) z^{-2} + ...$ (*i.e.* near $z = \infty$) conjugating ∇_τ to ∇_0 and such that $S^*_\tau(-z) S_\tau(z) = \mathbf{1}$. It satisfies the homogeneity condition $(z\partial_z + L_E) S_\tau = [\mu, S_\tau]$.

On the other hand, let τ be semisimple. Then the functions f_τ have N non-degenerate critical points $x^{(a)}(\tau)$ with the critical values $u^a(\tau)$ and the Hessians

$\Delta_a(\tau)$. The local coordinate system $\{u^a\}$ (called *canonical*) diagonalizes the product $\bullet$ and the residue metric:

$$\partial/\partial u^a \bullet \partial/\partial u^b = \delta_{ab}\partial/\partial u^b, \quad (\partial/\partial u^a, \partial/\partial u^b)_\tau = \delta_{ab}\Delta_a^{-1}\partial/\partial u^b.$$

Define an orthonormal coordinate system

$$\Psi(\tau) : \mathbb{C}^N \to T_\tau\mathcal{T} = H, \quad \Psi(q^1, ..., q^N) = \sum q^a\sqrt{\Delta_a}\partial/\partial u^a,$$

and put $U_\tau = \operatorname{diag}[u^1(\tau), ..., u^N(\tau)]$. Stationary phase asymptotics of the oscillating integrals $J_{\mathfrak{B}_a}, a = 1, ..., N$, near the corresponding critical points $x^{(a)}$ yield a fundamental solutions to the system (7),(8) in the form

$$\Psi(\tau)R_\tau(z)e^{U_\tau/z}, \quad R_\tau = \mathbf{1} + R_1(\tau)z + R_2(\tau)z^2 + ..., \quad R_\tau^t(-z)R_\tau(z) = \mathbf{1}.$$

The matrix series R_τ satisfies the homogeneity condition $(z\partial_z + L_E)R_\tau = 0$ and, according to [11], an asymptotical solution with this property is unique up to reordering or reversing the basis vectors in $\mathbb{C}^N$.

Define

$$c(\tau) := \frac{1}{2}\int^\tau \sum_a R_1^{aa}(\tau)du^a$$

as the local potential of the 1-form $\sum R_1^{aa}du^a/2$ (which is known to be closed [7]).

In the above notations, the total descendent potential of the ADE-singularity assumes the form

$$\mathcal{D} = e^{c(\tau)}\ \hat{S}_\tau^{-1}\ \Psi(\tau)\ \hat{R}_\tau\ e^{\frac{\hat{U_\tau}}{z}}\ \mathcal{D}_{A_1}^{\otimes N}. \tag{9}$$

The R.H.S. is known to be independent of τ (see [11]) and defines $\mathcal{D}$ (up to a constant factor) as an asymptotical function of $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 z + \mathbf{q}_2 z^2 + ...$ in the formal neighborhood of $\mathbf{q} = \tau - z$ with semisimple τ.

Our main result is the following theorem.

Theorem 1. *The total descendent potential (9) of a simple singularity satisfies the corresponding Hirota quadratic equation (1–5).*

In Section 4, we discuss Hirota quadratic equations of the KdV-hierarchy. The plan for the proof of Theorem 1 is to reduce the Hirota quadratic equations for $\mathcal{D}$ to those for $\mathcal{D}_{A_1}$ by conjugating the vertex operators in (1,2) past the quantized symplectic transformations from (9). In Section 5, we describe the results of such conjugations by quoting corresponding theorems from [12]. The residue in (1) is computed in Section 6 and is compared with (2) in Section 7. The case-by-case tables for the coefficients a_α are presented in Section 8. A key to all our computations is the phase form and its properties discussed in next section.

3. The phase forms and the root systems. Consider a flat family of cycles $\phi \in H_2(f_\tau^{-1}(\lambda))$ in the non-singular Milnor fibers and define the *period vector* $I_\phi^{(0)}(\lambda, \tau) \in H$ by

$$(I_\phi^{(0)}(\lambda, \tau), \partial_a) := \partial_a\ \frac{(-1)}{2\pi}\int_{\phi\subset f_\tau^{-1}(\lambda)} \frac{\omega}{df_\tau}. \tag{10}$$

It is a multiple-valued vector function on the complement to the discriminant which turns into $I_\phi^{(0)}(\lambda)$ from Section 1 at $\tau = 0$.

The *phase form* $\tilde{\mathcal{W}}_{\alpha,\beta}$ (defined in [12], Section 7) is given by the formula

$$\tilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\tau) := \sum_{i=1}^{N} (I^{(0)}_{\alpha}(\lambda,\tau) \bullet I^{(0)}_{\beta}(\lambda,\tau), \partial_{\tau^a})\ d\tau^a.$$

It is a multiple-valued 1-form on the complement to the discriminant and depends bilinearly on the cycles α, β (to be chosen in $(f_0^{-1}(1)$ and transported to $f_\tau^{-1}(\lambda)$). According to [12], the phase forms have the following properties.

(1) $d\tilde{\mathcal{W}}_{\alpha,\beta} = 0$.
(2) $L_{\partial_\lambda + \partial_N}\tilde{\mathcal{W}}_{\alpha,\beta} = 0$, *i.e.* $\tilde{\mathcal{W}}$ is determined by the restriction
$$\mathcal{W}_{\alpha,\beta}(\tau) := \tilde{\mathcal{W}}_{\alpha,\beta}(0,\tau), \quad \tilde{\mathcal{W}}_{\alpha,\beta}(\lambda,\tau) = \mathcal{W}_{\alpha,\beta}(\tau - \lambda\mathbf{1}).$$
(3) $L_E\mathcal{W}_{\alpha,\beta} = 0$.
(4) Near a generic point of the discriminant $\Delta \subset \mathcal{T}$ the form $\mathcal{W}_{\alpha,\beta}$ becomes single-valued on the double cover and has a pole of order ≤ 1 on $\mathcal{D}$ (since $I^{(0)}_{\alpha}$ have a pole of order $\leq 1/2$).
(5) $\oint_{\delta_\gamma} \mathcal{W}_{\alpha,\beta} = -2\pi i\langle\alpha,\gamma\rangle\langle\beta,\gamma\rangle$, where γ is the cycle vanishing over a generic point of the discriminant, and δ_γ is a small loop going *twice* (in the positive direction defined by complex orientations) around the discriminant near this point.

Proposition 1.
$$\mathcal{W}_{\alpha\beta} = -\frac{1}{2}\sum_{\gamma\in A}\langle\alpha,\gamma\rangle\langle\beta,\gamma\rangle\frac{d\langle\gamma,x\rangle}{\langle\gamma,x\rangle}.$$

Proof. The phase form $\mathcal{W}_{\alpha,\beta}$ becomes single-valued on the *Chevalley cover* representing $\mathcal{T}$ as the quotient of $\mathbb{C}^N = H^2(f_0^{-1}(1),\mathbb{C})$ by the monodromy group. The properties (1) and (4) show that it has at most logarithmic pole on the mirrors $\langle\gamma,x\rangle = 0$. The property (5) controls the residues on the mirrors. The difference of the L.H.S. and the R.H.S. has to be a holomorphic 1-form, homogeneous of degree 0 by to the property (3), and therefore vanishes identically. □

Corollary 1. $i_E\mathcal{W}_{\alpha,\beta} = -\langle\alpha,\beta\rangle$.

Indeed, the Euler vector field becomes $h^{-1}\sum x_a\partial_{x_a}$ on the Chevalley cover, so that the equality follows from $\sum_{\gamma\in A}\langle\alpha,\gamma\rangle\langle\beta,\gamma\rangle = 2h\langle\alpha,\beta\rangle$. This is one more general property of phase forms established in [12].

Corollary 2. *The phase form $\mathcal{W}_\beta$ of Section 1 coincides with $\mathcal{W}_{\beta,\beta}$.*

Remark. The inverse to the Chevalley quotient map is given by the period map

$$\tau \mapsto [\omega/df_\tau] \in H^2(f_\tau^{-1}(0),\mathbb{C}) \rightsquigarrow H^2(f_0^{-1}(1),\mathbb{C}) \simeq \mathbb{C}^N.$$

The periods $I^{(0)}_{\alpha}$ are defined via the differential of the inverse Chevalley map and therefore represent parallel translations of the cycles α considered as *co*vectors in $\mathbb{C}^N$. The value of phase form $\mathcal{W}_{\alpha,\beta}$, which is also a covector, is constructed as the Frobenius product $\alpha \bullet \beta$ of *co*vectors (defined by the isomorphisms $T_\tau\mathcal{T} \simeq T^*_\tau\mathcal{T}$ based on the residue metric). Thus the formula

$$\alpha \bullet \beta := \frac{1}{2}\sum_{\gamma\in A}\frac{\langle\alpha,\gamma\rangle\langle\beta,\gamma\rangle}{\langle\gamma,x\rangle}\ \gamma$$

defines on $(\mathbb{C}^N)^*$ a family of commutative associative multiplications depending on the parameter x. [5] It would be interesting to find a representation-theoretic interpretation of this structure defined entirely in terms of the root system A.

We prove several further properties of phase forms needed in our computations.

Proposition 2. *In the case of ADE-singularities, suppose that β has integer intersection indices with all $\alpha \in A$ and is invariant under the monodromy around a discriminant-avoiding loop γ. Then $\oint_\gamma \mathcal{W}_{\beta,\beta} \in 2\pi i\ \mathbb{Z}$.*

Proof. This is Proposition 1 from Section 7 of [12]. □

It would be interesting to find out if the property remains valid for non-simple singularities.

