Math H117 – Mathematical Problem Seminar – class log

Spring 1988 MWF 10-11, Room 4, Evans Hall George M. Bergman

The purpose of this course, from my point of view, is to give you a taste of what research in mathematics is, using problems that can be stated and (in most cases) solved using the material you will have learned in our undergraduate courses.

I will give several problems each day, ranging from the very easy, through harder problems, to outstanding open questions that would merit publication if solved. I will expect you to think about *all* these questions, so that we can discuss them in class, and to hand in solutions to about three medium-difficulty problems per week, or the equivalent in harder or easier problems. Partial solutions will be welcomed if you make it clear that you are aware that they are partial; e.g. if you say, "The result would follow if we knew X and Y, and here is a proof of X". Grades will be based on work done on these problems, i.e. homework handed in and classroom presentations.

Members of the class are also invited to suggest problems.

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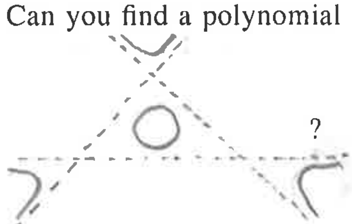
20 January, 1988

1. Suppose $s_1, \ldots, s_{100}$ is a sequence of 100 distinct integers. Must it contain either a monotone increasing sequence of 10 terms or a monotone decreasing sequence of 10 terms? If so, is this a "best result"?

2. Prove: $10 \mid 7^{7^{7^7}} - 7^{7^7}$.

3. Can you find a polynomial $f(x, y)$ such that the graph of the equation $f(x, y) = 0$ has the form

(a)  ? (b)  ? (c)  ?

4. Does there exist a polynomial $f(x, y)$, such that for each value of x in the interval $[0, 1]$ there exists a unique real number y such that $f(x, y) = 0$, while for each x in $[-1, 0)$ there is no such y?

5. (Open question of Jeff Lagarias and Mike Saks.) Consider the following game, played on a board consisting of the vertices of an n-cube, i.e. the set $\{0, 1\}^n$. The first player's only move is to distribute 2^n counters among these 2^n vertices; i.e., to place none, or one, or more than one on each vertex, till they are all used. The second player may perform any number of moves of the following sort: Remove *two* counters from some vertex, and replace them by *one* counter on a vertex "immediately above" it. Here we say that vertex w is "immediately above" vertex v if the n-tuple w can be obtained from v by replacing a single "0" in v by a 1. The second player's goal is to get a counter to the top vertex, $(1,1,\ldots,1)$. Is it possible to do this no matter how the first player has distributed the counters?

We also discussed the following three problems, which are not assigned because some of you saw them when I spoke to Dubins' 191 last Semester, and so could have an unfair advantage.

6. (a) (easy) True or False: $|2^{19} - 3^{12}| > 1/2$?

(b) (medium difficulty) Prove: If f and g are polynomials with integer coefficients such that for all integers n, $f(n) \mid g(n)$, then $f \mid g$ in $\mathbf{Q}[x]$.

(c) (hard) Prove: If c is a real number and for all positive integers n, n^c is an integer, then c is an integer.

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7. Suppose $a_1, a_2, \ldots, a_n, \ldots$ are positive real numbers such that the sum $\Sigma\, a_n$ converges. Let S be the set of all sums of subseries $\Sigma\, a_{n(i)}$ $(n(1) < n(2) < \ldots)$. Show that S is a closed subset of $\mathbf{R}$.

8. If C is a closed convex curve, we shall say that two triangles T and T' *coscribe* C if one is inscribed in C, the other is circumscribed about C, and the vertices of the inscribed triangle lie on the sides of the circumscribed triangle.

If T is a triangle, then $m(T)$ will denote the triangle (similar to T) whose vertices form the *midpoints* of the sides of T. We shall write M for m^{-1}, i.e. the construction which takes a triangle T to the triangle having the vertices of T at the midpoints of its sides.

(a) Show that if C is a closed convex curve in the plane, then among the triangles lying within C there is at least one such triangle, T, of maximum area. Prove that T and $M(T)$ coscribe C.

9. (a) Find all subrings of the ring $\mathbf{Z}\times\mathbf{Z}$. (A subring of a ring R is a subset of R closed under addition, additive inverse, and multiplication operations, and containing the additive and multiplicative identity elements, 0 and 1 of R.)

(b) Investigate subrings of $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$.

10. (a) Give an example of a group G with nonidentity elements a, b such that $ab = ba^2$.