We will see in Section 7 that the coefficients a_α introduced in Section 1 can be equivalently defined via the following limits b_α. Start with choosing $(\tau_1, ..., \tau_N) = -\mathbf{1} = (0, ..., 0, -1)$ in the role of the base point in $\mathcal{T}$ and identify A with the set of vanishing cycles in $H_2(f_{-\mathbf{1}}^{-1}(0)) = H_2(f_0^{-1}(1))$. Let us also fix $\tau \in \mathcal{T}$ such that f_τ is a Morse function, and let u will be one of the critical values of f_τ so that $\tau - u\mathbf{1} \in \Delta$. We may assume that $\tau - (u+1)\mathbf{1} \notin \Delta$ and that the straight segment connecting $\tau - (u+1)\mathbf{1}$ with $\tau - u\mathbf{1}$ does not intersect Δ. For each $\alpha \in A$, pick a discriminant-avoiding path γ_α connecting -1 with $\tau - (u+1)\mathbf{1}$ and further with $\tau - u\mathbf{1}$ along the straight segment and such that α becomes the vanishing cycle when transported along γ_α from $\mathbf{1}$ to $\tau - u\mathbf{1}$. Assuming that integration of the phase form is performed along this path we put

$$b_\alpha := \lim_{\varepsilon \to 0}\ \exp\left\{ -\int_{-\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\alpha,\alpha} - \int_{-1}^{-\varepsilon} \frac{2\ dt}{t} \right\} \tag{11}$$

Proposition 3. *The limit exists and does not depend on the choice of the path of integration provided that the path terminates at a generic point of the discriminant and that the cycle α transported along the path vanishes over this point.*

Proof. We may assume that $u = u^1$ is the first of the canonical coordinates $U = (u^1, ..., u^N)$, and therefore $u^1 = 0$ is the local equation of the discriminant branch. Since α is vanishing at the end of the path, the period vector $I_\alpha^{(0)}$ has the following expansion (here $\mathbf{1}_i$ stand for the standard basis vectors in $\mathbb{C}^N$):

$$\Psi_\tau^{-1} I_\alpha^{(0)}(\lambda, \tau) = \frac{\pm 2}{\sqrt{2(\lambda - u^1)}} \left(\mathbf{1}_1 + (\lambda - u^1) \sum a^i(U) \mathbf{1}_i + o(\lambda - u^1) \right).$$

Since $\Psi(\mathbf{1}_i) = \sqrt{\Delta_i}\partial/\partial u^i$, we have $(\mathbf{1}_i \bullet \mathbf{1}_j, \partial/\partial u^k) = \delta_{ij}\delta_{ik}$, and therefore

$$\mathcal{W}_{\alpha,\alpha} = \sum (I_\alpha^{(0)} \bullet I_\alpha^{(0)}, \partial/\partial u^k) du^k\ |_{\lambda=0} = \frac{2du^1}{-u^1} + 4a^1(U)du_1 + \mathcal{O}(-u_1).$$

We see that the integral $\int_{u_1=-1}^{u_1=0} \mathcal{W}_{\alpha,\alpha}$ diverges the same way as $-\int_{-1}^{0} 2dt/t$ so that the difference converges. This proves the existence of the limit. Removing this singular term we find that the integral $\int [4a_1(U)du^1 + \mathcal{O}(-u_1)]$ vanishes along any path inside the discriminant branch $u_1 = 0$. This shows that the limit b_α is locally constant as a function of the path's endpoint on the discriminant, and therefore — globally constant due to the irreducibility of the discriminant. Finally,

[5] We are thankful to V. A. Ginzburg who explained to us that this is a special case of a family of Frobenius structures constructed by A. P. Veselov [21].

precomposing a path with a discriminant-avoiding loop γ with trivial monodromy of the cycle α does not change b_α thanks to Proposition 2. $\square$

Corollary. *$a_\alpha/a_\beta = b_\alpha/b_\beta$ for all $\alpha, \beta \in A$.*

Proposition 4. *Let δ_ε be a small loop of radius ε around the discriminant near a generic point $\tau - u\mathbf{1}$, and let $\langle \alpha, \beta\rangle = \pm 1$, where β is the cycle vanishing at this point. Then $\lim_{\varepsilon\to 0}\oint_{\delta_\varepsilon} \mathcal{W}_{\alpha,\alpha} = -\pi i$.*

Proof. We have $\alpha = \pm\beta/2 + \alpha'$ where α' is invariant under the monodromy around δ_ε. Expanding $I^{(0)}_\alpha = I^{(0)}_{\alpha'} \pm I^{(0)}_{\beta/2}$ near $\lambda = u$ as in the proof of Proposition 3 we find

$$\oint_{\delta_\varepsilon} \mathcal{W}_{\alpha,\alpha} = \oint_{\delta_\varepsilon} \frac{du}{-2u} + \oint_{\delta_\varepsilon} \mathcal{O}(\sqrt{u})\ du = -\pi i + \mathcal{O}(\sqrt{\varepsilon}) \to -\pi i. \qquad \square$$

In fact this property has been already used in [12].

4. Two forms of the KdV-hierarchy. Consider the miniversal deformation of the A_1-singularity in the form $f_u(x) := (x_1^2 + x_2^2 + x_3^2)/2 + u$. The vanishing cycle α can be identified with the real sphere $(x_1^2 + x_2^2 + x_3^2) = 2(\lambda - u)$. The period is

$$\int_\alpha \frac{dx_1 \wedge dx_2 \wedge dx_3}{df_u} = \frac{d}{d\lambda}\frac{4}{3}\pi(2(\lambda-u))^{3/2} = 4\pi\sqrt{2(\lambda-u)}.$$

Since $(1,1) = \operatorname{Res} dx_1 \wedge dx_2 \wedge dx_3/x_1x_2x_3 = 1$, we have $I^{(-1)}_\alpha(\lambda, u) = 2\sqrt{2(\lambda-u)}$, and more generally, $I^{(k)}_{\pm\alpha}(\lambda, u) = \pm 2(d/d\lambda)^k(2(\lambda-u))^{-1/2}$, $k \in \mathbb{Z}$. The Coxeter transformation swaps α and $-\alpha$ and so $a_\alpha = a_{-\alpha} = (h+1)/12h^2 = 1/16$. The equation (1,2) in this example assumes the form

$$\operatorname{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda}\left[\sum_{\pm}\ {}^{A_1}\Gamma^{\pm\alpha}(\lambda) \otimes\ {}^{A_1}\Gamma^{\mp\alpha}(\lambda)\right](\Phi\otimes\Phi) = 16\ (l + \frac{1}{8})\ (\Phi\otimes\Phi), \tag{12}$$

where

$$l = \sum_{k\geq 0} \frac{2k+1}{2}(q_k \otimes 1 - 1 \otimes q_k)\ (\partial_{q_k} \otimes 1 - 1 \otimes \partial_{q_k}). \tag{13}$$

Here we use the notation ${}^{A_1}\Gamma^\phi(\lambda)$ to single out the vertex operators $\Gamma^\phi(\lambda)$ of the A_1-singularity. In order to identify the condition (12) for Φ with the KdV hierarchy in [16, 17] corresponding to the root system A_1, we denote $\sqrt{2\lambda}$ by ζ, rescale the variables by $q_k = (2k+1)!!t_{2k+1}$ and put $x_m = (t'_m + t''_m)/2$, $y_m = (t'_m - t''_m)/2$ where $m = 1, 3, 5, ...$ In this notation $l = \sum m y_m \partial_{y_m}$, and (12,13) becomes

$$\left[\operatorname{Res}\frac{d\zeta}{\zeta}\ e^{4\sum \zeta^m \frac{y_m}{\sqrt{\hbar}}}\ e^{-2\sum \frac{\zeta^{-m}}{m}\sqrt{\hbar}\partial_{y_m}} - 1 - 8\sum m y_m \partial_{y_m}\right]\ \Phi(\mathbf{x}+\mathbf{y})\ \Phi(\mathbf{x}-\mathbf{y}) = 0.$$

This coincides with the equation (14.13.1) in [16] characterizing tau-functions Φ of the KdV hierarchy.

Another form of the Hirota quadratic equation for Φ is based on the representation of the KdV-hierarchy as the mod 2-reduction of the KP-hierarchy (see [16], Section 14.11). It can be rephrased (see [12]) as the condition

$$\left[\sum_{\pm}\ {}^{A_1}\Gamma^{\pm\alpha/2}(\lambda) \otimes\ {}^{A_1}\Gamma^{\mp\alpha/2}(\lambda)\frac{d\lambda}{\pm\sqrt{\lambda}}\right]\ (\Phi\otimes\Phi)\ \textit{has no pole in } \lambda. \tag{14}$$

Indeed, in the previous notations this can be rewritten as the property

$$e^{2\sum \zeta^m \frac{y_m}{\sqrt{\hbar}}}\ e^{-\sum \frac{\zeta^{-m}}{m}\sqrt{\hbar}\partial_{y_m}}\ \Phi(\mathbf{x}+\mathbf{y})\Phi(\mathbf{x}-\mathbf{y})\ \textit{contains no}\ \zeta^{-m}\ \textit{for odd}\ m>0.$$

This coincides with the mod 2-reduction of the KP-hierarchy of the Hirota equation (14.11.5) in [16]. According to a result from [16], Section 14.13, this condition is actually equivalent to (12).

In Section 6 we will use the fact that (according to Kontsevich's theorem) the function $\Phi = \mathcal{D}_{A_1}$ satisfies both forms (12) and (14) of the KdV-hierarchy.