(b) Find all possible values for the pair $(\mathrm{ord}(a), \mathrm{ord}(b))$ in this situation.

The significance in music of the fact that 2^{19} and 3^{12} are very close (noted in (6)(a)) was discussed. (6)(b) was solved.

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25 January, 1988

Clarification: in (7), ''closed set'' is meant in the topological sense.

11. The ancient Egyptians had a notational system for writing positive integers, and a way of forming, from the symbol for an integer n, a symbol which meant $1/n$. To represent an arbitrary rational number $r \in (0, 1)$, they would write it as the sum of the largest ''$1/n$'' it contained, then the largest such number in the remainder, etc.. For example, they would write ''2/5'' *not* as $1/5 + 1/5$ but as $1/3 + 1/15$.

(a) Will the indicated process of breaking a rational number $r \in (0, 1)$ into reciprocals of integers always terminate?

12. Suppose $f\colon \mathbf{R}^2 \to \mathbf{R}$ is a C^∞ (or C^1, or polynomial) function having exactly one critical point (point where the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are both zero), and suppose this point is a local maximum of f. Must it be a global maximum of f?

13. If G is a finite connected nonempty graph, show that one can remove one vertex from G without disconnecting it.

Solutions were given to (2) and (3).

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27 January, 1988

14. If in a tetrahedron all pairs of opposite edges are equal, prove that for each pair of opposite edges, the angles between the pairs of faces meeting at these edges are equal.

15. (Algebraicity of Lissajous figures) For real numbers a and b, consider the set

$$L = \{(\sin t, \cos at+b) \mid t \in \mathbf{R}\}.$$

(a) Show that if a is rational, then L satisfies a nontrivial polynomial equation $f(x, y) = 0$; i.e., there exists $f \in \mathbf{R}[x, y]$ such that $f(x, y) = 0$ for all $(x, y) \in L$.

(b) Show that if a is irrational, then L does not satisfy any nontrivial polynomial equation.

(c) In situation (a), what can you say about whether L will in fact be the set of all solutions of $f = 0$, for f appropriately chosen?

16. (Peter Fraenkl's question) Let S be a finite set, and X a set of subsets of S. Assume that $A, B \in X \Rightarrow A \cap B \in X$, and that $\emptyset \in X$. Does it follow that there exists $s \in S$ which is a member of not more than half of the members of X?

Peter Marshall presented a solution to problem (1), showing that any sequence of n^2+1 distinct integers contains a monotone increasing or decreasing sequence of length $n+1$. Hence any sequence of $9^2+1 = 82$ distinct integers contains a monotone increasing or decreasing sequence of length $9+1 = 10$; a strengthening of the statement about sequences of length 100.

The following generalization of the problem was given:

1. (continued) (b) For what integers N, r, s is it true that every sequence of N terms contains either a monotone increasing sequence of r terms, or a monotone decreasing sequence of s terms?

We discussed (6)(c), and I indicated an idea for a proof.

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29 January, 1988

11. (continued) (b) Does the expression for a rational number r in Egyptian form as described above always use the smallest possible number of inverse-integer summands?

(c) Given a sum of inverses of positive integers, how can one tell whether it is in the correct form? (E.g., $1/5 + 1/5$ is not.)

(d) Given two expressions in Egyptian form, how can one tell which is larger?

17. Consider expressions formed from two symbols x and y using symbolic group-theoretic operations of multiplication, inverse, and identity-element e. (For example, $((x \cdot y^{-1}) \cdot e) \cdot (y \cdot (x \cdot x))^{-1}$ is such an expression.) Call two such expressions f and g "S_3-equivalent" if $f(x, y) = g(x, y)$ for all values of x and y in S_3. (E.g., x^6 is S_3-equivalent to e; x^2y^2 is S_3-equivalent to y^2x^2.)

(a) What can you say about the number of S_3-equivalence classes of such expressions? Is it finite? If so, what is its value?

18. Prove that the abelian group $\mathbf{Z}^\omega$ of all sequences of integers, with componentwise addition, is not a free abelian group. (The concept of free abelian group, i.e. abelian group with an additive basis, was explained.)

A proof of (13)(a) was given. (Should have noted under 29 Jan.: A brief solution to (0)(a) was given.)

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1 February, 1988

13. (continued) Let us call a vertex of a connected graph "removable" if it can be removed without disconnecting the graph. Determine all connected graphs with exactly 1, respectively 2, respectively 3, respectively 4 removable points. (This constitutes four subproblems, of ascending difficulty. You can hand in any subset of them.)