5. Symplectic transformations of vertex operators. Generalizing the construction of Section 1, introduce the vertex operator $\Gamma^\phi_\tau(\lambda)$ corresponding to the vector $\mathbf{f} \in H[[z, z^{-1}]]$ of the form

$$\mathbf{f}^\phi_\tau(\lambda) := \sum_{k\in\mathbb{Z}} I^{(k)}_\phi(\lambda,\tau)(-z)^k.$$

Here $I^{(0)}_\phi$ is the period vector introduced in Section 3, and $I^{(k)}_\phi := d^k I^{(0)}_\phi/d\lambda^k$ as before. For $k<0$ the integration constants are taken "equal 0" so that $I^{(k)}_\phi$ satisfy the homogeneity conditions:

$$(\lambda\partial_\lambda + L_E)\ I^{(k)}_\phi(\lambda,\tau) = (\mu - \frac{1}{2} - k)\ I^{(k)}_\phi(\lambda,\tau).$$

In particular Γ^ϕ_0 coincides with the vertex operator Γ^ϕ from Section 1. We state below several results about behavior of the vertex operators under conjugation by some symplectic transformations and refer to Sections $5, 6, 7$ of [12] for the proofs.

Theorem A (see Proposition 2 in [12]).

$$\hat{S}_\tau\ \Gamma^\phi_0(\lambda)\ \hat{S}_\tau^{-1}\ =\ \exp\left\{\frac{1}{2}\int_{-\lambda\mathbf{1}}^{\tau-\lambda\mathbf{1}} \mathcal{W}_{\phi,\phi}\right\}\ \Gamma^\phi_\tau(\lambda)$$

We have to stress here that in order to compare the vertex operators $\Gamma^\phi_0(\lambda)$ and $\Gamma^\phi_\tau(\lambda)$ one needs to transport the cycle ϕ from $f_0^{-1}(\lambda)$ to $f_\tau^{-1}(\lambda)$ along a path in $\mathcal{T}$ connecting $-\lambda\mathbf{1} = (0, ..., -\lambda)$ with $\tau - \lambda\mathbf{1} = (\tau_1, ..., \tau_N - \lambda)$ and avoiding the discriminant Δ corresponding to singular levels $f_\tau^{-1}(0)$. It is assumed in the formulation of the theorem that the integral of the phase form is taken along this very path. Similar conventions apply to other formulas of this and following sections involving integration of phase forms.

Now let the cycle $\phi \in H_2(f_\tau^{-1}(\lambda)$ be written as the sum $\phi = \langle\phi,\beta\rangle\beta/2 + \phi'$ where $\langle\phi',\beta\rangle = 0$. Here β is the cycle *vanishing* at a non-degenerate critical point of the function f_τ with the critical value u and transported to $f_\tau^{-1}(\lambda)$ along a discriminant-avoiding path connecting $\tau - \lambda\mathbf{1}$ and $\tau - u\mathbf{1}$.

Theorem B (see Proposition 4 in [12]).

$$\Gamma^\phi_\tau(\lambda) = \exp\left\{\frac{\langle\phi,\beta\rangle}{2}\int_{\tau-\lambda\mathbf{1}}^{\tau-u\mathbf{1}} \mathcal{W}_{\beta,\phi'}\right\}\ \Gamma^{\phi'}_\tau(\lambda)\ \Gamma_\tau^{\frac{\langle\phi,\beta\rangle}{2}\beta}(\lambda)$$

The integral here is taken along the path terminating on the discriminant where the phase form is singular. However the singularity is proportional to $(\lambda-u)^{-1/2}$ and is therefore integrable.

Let us recall that the columns of the matrix R_τ in the asymptotical expansion $\Psi(\tau)R_\tau(z)\exp(U/z)$ correspond to non-degenerate critical points of the Morse function f_τ with the critical values $u^i(\tau)$. Let β_i be the cycle vanishing over u^i.

Theorem C (see Proposition 3 in [12]).

$$(\Psi(\tau)\hat{R}_\tau)^{-1}\ \Gamma_\tau^{c\beta_i}(\lambda)\ (\Psi(\tau)\hat{R}_\tau) = e^{c^2 W_i/2}\ [\cdots \mathbf{1}\otimes(\ {}^{A_1}\Gamma_{u^i}^{c\beta}(\lambda))^{(i)}\otimes\mathbf{1}\cdots],$$

where

$$W_i := \int_{\tau-\lambda\mathbf{1}}^{\tau-u^i\mathbf{1}} \left(\mathcal{W}_{\beta_i,\beta_i} - \frac{2\ dt^N}{\tau^N - u^i - t^N}\right),$$

and ${}^{A_1}\Gamma_u^{c\beta}(\lambda)$ *is the vertex operator of the A_1-singularity with the miniversal deformation $\frac{x_1^2}{2}+\frac{x_2^2}{2}+\frac{x_3^2}{2}+u$ corresponding to the c-multiple of the vanishing cycle.*

The behavior of $I_{\beta_i}^{(0)}$ near $\lambda = u^i$ is described by the asymptotics

$$\Psi^{-1}(\tau)\ I_{\beta_i}^{(0)}(\lambda,\tau) = \frac{2}{\sqrt{2(\lambda-u^i)}}\ (\mathbf{1}_i + ...)$$

where $\mathbf{1}_i$ is the i-th basis vector in $\mathbb{C}^N$, and the dots mean higher order powers of $\lambda - u^i$. Respectively, the vertex operator of the A_1-singularity is more explicitly defined by the series $\mathbf{f}\in\mathbb{C}[[z,z^{-1}]]$ of the form

$$\mathbf{f} = \sum_{k\in\mathbb{Z}} \frac{d^k}{d\lambda^k}\frac{2c}{\sqrt{2(\lambda-u)}}\ (-z)^k,$$

where the branch of the square root should be the same as in the above asymptotics. The subscript (i) indicates the position of the vertex operator in the tensor product operator acting on the Fock space of functions of $(\mathbf{q}^{(1)},...,\mathbf{q}^{(N)}) = \Psi^{-1}(\tau)\mathbf{q}$. The integrand in the formula for W_i considered as a 1-form in the space with coordinates $(t^1,...,t^N)$ identical to parameters of the miniversal deformation, while the notation $\tau = (\tau^1,...,\tau^N)$ is reserved for expressing the limits of integration. The phase form $\mathcal{W}$ has a non-integrable singularity at $t = \tau - u^i\mathbf{1}$ which happens to cancel out with that of the subtracted term so that the difference is integrable.

Finally, the following result is the special case of Theorem A corresponding to the A_1-singularity.

Theorem D (see Proposition 3 in [12]).

$$e^{-(u/z)^\wedge}\ {}^{A_1}\Gamma_u^{c\beta}(\lambda)\ e^{(u/z)^\wedge} = \exp\left\{-\frac{c^2}{2}\int_{\lambda-u^i}^{\lambda}\frac{2\ dt}{t}\right\}\ {}^{A_1}\Gamma_0^{c\beta}(\lambda)$$

In fact this result can be obtained more directly using Taylor's formula. Indeed, for any analytic function $I^{(0)}$ we have

$$e^{-u/z}\left[\sum_{k\in\mathbb{Z}} I^{(k)}(\lambda)(-z)^k\right] e^{u/z} = \sum_{k\in\mathbb{Z}} I^{(k)}(\lambda+u)(-z)^k$$

provided that $|u|$ does not exceed the convergence radius of $I^{(0)}$ at λ. Thus the transformation in the theorem effectively consists in the translation $\sqrt{\lambda-u}\rightsquigarrow\sqrt{\lambda}$ along an origin-avoiding path. The integral in the exponent should be taken along this path.

6. The residue sum. In this section, we compute the residue sum

$$\mathrm{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda} \left[\sum_{\alpha\in A} b_\alpha \Gamma_0^\alpha(\lambda) \otimes \Gamma_0^{-\alpha}(\lambda) \right] \mathcal{D}^{\otimes 2} \tag{15}$$

assuming that the coefficients b_α are defined as in Proposition 3.

Introduce the *total ancestor potential*

$$\mathcal{A}_\tau := \hat{S}_\tau \mathcal{D} = e^{c(\tau)} \Psi(\tau) \hat{R}_\tau e^{(U_\tau/z)\hat{}} \mathcal{D}_{A_1}^{\otimes N}.$$

Applying Theorem A of the previous section we find that (15) can be rewritten as

$$\mathrm{Res}_{\lambda=\infty} \lambda d\lambda \left[\sum_{\alpha\in A} c_\alpha \Gamma_\tau^\alpha(\lambda) \otimes \Gamma_\tau^{-\alpha}(\lambda) \right] \mathcal{A}_\tau^{\otimes 2}, \tag{16}$$

where

$$c_\alpha = \lim_{\varepsilon\to 0} \exp \left\{ -\int_{\tau-\lambda\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\alpha,\alpha} - \int_{-1}^{-\varepsilon} \frac{2\ dt}{t} \right\}.$$

assuming that $\alpha \in H_2(f_\tau^{-1}(\lambda))$ vanishes at $\lambda = u$ when transported along the path of integration of the phase form. Note that the factor $\lambda^{-1}d\lambda$ in (15) is replaced by $\lambda d\lambda$ in (16) due to Corollary 1 from Section 3 which shows that

$$\exp\left\{ -\int_{-\mathbf{1}}^{-\lambda\mathbf{1}} \mathcal{W}_{\alpha,\alpha} \right\} = \exp\left\{ \langle\alpha,\alpha\rangle \int_{-1}^{-\lambda} \frac{dt}{t} \right\} = \lambda^2.$$

The ancestor potential $\mathcal{A}_\tau = \exp \sum \hbar^{(g-1)} \mathcal{F}_\tau^{(g)}$ is a *tame* asymptotical function in the following sense: $\mathcal{F}_\tau^{(g)}$ considered as a formal function of $t_k^a = q_k^a + \delta_{k1}\delta_{aN}$ satisfy

$$\frac{\partial^r \mathcal{F}_\tau^{(g)}}{\partial t_{k_1}^{a_1} \ldots \partial t_{k_r}^{a_r}}|_{\mathbf{t}=0} = 0 \quad \text{whenever} \quad k_1 + \ldots + k_r > 3g - 3 + r.$$

This follows from the analogous property of $\mathcal{D}_{A_1}^{\otimes N}$, from the invariance of $\mathcal{D}_{A_1}$ under the string flow $\exp(u/z)\hat{}$ and from the "upper-triangular" property of $\hat{R}_\tau$. We refer to Proposition 5 in [12] for the proof. It is also shown in Section 8 of [12] that for tame asymptotical functions Φ the vertex operator expressions $\Gamma_\tau^\phi(\lambda) \otimes \Gamma_\tau^{-\phi}(\lambda)\ \Phi^{\otimes 2}$ can be considered not only as series expansions in fractional powers of λ near $\lambda = \infty$, but also as multiple-valued analytical functions defined over the entire range of λ and ramified only on the discriminant. Moreover, the sum in (16) is manifestly invariant under the entire monodromy group (= the ADE-reflection group). Therefore the sum is actually a single-valued differential 1-form on the complement to $\mathcal{D}$. Thus the residue (16) at $\lambda = \infty$ coincides with the sum of residues at the critical values $\lambda = u_i$ of the function f_τ. Our next goal is to take $u = u_i$ and compute the residue.

In a neighborhood of $\lambda = u$, the monodromy group reduces to $\mathbb{Z}_2$ generated by the reflection σ in the hyperplane orthogonal to two vanishing cycles which we denote $\pm\beta$.

First, consider the summand in (16) corresponding to a σ-invariant cycle $\alpha \in A$. The period vectors $I_\alpha^{(k)}(\lambda, \tau)$ are therefore single-valued analytic functions near $\lambda = u$. In particular, $\ln c_\alpha$, which differs from a constant by $\int_{\tau-(u+1)\mathbf{1}}^{\tau-\lambda\mathbf{1}} \mathcal{W}_{\alpha,\alpha}$, is analytic too. We conclude that $\lambda\ c_\alpha\ \Gamma_\tau^\alpha(\lambda) \otimes \Gamma_\tau^{-\alpha}(\lambda)\ \mathcal{A}^{\otimes 2}$ has no pole at $\lambda = u$.

Next, consider a pair of cycles $\alpha_\pm \in A$ transposed by σ and having intersection indices ± 1 with β. We have $\alpha_\pm = \alpha' \pm \beta/2$ where $\sigma\alpha' = \alpha'$. We use Theorem B to replace $\Gamma_\tau^{\alpha_\pm}$ with $\Gamma_\tau^{\alpha'}\Gamma_\tau^{\pm\beta/2}$ and then commute $\Gamma_\tau^{\pm\beta/2}$ across $\Psi\hat{R}\exp(U/z)\hat{}$ using Theorems C and D. The terms from (16) corresponding to $\alpha = \alpha_\pm$ turn into

$$\lambda\, d\lambda\, \left[\Gamma_\tau^{\alpha'}(\lambda) \otimes \Gamma_\tau^{-\alpha'}(\lambda)\right] \left(\Psi(\tau)\hat{R}_\tau e^{(U_\tau/z)\hat{}} \otimes \Psi(\tau)\hat{R}_\tau e^{(U_\tau/z)\hat{}}\right) \tag{17}$$
$$\times \left[...\mathbf{1} \otimes \left(\sum_\pm d_\pm \ {}^{A_1}\Gamma_0^{\pm\beta/2}(\lambda) \otimes \ {}^{A_1}\Gamma_0^{\mp\beta/2}(\lambda) \right)^{(i)} \otimes \mathbf{1}... \right] \left(\mathcal{D}_{A_1}^{\otimes N} \otimes \mathcal{D}_{A_1}^{\otimes N}\right).$$

The coefficients $d_\pm$ here are

$$d_\pm = \lim_{\varepsilon\to 0} \exp\left\{ -\oint_{\tau-\lambda\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\alpha_\pm,\alpha_\pm} - \int_{-1}^{-\varepsilon} \frac{2dt}{t} \pm \int_{\tau-\lambda\mathbf{1}}^{\tau-u\mathbf{1}} \mathcal{W}_{\alpha',\beta} + \right.$$
$$\left. \int_{\tau-\lambda\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\beta/2,\beta/2} + \int_{u-\lambda}^{-\varepsilon} \frac{dt}{2t} - \int_{u-\lambda}^{-\lambda} \frac{dt}{2t} \right\}. \tag{18}$$

We have to emphasize that all integrals here except the first one are taken along a short path near $\lambda = u$ making β vanish while in the first integral this path is precomposed with a loop transforming $\alpha_\pm$ to β.

Let us take $\lambda = u + 1$ for the base point for such a loop $\gamma_\pm$ and rearrange the first integral as

$$-\int_{\gamma_\pm} \mathcal{W}_{\alpha_\pm,\alpha_\pm} - \int_{\tau-(u+1)\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\beta,\beta} + \int_{\tau-(u+1)\mathbf{1}}^{\tau-\lambda\mathbf{1}} \mathcal{W}_{\alpha_\pm,\alpha_\pm}.$$

Combining this with $\mathcal{W}_{\alpha_\pm,\alpha_\pm} = \mathcal{W}_{\alpha',\alpha'} \pm \mathcal{W}_{\alpha',\beta} + \mathcal{W}_{\beta/2,\beta/2}$ we can rewrite the exponent in (18) as

$$-\int_{\gamma_\pm} \mathcal{W}_{\alpha_\pm,\alpha_\pm} - \int_{\tau-(u+1)\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\beta,\beta} - \int_{-1}^{-\varepsilon} \frac{2dt}{t} \pm \int_{\tau-(u+1)\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\alpha',\beta} \tag{19}$$
$$\int_{\tau-(u+1)\mathbf{1}}^{\tau-\lambda\mathbf{1}} \mathcal{W}_{\alpha',\alpha'} + \int_{\tau-(u+1)\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}} \mathcal{W}_{\beta/2,\beta/2} + \int_{-1}^{-\varepsilon} \frac{dt}{2t} \tag{20}$$
$$-\int_{-1}^{-\varepsilon} \frac{dt}{2t} - \int_{-\varepsilon}^{u-\lambda} \frac{dt}{2t} - \int_{u-\lambda}^{-\lambda} \frac{dt}{2t}. \tag{21}$$

The integrals in (21) add up to $-\int_{-1}^{-\lambda} dt/2t$ and contribute $\lambda^{-1/2}$ to the coefficients $d_\pm$. The sum in (20) is a function of λ analytic near $\lambda = u$ (since α' is σ-invariant) and is the same for both cycles $\alpha_\pm$. The values of (19) may depend on the cycle $\alpha_\pm$ but are independent of λ. We claim that in the limit $\varepsilon \to 0$ the difference is an odd multiple of πi. Indeed, transporting α_- along the composition $\gamma_-\gamma_+^{-1}$ yields α_+. On the other hand $2\mathcal{W}_{\alpha',\beta} = \mathcal{W}_{\alpha_+,\alpha_+} - \mathcal{W}_{\alpha_-,\alpha_-}$. Thus the difference of the two values of (19) can be interpreted as $\oint \mathcal{W}_{\alpha_-,\alpha_-}$ along a loop γ_ε starting and terminating at $\tau - (u+\varepsilon)\mathbf{1}$ and transporting α_- to α_+. Let us compose it with a small loop δ_ε of radius ε around $\lambda = u$. Since α_+ transports along this loop back to α_-, the composite integral $\int_{\gamma_\varepsilon\delta_\varepsilon} \mathcal{W}_{\alpha_-,\alpha_-} \in 2\pi i\mathbb{Z}$ due to Proposition 2 and does not depend on ε. Our claim follows therefore from Proposition 4.

We conclude that $d_\pm = \pm d_0(\lambda)\lambda^{-1/2}$ where d_0 is a non-vanishing analytic function near $\lambda = u$. Now we use the fact that $\mathcal{D}_{A_1}$ is a tau-function of the KdV-hierarchy (14) to conclude that the factor in (17) of the form

$$\sum_{\pm} \pm \frac{d\lambda}{\sqrt{\lambda}} \left[{}^{A_1}\Gamma_0^{\pm\beta/2}(\lambda) \otimes {}^{A_1}\Gamma_0^{\mp\beta/2}(\lambda) \right] (\mathcal{D}_{A_1} \otimes \mathcal{D}_{A_1})$$

is everywhere analytic in λ. The same remains true after application of the operator $(\Psi \hat{R} e^{(U/z)\hat{}})^{\otimes 2}$. The vertex operator $\Gamma_\tau^{\alpha'} \otimes \Gamma_\tau^{-\alpha'}$ is analytic near $\lambda = u$ since α' is σ-invariant. Thus (17) has no pole at $\lambda = u$ and contributes 0 to the residue sum.