19. If S is a subset of a group G, define the *commutativity graph* $C(G,S)$ of S to be the graph having for vertices the elements of S, and an edge connecting distinct vertices s and t if and only s and t commute in G. Is every finite graph isomorphic to the commutativity graph of a subset of a finite group?

20. (Emil Artin) In an election, candidate A received a votes and candidate B received b votes. The ballots were counted in random order. Find, as a function of a and b, the probability that at some time during the counting, A had tallied more votes than B.

Discussion of (5): Call a subset of $\{0,1\}^n$ a "lower face" if it is the set of all n-tuples whose ith term is 0, for some i. Then the desired result is true if and only if whenever we distribute 2^n counters over the vertices of $\{0,1\}^n$, either we put a counter on the top vertex, or there is some lower face within which we can bring two counters to the top of that face. Thus, we want a statement to the effect that however the counters are distributed over the non-top vertices, we will have a "good supply" of them on some lower face. If we ask how many we must have on at least one lower face, the best we can say is $\geq 2^n/n$. A more useful result is that if we consider a counter at distance r from the top has having "weight" $1/r$, at least one lower face must have counters of weight totaling $\geq 2^n/n$.

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3 February, 1988

I discussed the importance of distinguishing between an assertion that two objects are isomorphic (i.e., that there exists an isomorphism between them) and asserting that a specific correspondence between them is an isomorphism, using the example of the *pons asinorum* of elementary geometry, in preparation for:

21. (Open question on graph reconstruction.) Suppose G is a graph with vertices $x_1, \ldots, x_n$ $(n \geq 3)$. Does the list of isomorphism-classes of the graphs $G-\{x_1\}, \ldots, G-\{x_n\}$, counting multiplicities, determine the isomorphism-class of G? Equivalently, given two graphs: G with vertices $x_1, \ldots, x_n$, and H with vertices $y_1, \ldots, y_n$ $(n \geq 3)$, such that for each i we have $G-\{x_i\} \cong H-\{x_i\}$, must $G \cong H$? (Not necessarily by $x_i \longleftrightarrow y_i$.)

The next problem consists of a pair of recent results of mine and Lenstra's. Note that for elements g and h of a group G, g^h denotes the conjugate $h^{-1}gh$, and that for S a subset of the group, S^h denotes $\{g^h \mid g \in S\}$.

22. Suppose G is a group and H a subgroup. Prove:

(a) The inequality $[H:H \cap H^g] \leq 2$ holds for all $g \in G$, if and only if there exists a normal subgroup N of G such that either $N \supseteq H$ with $[N:H] \leq 2$, or $H \supseteq N$ with $[H:N] \leq 2$.

(b) The inequality $[H:H \cap H^g] \leq n$ holds for all $g \in G$ for some fixed integer n if and only if there exists a normal subgroup N of G which is *commensurable* with H, i.e., such that $[H:H \cap N]$ and $[N:H \cap N]$ are both finite.

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22. (continued) (c) Suppose G is a group, and S a subset which is ''near to'' being a subgroup, in the sense that for each $s \in S$, one has $st \in S$ for all but at most n elements $t \in S$, and $ts \in S$ for all but at most n elements $t \in S$, and such that $s^{-1} \in S$ for all but at most n elements $s \in S$. Must there exist a subgroup $H \subseteq G$ which differs from S in only finitely many elements; i.e. such that there are only finitely many elements that lie in S but not in H and finitely many elements that lie in H but not in S?

For the next problem, we note that a *preordering* on a set A means a relation ''$\leq$'' which is reflexive and transitive, but not necessarily antisymmetric. For example, ''at least as old as'' is a preordering. More generally, if f is a function from a set A to a set S, and S has a partial ordering $\leq$, and we define a relation $\leq_f$ on A by writing $a \leq_f b$ to mean $f(a) \leq f(b)$, this relation is a preordering on A.

23. (a) Show that every preordering on a set A has the form $\leq_f$ for an appropriate map f into a partially ordered set S.

(b) Suppose A is a finite set given with two preorderings, $\leq$ and $\preccurlyeq$, such that for any two elements a and b in A, either $a \leq b$ or $b \preccurlyeq a$. Show that A has a ''highest'' element with respect to one of these preorderings; i.e. that there is an element x having either the property that all elements of A are $\leq x$, or the property that all elements are $\preccurlyeq x$.