Finally, consider the summands in (15) with $\alpha = \pm\beta$. Applying Theorems C and D, we transform the corresponding summands from (16) to the form

$$\left(\Psi(\tau)\hat{R}_\tau e^{(U/z)\hat{}}\right)^{\otimes 2} \left[...\mathcal{D}_{A_1}^{\otimes 2} \otimes \left(\sum_{\pm} e\lambda d\lambda \left({}^{A_1}\Gamma_0^{\pm\beta} \otimes {}^{A_1}\Gamma_0^{\mp\beta} \right) \mathcal{D}_{A_1}^{\otimes 2} \right)^{(i)} \otimes \mathcal{D}_{A_1}^{\otimes 2} ... \right],$$

where

$$e = \exp\left\{ -\int_{-1}^{-\varepsilon} \frac{2dt}{t} - \int_{-\varepsilon}^{u-\lambda} \frac{2dt}{t} - \int_{u-\lambda}^{-\lambda} \frac{2dt}{t} \right\} = \exp\left\{ -\int_{-1}^{-\lambda} \frac{2dt}{t} \right\} = \lambda^{-2}.$$

The contribution of these terms to the residue sum (16) at $\lambda = u^i$ can be calculated using the form (12) of the KdV-hierarchy for $\mathcal{D}_{A_1}$ and is equal to

$$16\ (\Psi \hat{R} e^{(U/z)\hat{}})^{\otimes 2}\ l^{(i)}\ (\mathcal{D}_{A_1}^{\otimes N})^{\otimes 2}, \text{ where } l^{(i)} = (...1 \otimes l \otimes 1...).$$

In order to justify this conclusion, recall from the end of Section 5 that conjugation by $\exp(u^i/z)$ act as translation $\lambda - u^i \mapsto \lambda$. Also, since $\mathcal{D}_{A_1}$ is tame, the vertex operator expression in (12) yields a meromorphic 1-form in λ with a singularity only at $\lambda = 0$. Thus the residue in (12) at $\lambda = \infty$ is the same as at $\lambda = 0$.

Let us summarize our computation.

Proposition 5. *The residue sum (15) is equal to*

$$16\ \left(e^c \hat{S}^{-1} \Psi \hat{R} e^{(U/z)\hat{}}\right)^{\otimes 2} \left(\frac{N}{8} + \sum_{i=1}^{N} l^{(i)} \right) (\mathcal{D}_{A_1}^{\otimes N})^{\otimes 2}. \tag{22}$$

7. The Virasoro operator. Functions of the form $\Phi \otimes \Phi$ belong to a Fock space which is the quantization of the symplectic space $\mathcal{H} \oplus \mathcal{H}$, the direct sum of two copies of $(\mathcal{H}, \Omega)$. Respectively the operator

$$\sum_{k \geq 0} \sum_a (\frac{m_a}{h} + k)(q_k^a \otimes 1 - 1 \otimes q_k^a)(\frac{\partial}{\partial q_k^a} \otimes 1 - 1 \otimes \frac{\partial}{\partial q_k^a}) \tag{23}$$

in (2) is the quantization of a certain quadratic hamiltonian $\Omega(D\mathbf{f}, \mathbf{f})/2$ on $\mathcal{H} \oplus \mathcal{H}$. Let us describe the infinitesimal symplectic transformation D explicitly.

Introduce the *Virasoro operator* $l_0 := z\partial_z + 1/2 - \mu$. [6] Since $\mu^* = -\mu$, the operator $l_0 : \mathcal{H} \to \mathcal{H}$ is anti-symmetric with respect to Ω, and the corresponding

[6] The name comes from the property of the operators $l_m := l_0 z l_0 z ... z l_0$, ($z$ repeated m times, $m = -1, 0, 1, 2, ...$) to form a Lie algebra isomorphic to the algebra of formal vector fields $x^{m+1}\partial/\partial x$ on the line and participating in the formulation of the Virasoro constraints (see [11, 14]).

quadratic hamiltonian reads

$$\frac{1}{4\pi i}\oint(l_0\mathbf{f}(-z),\mathbf{f}(z))\,dz=\sum_{k\geq 0}((k+\frac{1}{2}-\mu)f_k,(-1)^kf_{-1-k})=-\sum_{k\geq 0}\sum_{a=1}^{N}(\frac{m_a}{h}+k)q_k^ap_k^a.$$

Comparing this with (23) we conclude that

$$D=\begin{bmatrix}-l_0 & l_0\\ l_0 & -l_0\end{bmatrix}\ \in\ \mathrm{End}(\mathcal{H}\oplus\mathcal{H}). \tag{24}$$

The expression (2) on the R.H.S. of the Hirota equation is proportional to

$$\hat{D}\mathcal{D}^{\otimes 2}=\hat{M}^{\otimes 2}\ (\hat{M}^{\otimes 2})^{-1}\hat{D}(\hat{M}^{\otimes 2})(\mathcal{D}_{A_1}^{\times N})^{\otimes 2},$$

where $\hat{M}=e^{c(\tau)}\hat{S}_\tau^{-1}\Psi(\tau)\hat{R}_\tau e^{(U_\tau/z)\hat{}}$. Note that $\hat{M}^{\otimes 2}$ is the quantization of a block-diagonal operator

$$B:=\begin{bmatrix}M & 0\\ 0 & M\end{bmatrix},\ \text{and } B^{-1}DB=\begin{bmatrix}-M^{-1}l_0M & M^{-1}l_0M\\ M^{-1}l_0M & -M^{-1}l_0M\end{bmatrix}.$$

Proposition 6. $M^{-1}l_0M=\sum_{i=1}^N(\ ^{A_1}l_0)^{(i)}$.

Proof. We have

$$S(z\partial_z+\frac{1}{2}-\mu)S^{-1}=z\partial_z+\frac{1}{2}-\mu+\frac{E\bullet}{z},$$

since $(z\partial_z+L_E)S=\mu S-S\mu$ and $\partial_aS=z^{-1}\partial_a\bullet S$. Next, in the canonical coordinates $E=\sum u^i\partial/\partial u^i$, and therefore

$$\Psi^{-1}(z\partial_z+\frac{1}{2}-\mu+\frac{E\bullet}{z})\Psi=z\partial_z+\frac{1}{2}-V+\frac{U}{z},\ \text{where } V:=\Psi^{-1}\mu\Psi=\Psi^{-1}L_E\Psi.$$

Furthermore, the differential equations $\partial_a(\Psi Re^{U/z})=z^{-1}(\partial_a\bullet)(\Psi Re^{U/z})$ translate into $(d+\Psi^{-1}d\Psi)R=z^{-1}(dU\ R-R\ dU)$. This implies $(L_E+V)R=z^{-1}(UR-RU)$, which together with the homogeneity condition $(z\partial_z+L_E)R=0$ shows that

$$R^{-1}(z\partial_z+\frac{1}{2}-V+\frac{U}{z})R=z\partial_z+\frac{1}{2}+\frac{U}{z}.$$

Finally

$$e^{-U/z}(z\partial_z+\frac{1}{2}+\frac{U}{z})e^{U/z}=z\partial_z+\frac{1}{2}. \qquad \square$$

Proposition 7. $\hat{M}^{-1}\hat{l}_0\hat{M}=(M^{-1}l_0M)\hat{}+\mathrm{tr}\,\mu\mu^*/4$.

Proof. The quadratic hamiltonians for $z\partial_z+1/2,\mu,V,\ln S,(E\bullet)/z,U/z$ contain no p^2-terms, and the quadratic hamiltonians for $z\partial_z+1/2,\mu,V,\ln R$ contain no q^2-terms. Therefore, in the quantized version of the previous computation, the only point where the cocycle $\mathcal{C}$ makes a non-trivial contribution is:

$$\hat{R}^{-1}\ (\frac{U}{z})\hat{}\ \hat{R}=(R^{-1}\frac{U}{z}R)\hat{}\ +\ C.$$

Let $A=\ln R, B=U/z$. Then the quadratic hamiltonian of $BA-AB$ contains no q^2-terms (since $R|_{z=0}=1$). We have therefore

$$\frac{d}{dt}e^{-t\hat{A}}\hat{B}e^{t\hat{A}}=e^{-t\hat{A}}(\hat{B}\hat{A}-\hat{A}\hat{B})e^{t\hat{A}}=e^{-t\hat{A}}\ (BA-AB)\hat{}\ e^{t\hat{A}}+\mathcal{C}(B,A)=$$

$$\left[e^{-tA}(BA-AB)e^{tA}\right]\hat{}\ +\ \mathcal{C}(B,A)=\frac{d}{dt}\left[e^{-tA}Be^{tA}\right]\hat{}+\mathcal{C}(B,A).$$

Integrating in t from 0 to 1 we find $C = \mathcal{C}(B, A)$. Since $A = R_1 z + o(z)$, we compute explicitly $C = \operatorname{tr}(BA)/2 = \sum_i R_1^{ii} u^i/2$.

This expression, which seems to be a function of τ, has to be a constant, and the value of this constant is well-known to be $\operatorname{tr}\mu\mu^*/4$ (see for instance the last chapter in [15]). For the sake of completeness we include the computation. Namely, comparing the z^0- and z^1-terms in the equation $(L_E + V)R = z^{-1}(UR - RU)$ we find $V^{ij} = (u^i - u^j)R_1^{ij}$ and respectively

$$R_1^{ii} = -L_E R_1^{ii} = \sum_j (u^i - u^j) R_1^{ij} R_1^{ji} = \sum_j \frac{V^{ij}V^{ji}}{u^j - u^i}.$$

Thus we have

$$\frac{1}{2}\sum_i u^i R_1^{ii} = \sum_{ij} \frac{u^i V^{ij} V^{ji}}{2(u^j - u^i)} = \sum_{ij} \frac{u^j V^{ij} V^{ji}}{2(u^i - u^j)} = -\frac{1}{4}\sum_{ij} V^{ij}V^{ji} = \frac{1}{4}\operatorname{tr}\mu\mu^*,$$

since $V = \Psi^{-1}\mu\Psi$ and $V^t = \Psi^{-1}\mu^*\Psi$. $\square$

Remark. Slightly generalizing Propositions 6 and 7 one obtains the following transformation formula (see Theorem 8.1 in [11]) $\hat{M}^{-1}\hat{l}_m\hat{M} = \sum_i {}^{A_1}\hat{l}_m^{(i)}$ for the Virasoro operators with $m \neq 0$. Since $(\hat{l}_m - \delta_{m,0}/16)\mathcal{D}_{A_1} = 0$, this implies that $\mathcal{D} = \hat{M}\mathcal{D}_{A_1}^{\otimes N}$ satisfies the Virasoro constraints $[\hat{l}_m - \delta_{m,0}\operatorname{tr}(\mu\mu^*/4 + 1/16)]\mathcal{D} = 0$. In fact this is Corollary 8.2 in [11] specialized to Frobenius structures of weighted-homogeneous singularities.

Note that the conjugation $l_0 \mapsto M^{-1}l_0M$ of the *off*-diagonal blocks in the matrix D yields after quantization $\hat{M}^{-1}\hat{l}_0\hat{M} = (M^{-1}l_0M)\hat{}$ (since the cocycle $\mathcal{C}$ vanishes on pairs of quadratic hamiltonians corresponding to block-diagonal and block-off-diagonal operators.) Thus $\hat{B}^{-1}\hat{D}\hat{B} = \sum_i l^{(i)} - \operatorname{tr}\mu\mu^*/2$. Taking into account that

$$\frac{N(h+1)}{12h} = \sum_a \frac{m_a(h-m_a)}{2h^2} = \frac{1}{2}\sum_a (\frac{1}{2} + \mu_a)(\frac{1}{2} - \mu_a) = \frac{1}{2}\operatorname{tr}(\frac{1}{4} + \mu\mu^*)$$

we conclude that the R.H.S. of the Hirota equation (1,2) can be written as

$$\hat{M}^{\otimes 2}\ (\frac{N}{8} + \sum_i l^{(i)})\ (\mathcal{D}_{A_1}^{\otimes N})^{\otimes 2}.$$

Comparing this with Proposition 5 we arrive at the following result.

Proposition 8. *The function $\mathcal{D}$ satisfies the Hirota quadratic equation (1,2) with $a_\alpha = b_\alpha/16$.*

Since $\mathcal{D} \neq 0$, the Hirota equation is thus rendered consistent, and the following corollary completes the proof of Theorem 1.

Corollary. *The average value*

$$\frac{1}{Nh}\sum_{\alpha\in A} b_\alpha = \frac{4(h+1)}{3h^2}.$$

Note that in the proof of Theorem 1 we use neither the Virasoro constraints for $\mathcal{D}_{A_1}$ nor the fact that the N factors in $\mathcal{D}_{A_1}^{\otimes N}$ are the same. The only relevant conditions for $\mathcal{D}_{A_1}$ were both forms of the KdV-hierarchy and the tame property of $\exp(u/z)\hat{}\mathcal{D}_{A_1}$. Thus we have actually proved the following generalization of Theorem 1.

Theorem 2. *Suppose that tame asymptotical functions $\Phi_1, ..., \Phi_N$ are tau-functions of the KdV-hierarchy and remain tame under the string flow $\Phi_i \mapsto \exp(u/z)\hat{}\Phi_i$ for all u. Then*

$$\Phi := e^{c(\tau)}\ \hat{S}_\tau^{-1}\ \Psi(\tau)\ \hat{R}_\tau\ \ e^{(U_\tau/z)\hat{}}\ (\Phi_1 \otimes ... \otimes \Phi_N)$$

satisfies the corresponding Hirota quadratic equation (1 – 5).

Remark. Although the condition for Φ_i to remain tame under the string flow is quite restrictive, $\mathcal{D}_{A_1}$ is not the only tau-function satisfying it. A large class of examples consists of the shifts $\mathcal{D}_{A_1}(\mathbf{q}+\mathbf{a})$ where $\mathbf{a}(z) = a_0 + a_1 z + a_2 z^2 + ...$ is a series with coefficients a_k which are arbitrary series in $\hbar$ such that a_0 and a_1 are smaller than 1 in the $\hbar$-adic norm and $a_k \to 0$ in this norm as $k \to \infty$.

8. The Kac – Wakimoto hierarchies. Let us compare the ADE- hierarchies (1–5) with the *principal hierarchies* of the types $A_N^{(1)}, D_N^{(1)}, E_N^{(1)}$ described in Theorem 1.1 in [17]. The corresponding Hirota equation (1.14) in [17] has the form

$$\mathrm{Res}\,\frac{d\zeta}{\zeta}\sum_{i=1}^N g_i e^{\sum_{m\in E_+} 2\beta_{i,\overline{m}}\hbar^{-1/2}y_m\zeta^m}\, e^{-\sum_{m\in E_+}\beta_{i,\overline{-m}}\hbar^{1/2}\partial_{y_m}\zeta^{-m}/m}\Phi(\mathbf{x}+\mathbf{y})\Phi(\mathbf{x}-\mathbf{y})$$

$$= \left(2h\sum_{m\in E_+} m\ y_m\ + \langle\rho,\rho\rangle\right)\Phi(\mathbf{x}+\mathbf{y})\Phi(\mathbf{x}-\mathbf{y}). \tag{25}$$

Here ρ is the sum of the fundamental weights of the root system A, and the value $\langle\rho,\rho\rangle = Nh(h+1)/12$ can be found for instance from the tables in [5]. The index set $E_+ = \{m_a + kh | a = 1, ..., N,\ k = 0, 1, 2, ...\}$, and $\overline{m}$ denote the remainder modulo h. The vertex operators in the sum correspond to a set of roots $\alpha_i, i = 1, ..., N$, chosen one from each orbit of some Coxeter element M on the root system A. The coefficients $\beta_{i,\overline{m}}$ are coordinates of α_i with respect to a basis of eigenvectors $H_{\overline{m}}$ of the Coxeter transformation M with the eigenvalues $\exp(2\pi\sqrt{-1}m/h)$ satisfying the additional normalization condition $\langle H_{\overline{m}}, H_{\overline{-m}}\rangle = 1$. [7] The coefficients g_i are defined via representation theory of affine Lie algebras. The numerical values of g_i are computed in [17] in the cases A_N, D_4 and E_6.

In order to identify the vertex operators in (25) with those in (1,2) let us start with taking $\zeta = (h\lambda)^{1/h}$. Then the components of the period vector $I_{\alpha_i}^{(-1)}$ with respect to a suitable basis $[\psi_a] \in H$ will have the form

$$(I_{\alpha_i}^{(-1)}(\lambda), [\psi_a]) = \beta_{i,\overline{m_a}}\ m_a^{-1}\ (h\lambda)^{m_a/h}. \tag{26}$$

Indeed, the weighted - homogeneous forms $\psi_a\omega/df$ represent a basis of eigenvectors for the classical monodromy operator in $H^2(f^{-1}(1), \mathbb{C})$. Then it is straightforward to check that the relation

$$q_k^a = \prod_{r=0}^k (m_a + rh)\ t_{m_a+kh} \tag{27}$$

[7] As usual, there is a caveat in the case D_l with odd l when the eigenvalue -1 of the monodromy operator has multiplicity 2. The involution of the Dynkin diagram induces an automorphism of the root system which allows one in this case to single out one invariant and one anti-invariant eigenvector, $H_{l-1}^{(i)}$ and $H_{l-1}^{(a)}$, orthogonal to each other and normalized by $\langle H_{l-1}, H_{l-1}\rangle = 1$ each.

(together with the standard change $\mathbf{x}+\mathbf{y}=\mathbf{t}'$, $\mathbf{x}-\mathbf{y}=\mathbf{t}''$ as in Section 4) identifies the vertex operators in (25) with $\Gamma^{\alpha_i}\otimes\Gamma^{-\alpha_i}$. Note that replacing α_i with any of the h roots from the same M-orbit does not change the corresponding residue in (25) since the new vertex operator would differ from the old one only by the choice of the branch of $\zeta=(h\lambda)^{1/h}$. Thus we arrive at the following conclusion.

Proposition 9. *The choice of the basis $\{[\psi_a]\in H\}$ such that (26) holds true and the change of variables (27) identify the Hirota equation (1–5) with the corresponding hierarchy of the form (25) provided that $g_i=h^3a_{\alpha_i}=h^3b_{\alpha_i}/16$.*

Let us now compute the coefficients b_α. First, rewrite the definition (11) as

$$b_\alpha=\lim_{\varepsilon\to 0}e^{-\int_{-\varepsilon\mathbf{1}}^{\tau-(u+\varepsilon)\mathbf{1}}\mathcal{W}_{\alpha,\alpha}}=\lim_{\varepsilon\to 0}\prod_{\gamma\in A}\frac{\langle y(\varepsilon),\gamma\rangle^{\frac{\langle\alpha,\gamma\rangle^2}{2}}}{\langle\varepsilon^{1/h}\kappa,\gamma\rangle^{\frac{\langle\alpha,\gamma\rangle^2}{2}}}=\frac{\langle v,\alpha\rangle^4}{\langle\kappa,\alpha\rangle^4}\prod_{\langle\gamma,\alpha\rangle=1}\frac{\langle x,\gamma\rangle}{\langle\kappa,\gamma\rangle},$$

where $\varepsilon^{1/h}\kappa$, $y(\varepsilon)$ and x are inverse images under the Chevalley map of $-\varepsilon\mathbf{1}$, $\tau-(u+\varepsilon)\mathbf{1}$ and $\tau-u\mathbf{1}$ respectively, x is a generic point on the mirror $\langle\alpha,x\rangle=0$ and v is determined from the expansion $y(\varepsilon)=x+\varepsilon^{1/2}v+o(\varepsilon^{1/2})$. We will use this formula in the case of A and D series.