(c) Does (b) remain true if A is infinite? If not, can you prove some modification of that statement?

For the next problem we also need some definitions. If G is a group, an *action* of G on a set X means a map $G \times X \to X$, where the image of (g, x) is generally written gx, which satisfies the identities $g(hx) = (gh)x$ $(g, h \in G,\ x \in X)$ and $ex = x$ $(x \in X)$. This is equivalent to a group homomorphism from G into the permutation group of X. An action of a group G on a set X is called *transitive* if $(\forall x, y \in X)(\exists g \in G)\ gx = y$. A set given with an action of G on it is called a *G-set*. Given an element x of a G-set X, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *stabilizer* (or *isotropy subgroup*) of x. A homomorphism $X \to Y$ between G-sets means a map a such that $(\forall g \in G,\ x \in X)\ a(gx) = g(a(x))$. An invertible homomorphism is, as usual, called an *isomorphism*.

24. (a) If X is a transitive G-set and x any element of X, show that the G-set X is determined up to isomorphism by the subgroup G_x. Show that for any subgroup H of G there exists a G-set X with an element x such that $G_x = H$. Given transitive G-sets X and Y and elements $x \in X$, $y \in Y$, how can one tell from G_x and G_y whether $X \cong Y$?

(b) Describe up to isomorphism all nonempty transitive S_3-sets.

(c) Given two G-sets X and Y, show that the direct product set $X \times Y$ can be made a G-set in a unique way so that the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are homomorphisms of G-sets. We will from now on understand any direct product of G-sets to be made a G-set in this way.

(d) Consider the direct products of all the pairs of the transitive S_3-sets found in part (b). Make a table showing how each such product can be described as a disjoint union of isomorphic copies of the members of that list.

(e) Show that if x, y, z are members of a G-set X, then $(G_x)z \supseteq (G_y)z \iff (G_z)x \supseteq (G_y)x$ (where ''multiplication'' of a set and an element is defined in the obvious way).

(f) If x and y are elements of a G-set, must the sets $(G_x)y$ and $(G_y)x$ contain the same number of elements? What if we assume X finite, and/or transitive?

8 February, 1988

25. (a) An abelian group A is said to be the *direct sum* of subgroups B and C, written $A = B \oplus C$, if and only if every element of A can be written uniquely as the sum of an element of B and an element of C. Suppose P is an abelian group with two direct sum decompositions, $P = Q \oplus S'$ and $P = Q' \oplus S$, such that $Q \subseteq Q'$ and $S \subseteq S'$. Show that P has a subgroup R such that $Q' = Q \oplus R$ and $S' = R \oplus S$.

(b) Let us define a *direct product decomposition* of a set A to mean a pair of maps, $f: A \to B$ and $h: A \to C$ such that the induced map $(f, h): A \to B \times C$ is bijective. Suppose P is a set given with two direct product decompositions, corresponding to bijections $(f, h'): P \to Q \times S'$ and $(f', h): P \to Q' \times S$, such that for every pair of elements (p_1, p_2) of P, one has $f'(p_1) = f'(p_2) \Rightarrow f(p_1) = f(p_2)$, and $h'(p_1) = h'(p_2) \Rightarrow h(p_1) = h(p_2)$. Must such a pair of decompositions arise, in a way analogous to the one described in (a), from a 3-fold decomposition, i.e. a bijection $(f, g, h): P \to Q \times R \times S$?

A quick proof of (13) was noted, based on looking at a minimal possibly self-intersecting path that contains all vertices.

It was observed that our proof of (6)(c) is frustratingly uninformative in some ways – it says nothing about numbers c such that n^c is always rational, which we would expect also to be integers – but it can be extended very nicely in another direction, to show that if $a_1 n^{c_1} + \ldots + a_r n^{c_r}$ is always an integer (where the a's are nonzero and the c's are distinct), then the c's are all integers.

10 February, 1988

26. (Filip Machi) Show that the equation $x^n + y^n = z^n$ has no solution in positive integers satisfying $z \leq n$.

11. (continued) (e) (Open question) Suppose r/s is a rational number with odd denominator, and we attempt to decompose it as a sum of reciprocals of odd positive integers, subtracting first the largest reciprocal of an odd integer that is $\leq r/s$, then, if the remainder is nonzero, the largest reciprocal of an odd integer $\leq$ this remainder, and so on. Must this process terminate?