Case A_N. The root system consists of the vectors $\gamma_{ij}:=e_i-e_j$ in the space $\mathbb{C}^{N+1}$ with the standard orthonormal basis $e_0,...,e_N$ and coordinates $z_0,...,z_N$. Take

$$F(z,\tau)=\frac{z^{N+1}}{N+1}+t_1z^{N-1}+...+t_N=\frac{1}{N+1}\prod_{i=0}^{N}(z-z_i).$$

Let $\alpha=e_a-e_b$ and let $t=\tau-u\mathbf{1}$ be a generic point on the discriminant. Then the components $y_i(\varepsilon)=x_i+\varepsilon^{1/2}v_i+\varepsilon w_i+o(\varepsilon)$ (where $x_a=x_b$) satisfy $F(y_i,t-\varepsilon\mathbf{1})=0$ and therefore

$$\varepsilon=F(x_i,t)+F'(x_i,t)\varepsilon^{1/2}y_i+F'(x_i,t)\varepsilon w_i+F''(x_i,t)\varepsilon\frac{v_i^2}{2}+o(\varepsilon).$$

We have $F(x_i,t)=0$ for all i and $F'(x_i,t)=0$ for $i=a,b$. This implies that $v_i=\pm\sqrt{2/F''(x_a,t)}$ for $i=a,b$ and hence $\langle\alpha,v\rangle=\pm 2\sqrt{2/F''(x_a,t)}$. Thus $\langle\alpha,v\rangle^4=64/F''(x_a,t)^2$. On the other hand,

$$\prod_{\langle\gamma,\alpha\rangle=1}\langle x,\gamma\rangle=(-1)^{N-1}\prod_{i\neq a,b}(x_i-x_a)^2=(-1)^{N-1}\left(\frac{(N+1)F''(x_a,t)}{2}\right)^2.$$

The eigenvector $\kappa=(N+1)^{1/(N+1)}(1,\eta,\eta^2,...,\eta^{N-1})$ of the Coxeter transformation $(z_0,...,z_N)\mapsto(z_1,...,z_N,z_0)$ with the eigenvalue $\eta=\exp 2\pi i/(N+1)$ is a preimage of $t=-\mathbf{1}$ under the Chevalley map. We find [8]

$$\langle\kappa,\alpha\rangle^4\prod_{\langle\alpha,\gamma\rangle=1}\langle\kappa,\gamma\rangle=(N+1)^2(\eta^a-\eta^b)^2\prod_{j\neq a}(\eta^a-\eta^j)\prod_{i\neq b}(\eta^i-\eta^b)=$$
$$(-1)^N(N+1)^4(\eta^a-\eta^b)^2\eta^{N(a+b)}=(-1)^{N-1}(N+1)^4(2-\eta^{a-b}-\eta^{b-a}).$$

Collecting the results we find

$$b_\alpha=\frac{16}{(N+1)^2}\frac{1}{(2-\eta^{a-b}-\eta^{b-a})}.$$

[8] We use here the facts that the product $\prod_{k\neq a}(\zeta-e^{2\pi ik/n})$ over all n-th roots of unity except $\zeta=e^{2\pi ia/n}$ is equal to the derivative of z^n-1 at $z=\zeta$, *i.e.* to n/ζ.

This agrees with Theorem 1.2 in [17] where $g_i = (N+1)/(2-\eta^i-\eta^{-i})$ corresponds to $\alpha_i = e_0 - e_i$. In particular

$$\sum g_k = \frac{N+1}{4}\sum_{k=1}^{N}\sin^{-2}(\frac{\pi k}{N+1}) = \frac{N(N+1)(N+2)}{12}.$$

The middle expression is a special case of *Dedekind sums*, and the second equality, which follows from our results, is well known in number theory (see *i.g.* [4]).

Case D_N. The root system consists of the vectors $\pm e_i \pm e_j$, $i \neq j$, where $e_1, ...e_N$ is the standard orthonormal basis and $(z_1, ..., z_N)$ are the corresponding coordinates in $\mathbb{C}^N$. The parameters $(t_1, ..., t_N)$ in the following family of polynomials

$$F(z,t) = z^{2N} + t_2 z^{2N-2} + t_3 z^{2N-4} + ... + t_N z^2 + t_1^2 = \prod_{i=1}^{N}(z^2 - z_i^2)$$

are identified with coordinates on the Chevalley quotient $\mathbb{C}^N/W$. Note that the invariant t_N of degree $h = 2N-2$ is the coefficient at z^2. Let us assume that x is a generic point on the mirror $z_a \pm z_b$ orthogonal to the root $\alpha = e_a \mp e_b$, and that t is the corresponding point on the discriminant, so that $x_a = \pm x_b$ and $F(\pm x_a, t) = F'(\pm x_a, t) = 0$. Taking $y_i(\varepsilon) = x_i + \varepsilon^{1/2} v_i + \varepsilon w_i + o(\varepsilon)$ and expanding $F(y(\varepsilon), t_1, ..., t_N - 1, t_N - \varepsilon) = 0$ in ε we find

$$\varepsilon x_i^2 = F(x_i,t) + F'(x_i,t)(\varepsilon^{1/2}v_i + \varepsilon w_i) + F''(x_i,t)\varepsilon\frac{v_i^2}{2} + o(\varepsilon).$$

Thus $v_a = \sqrt{2x_a^2/F''(x_a,t)}$, $v_b = \mp\sqrt{2x_a^2/F''(x_a,t)}$ and $\langle\alpha, v\rangle^4 = 64x_a^4/F''(x_a,t)^2$. Furthermore,

$$F''(z,t)|_{z=x_a} = [2\sum_a\prod_{i\neq a}(z^2 - x_i^2) + 4z^2\sum_{a\neq b}\prod_{i\neq a,b}(z^2 - x_i^2)]_{z=x_a} = 8x_a^2\prod_{i\neq a}(x_a^2 - x_i^2).$$

Using this we find

$$\prod_{\langle\alpha,\gamma\rangle=1}\langle x,\alpha\rangle = \prod_{j\neq a,b}(x_a^2 - x_j^2)\prod_{i\neq a,b}(\mp 1)(x_i^2 - x_b^2) = (\pm 1)^{N-2}\frac{F''(x_a,t)^2}{64x_a^4}.$$

Next, the eigenvector $\kappa = (1, \eta, ..., \eta^{N-2}, 0)$ of the Coxeter transformation

$$(z_1, ..., z_N) \mapsto (z_2, ..., z_{N-1}, -z_1, z_N)$$

with the eigenvalue $\eta = \exp \pi i/(N-1)$ is mapped to $(t_1, ..., t_N) = (0, ..., 0, 1)$ under the Chevalley map. Assuming first that $\alpha = e_a \mp e_b$ with $a, b < N$ we find

$$\langle\kappa,\alpha\rangle^4\prod_{\langle\alpha,\gamma\rangle=1}\langle\kappa,\gamma\rangle = (\eta^a \mp \eta^b)^4(\eta^a)^2(\mp\eta^b)^2\prod_{i\neq a,b,N}(\eta^{2a} - \eta^{2i})(\pm 1)(\eta^{2b} - \eta^{2i}) =$$

$$-(\pm 1)^{N-2}(N-1)^2\frac{(\eta^a \mp \eta^b)^2}{(\eta^a \pm \eta^b)^2} = (\pm 1)^{N-2}(N-1)^2\frac{(2 \mp \eta^{a-b} \mp \eta^{b-a})}{(2 \pm \eta^{a-b} \pm \eta^{b-a})}.$$

Combining with the previous formulas we compute

$$b_\alpha = \frac{1}{(N-1)^2}\frac{(2 \pm \eta^{a-b} \pm \eta^{b-a})}{(2 \mp \eta^{a-b} \mp \eta^{b-a})} \quad \text{for} \quad \alpha = e_a \mp e_b.$$

Now let $\alpha = e_a \mp e_N$. Then

$$\langle\kappa,\alpha\rangle^4\prod_{\langle\alpha,\gamma\rangle=1}\langle\kappa,\gamma\rangle = \eta^{4a}\prod_{i\neq a,N}(\eta^{2a} - \eta^{2i})(-\eta^{2i}) = (-1)^{N-2}(N-1)$$

and therefore $b_\alpha = 1/(N-1)$.

Taking the representatives

$$\alpha_1 = e_{N-1} - e_1,\ \ldots,\ \alpha_{N_2} = e_{N-1} - e_{N-2},\ \alpha_{N-1} = e_{N-1} - e_N,\ \alpha_N = e_{N-1} + e_N$$

in the orbits of the Coxeter transformation on A, we find

$$g_i = \frac{(N-1)}{2}\frac{(2-\eta^i-\eta^{-i})}{(2+\eta^i+\eta^{-i})} \text{ for } i = 1, \ldots, N-2, \text{ and } g_i = \frac{(N-1)^2}{2} \text{ for } i = N-1, N.$$

The identity $\sum g_k = (N-1)N(2N-1)/6$, which follows from our general theory, agrees with the value of the Dedekind sum [9]

$$\sum_{k=1}^{N-2} \tan^2(\frac{\pi k}{2N-2}) = \frac{(N-2)(2N-3)}{3}.$$

In the case $N = 4$ the values $g_i = 1/2, 9/2, 9/2, 9/2$ agree with the values of g_i found in [17], Proposition 1.3(a).