27. Let $(S, \leq)$ be a finite partially ordered set.

(a) Show that there is a total ordering $\preccurlyeq$ on S which refines $\leq$, i.e., such that $s \leq t \Rightarrow s \preccurlyeq t$.

(b) (Fredman's conjecture) Let n_S denote the number of total orderings $\preccurlyeq$ of S which refine $\leq$, and for each pair of elements $s, t \in S$, let $n(s, t)$ denote the number of such total orderings for which $s \preccurlyeq t$. Prove or disprove: If S is not totally ordered, then there exist $s, t \in S$ such that $1/3 \leq n(s, t)/n_S \leq 2/3$.

A proof of (14) was given which suggested the following question:

14. (continued) What subgroups of the permutation group on n elements can be realized as the group of all isometries of an n-element metric space?

12 February, 1988

A way of seeing that the sum of the angles of a triangle is π will be given which suggests:

28. In a tetrahedron, is the sum of the angles between the faces, or of the solid angles at the vertices, or some combination of these, a constant?

29. A *rectangular band* means a set with a binary operation (which we will sometimes write as "multiplication", sometimes as a function, e.g. $a(s,t)$) which, in multiplicative notation, satisfies the identities $s(tu) = (st)u$ (associativity), $ss = s$ (idempotence), and $stu = su$.

(a) Show that given any two sets S and T, the direct product set $S \times T$ has a structure of rectangular band defined by $(s,t)(s',t') = (s,t')$; and that conversely, if P is a rectangular band, then it has a bijection with a direct product of two sets such that the given rectangular band structure arises from the rectangular band structure constructed on this product set as above.

(b) If S is a direct product of four sets, $S = S_1 \times S_2 \times S_3 \times S_4$, observe that one has natural bijections of S with $(S_1 \times S_2) \times (S_3 \times S_4)$ on the one hand, and $(S_1 \times S_3) \times (S_2 \times S_4)$ on the other. Write explicit formulas for the rectangular band operations on S corresponding to these two pairwise product decompositions, and verify that if these binary operations are denoted a and b, they satisfy the identity

$$a(b(s,t),\, b(u,v)) = b(a(s,u),\, a(t,v)).$$

Two binary operations satisfying the above identity are said to *commute*. Show that any pair of commuting rectangular band operations on a set S arises as above from a fourfold decomposition of S.

(c) If a and b are two commuting rectangular band operations, let their "intersection" mean the binary operation ab defined by $ab(s,t) = a(b(s,t),t)$. Show that this will again be a rectangular band operation, and will commute with any rectangular band operation with which a and b commute.

(d) For students familiar with the concept of a Boolean algebra: Let us define the "complement" of a rectangular band operation a on a set S to be the operation $a'(s,t) = a(t,s)$, and the "trivial" rectangular band operation to be the operation $0(s,t) = t$. Show that any family of pairwise commuting rectangular band operations which is closed under the operations of "intersection" and "complement" and contains the "trivial" operation forms a Boolean ring under these operations.

(e) If B is a Boolean ring and S a set, let an "action" of B on S mean a map $B \times S \times S \to S$, the value of which at $b \in B$, $s, t \in S$ will be written $b(s,t) \in S$, such that for each $b \in B$ the binary operation $b(-,-)$ on S is an operation of rectangular band, the operation associated to the zero element of B is the trivial rectangular band operation, the operation associated to the complement of an element is the complement of the corresponding rectangular band operation, and the operation associated to the intersection of two elements is given by the formula $(ab)(s,t) = a(b(s,t),t)$. Show that for such an action, the operations on S induced by the elements of B all commute.

30. Let $M_3\colon \{0,1\}^3 \to \{0,1\}$ denote the ternary majority vote operation:

$$M_3(x,y,z) = \begin{cases} 0 & \text{if at least two of } x, y, z \text{ are } 0 \\ 1 & \text{if at least two of } x, y, z \text{ are } 1 \end{cases} \qquad (x, y, z \in \{0,1\}).$$

Characterize the *clone* of operations on $\{0,1\}$ generated by M_3. (Meaning explained in class.) In particular, does this clone contain:

The unary "reversal" operation, $r(x) = 1-x$?

The binary operation max ?

The 5-ary majority vote function M_5?

The 4-ary function N_4 of "majority vote with first voter having tie-breaking power"?

(Warning: you can't get $\max(x,y)$ as $M_3(1,x,y)$, because the zeroary operation 1 was not given as one of the generators of the clone. So if you get it, it will have to be in a different way.)