Cases E_N. We find g_i using the packages *LiE* and *MAPLE* to compute the ratios via (5) and then apply the normalizing relation (3). In each case E_N, let $\alpha_1, \ldots, \alpha_N$ be the simple roots and M be the Coxeter transformation described by the following diagrams:

$$\begin{array}{ccccccccc} \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\ & & & & | & & & & \\ & & & & \bullet & & & & M = \sigma_1\sigma_4\sigma_6\sigma_2\sigma_3\sigma_5, \\ & & & & \alpha_2 & & & & \end{array}$$

$$\begin{array}{ccccccccccc} \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\ & & & & | & & & & & & \\ & & & & \bullet & & & & & & M = \sigma_1\sigma_4\sigma_6\sigma_2\sigma_3\sigma_5\sigma_7, \\ & & & & \alpha_2 & & & & & & \end{array}$$

$$\begin{array}{ccccccccccccc} \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \\ \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\ & & & & | & & & & & & & & \\ & & & & \bullet & & & & & & & & M = \sigma_1\sigma_4\sigma_6\sigma_8\sigma_2\sigma_3\sigma_5\sigma_7. \\ & & & & \alpha_2 & & & & & & & & \end{array}$$

One can check (*i.g.* using *LiE*) that all simple roots $\alpha_1, \ldots, \alpha_N$ belong to different M-orbits. The following tables represent the values of the corresponding coefficients g_i while the values of b_{α_i} can be obtained from them as in Proposition 9.

Case E_6. We have $b_{\alpha_i} = g_i/108$, where

$$g_1 = g_6 = 16 + 8\sqrt{3},\ g_3 = g_5 = 16 - 8\sqrt{3},\ g_2 = 7 + 4\sqrt{3},\ g_4 = 7 - 4\sqrt{3}.$$

This agrees with the values of g_i found in [17], Proposition 1.3(b).

Case E_7. We have $b_{\alpha_i} = 2g_i/729$. Put $u = \cos(\pi/9)$. Then

$$g_1 = \frac{27}{2} + 36u + 24u^2,\ g_2 = \frac{225}{2} + 36u - 144u^2,\ g_3 = \frac{3}{2},\ g_4 = \frac{147}{2} + 12u - 96u^2,$$

$$g_5 = \frac{9}{2} - 72u + 72u^2,\quad g_6 = -\frac{21}{2} - 48u + 72u^2,\quad g_7 = \frac{9}{2} + 36u + 72u^2\ .$$

[9] It is essentially the same one as in the A-case since $\sin^{-2} x - 1 = \cot^2 x = \tan^2(\pi/2 - x)$.

Case E_8. We have $b_{\alpha_i} = 2g_i/3375$. Let $u = \cos(\pi/15)$. Then

$$g_1 = \frac{33}{2} + 80u + 72u^2 - 16u^3, \quad g_2 = \frac{273}{2} + 132u - 136u^2 - 128u^3,$$
$$g_3 = -\frac{123}{2} + 568u + 376u^2 - 912u^3, \quad g_4 = \frac{109}{2} - 368u - 72u^2 + 400u^3,$$
$$g_5 = \frac{745}{2} + 584u - 376u^2 - 624u^3, \quad g_6 = \frac{257}{2} - 1220u - 232u^2 + 1376u^3,$$
$$g_7 = -\frac{35}{2} + 156u + 136u^2 - 256u^3, \quad g_8 = -\frac{19}{2} + 68u + 232u^2 + 160u^3 .$$

9. Open questions. (a) The formula (9) defines the total descendent potential $\mathcal{D}$ as an asymptotical function of $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 z + ...$ with semisimple $\mathbf{q}_0$. As it is shown in [12], Theorem 5, the function $\mathcal{D}_{A_N}$ extends to arbitrary values of $\mathbf{q}_0$ without singularities. We expect the same for $\mathcal{D}_{D_N}$ and $\mathcal{D}_{E_N}$ but leave this issue open.

(b) B. Dubrovin [7] associates to a Frobenius manifold a *dispersionless* integrable hierarchy. In particular, the hierarchy (1,2) for asymptotical functions $\Phi = \exp(\mathcal{F}^{(0)}/\hbar + \mathcal{F}^{(1)} + ...)$ admits the *dispersionless limit* as $\hbar \to 0$ which is an infinite system of equations for $\mathcal{F}^{(0)}$. It is not hard to show that $\mathcal{F}$ *satisfies the dispersionless hierarchy if and only if the Gaussian distributions* $\Phi := \exp\{d^2_{\mathbf{x}}\mathcal{F}(\mathbf{q})/2\hbar\}$ *(where* $d^2_{\mathbf{x}}\mathcal{F}$ *is the quadratic differential of* $\mathcal{F}$ *at* $\mathbf{x}$*) satisfy the original hierarchy (1,2) for all* $\mathbf{x}$. An elegant explicit characterization in terms of the semi-infinite Grassmannian of those Gaussian distributions which satisfy the hierarchy of the type A_N is given in the appendix to [12]. It would be interesting to generalize the characterization to the cases D_N, E_N.

(c) Theorem 1 implies that the genus 0 descendent potential $\mathcal{F}^{(0)} = \lim_{\hbar\to 0} \hbar \ln \mathcal{D}$ satisfies the corresponding dispersionless hierarchy. The quadratic forms $d^2_{\mathbf{x}}\mathcal{F}^{(0)}(\mathbf{q})$ depend only on N parameters $\tau = \tau(\mathbf{x})$ (due to the property $(*)$ of the cone $\mathcal{L} = \operatorname{graph} d\mathcal{F}^{(0)}$) and have the following explicit description (see Appendix in [12]):

$$\int_{\mathbf{0}}^{\tau} \sum_a ([S_t(z)\mathbf{q}(z)]_0 \bullet [S_t(z)\mathbf{q}(z)]_0, \partial_{t^a})\ dt^a,$$

where $[S(z)\mathbf{q}(z)]_0 = S_0\mathbf{q}_0 + S_1\mathbf{q}_1 + ...$ denotes the z^0-mode. The corresponding Gaussian distributions satisfy therefore the hierarchy $(1-5)$. Taking $\tau = u\mathbf{1}$ so that $[S_\tau \mathbf{q}]_0 = \sum \mathbf{q}_k u^k/k!$ we conclude that in particular the hierarchy has the 1-parametric family of Gaussian solutions

$$\Phi = \exp\left\{ \frac{1}{2\hbar}\int_0^u (\sum_{k\geq 0} \mathbf{q}_k \frac{v^k}{k!}, \sum_{l\geq 0} \mathbf{q}_l \frac{v^l}{l!})\ dv \right\}.$$

This imposes non-trivial constraints on the coefficients a_α in the Hirota equation (1). It would be interesting to find out if these constraints are sufficient in order to determine the coefficients unambiguously.

(d) Our computations in Section 8 confirm the Conjecture from Section 1 in the cases A_N, D_4, E_6 and leave it open in the cases D_N with $N > 4$, E_7 and E_8 — mostly because the values of the coefficients g_i in the Kac – Wakimoto theory remain unknown. A more conceptual approach to the identification of the Hirota equations should rely on the definition of the coefficients g_i given in [17] in terms of representation theory. Namely, the vertex operators $C_i^{\pm}\Gamma^{\pm\alpha_i}$ participate in the so called *principal* construction of the basis representation of the affine Lie algebra $\hat{A}_N, \hat{D}_N$ or $\hat{E}_N$, and $g_i = C_i^+C_i^-$. Here $C_i^{\pm}$ are certain structure constants whose values remain generally speaking unknown. Our successful description of the products $C_i^+C_i^-$ via the phase forms suggests that one should look for the intrinsic role of the phase forms in representation theory and for a description of the individual coefficients $C_i^{\pm}$ in terms of the phase forms or their generalizations.

(e) In representation theory, the hierarchies of the ADE-type form only a part of a larger list of examples including twisted versions of the affine Lie algebras and non-simply laced Dynkin diagrams. It would be interesting to find the corresponding constructions in singularity theory and, in particular, to associate the Hirota equations to the boundary singularities B_N, C_N, F_4.

(f) B. Dubrovin and Y. Zhang [9] associate an integrable hierarchy to any semisimple Frobenius manifold. In a sense their construction is parallel to the definition (9) of the total descendent potential $\mathcal{D}$ (see [11]) and in particular yields objects defined in the complement to the caustic. In this regard the vertex operator description of the hierarchies seems more attractive as it is free of this defect. Of course, the ADE-hierarchies (1 – 5) are expected to be equivalent to the hierarchies of Dubrovin – Zhang. It would be interesting to confirm this expectation.

(g) Conjecturally, the total descendent potential $\mathcal{D}$ extends analytically across the caustic values of $\mathbf{q}_0$ in the case of K. Saito's (semisimple) Frobenius structure corresponding to any isolated singularity. (By the way, this is known to be false for, say, boundary singularities or for finite reflection groups other than A_N, D_N, E_N.) Respectively, one should expect the same for the hierarchies of Dubrovin – Zhang. It would be very interesting to give a vertex operator description of the hierarchies together with Theorem 1 for arbitrary (or at least weighted - homogeneous) isolated singularities of functions. The most obvious difficulty is that the vertex operator sum (1) over the set of all vanishing cycles (or even orbits of the classical monodromy operator on this set) becomes infinite beyond the ADE list. Nevertheless we believe that the obstructions can be removed by an appropriate generalization of the concepts involved. The first examples to study here would be the unimodal singularities P_8, X_9, J_{10} (see [1]). Their miniversal deformations are closely related to the complex crystallographic reflection groups $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ (see [19]). Moreover, the question can be extrapolated to the complex crystallographic groups $\tilde{A}_N, \tilde{D}_N$, and the 3-dimensional Frobenius manifold to be called $\tilde{A}_1$ represents the first challenge.

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