(10) was solved; the solution led to the concept of a *semidirect product* of groups.

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17 February, 1988

In the following problem, and several more that will be given later, [SUPPG] stands for the paper *Some Unsolved Problems in Plane Geometry* by Victor Klee, *Mathematics Magazine* **52** (1979) 131-145. That paper has further references to the origins of the problems, and work that has been done on them. I will often state the problem somewhat differently from the way Klee does.

31. (Open question: [SUPPG], Problem (H)) Let us write $d(u, v)$ for the distance between points u and v of the plane. Suppose $p_1, \ldots, p_n$ are points of the plane, and P the set of all points having distance ≤ 1 from at least one p_i, i.e. the union of the n unit disks centered at $p_1, \ldots, p_n$. Suppose $q_1, \ldots, q_n$ is another family of n points, such that for all i, j one has $d(q_i, q_j) \leq d(p_i, p_j)$, and let Q be the union of the n unit disks centered at the q's. Must the area of Q be $\leq$ that of P?

32. The Fibonacci sequence is defined by the conditions $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$. Thus, the terms $f_0, \ldots, f_{12}$ are

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144.$$

(a) Show that for every integer d there exists a positive integer $a(d)$ such that for all n, $d \mid f_n \Leftrightarrow a(d) \mid n$.

(b) Show that for every integer d, the Fibonacci sequence is periodic $\bmod d$. Let us denote its least period $\bmod d$ by $b(d)$.

(c) Investigate the numerical behavior of the functions a and b defined above.

A proof of (1) based on the concepts of chains and antichains in a partially ordered set was given.

As a step toward a solution of (4), it was shown that given polynomials f and g in two variables x and y, there exists a polynomial h whose degree in y is $\leq$ those of f and g, and such that for all but finitely many values of x one has

$$\{y \in \mathbf{R} \mid h(x, y) = 0\} = \{y \in \mathbf{R} \mid f(x, y) = 0\} \cap \{y \in \mathbf{R} \mid g(x, y) = 0\}.$$

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19 February, 1988

33. (a) Show that for every positive integer n, the congruence $x^2 \equiv -1 \pmod{5^n}$ has a solution. (How many solutions, precisely, will it have?)

(b) Investigate similarly the congruences

$$x^2 \equiv 2 \pmod{5^n},\quad x^3 \equiv 2 \pmod{5^n},\quad x^2 \equiv 5 \pmod{5^n}.$$

(c) Investigate similarly the congruences

$$x^2 \equiv -7 \pmod{2^n},\quad x^2 \equiv 17 \pmod{2^n},\quad x^3 \equiv 5 \pmod{2^n}.$$

(d) Try to find as general a result as you can along the lines suggested by the above questions.

34. If S is a finite set of points in the plane, not all collinear, show that there is a line which contains exactly two points of S.

A handout "The Fibonacci Sequence Meets the Shift Operator" (prepared for Math 113, Spring 1976) was given out a supplementary material to (32).

A solution was given to (4), which in fact showed that if f is a polynomial in x and y such that for infinitely many values of x, the equation $f(x,y)=0$ is satisfied by a unique value of y, then for all but finitely many values of x, this equation has a solution.

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22 February, 1988

35. The *Fourier transform* of a complex-valued function g on the real line is the function $\hat{g}$ defined by $\hat{g}(y)=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ixy}g(x)dx$. Although the elementary theory of the Fourier transform concerns the case where the integral in question is absolutely convergent (or where the square of the integrand gives an absolutely convergent integral) it is also studied under weaker hypotheses. A function for which this integral converges, but not absolutely, is $g(x)=e^{ix^2}$. This function is its own Fourier transform.

Paul Chernoff wonders whether this is ''essentially'' the only case of a function of constant absolute value with Fourier transform also having constant absolute value. Decide what ''essentially'' should mean here, and investigate the question. Though you probably don't have the tools to give a rigorous proof if the answer is ''yes'', you may be able to come up with an example showing a negative answer, or alternatively, get some good ideas on how to prove an affirmative answer.

36. (a) Given positive real numbers $x_1,\ldots,x_n$, show that it is possible to delete from this sequence a family of terms, the sum of whose reciprocals is $\leq n$, so that on each *interval* of consecutive terms not deleted, the sum of the terms is ≤ 1.

(b) Examine the analogous situation in which instead of a *sequence* of terms, one has a finite graph, with each vertex labeled by a positive real number, and one wishes to delete a family of vertices so that the sum of the values at the vertices of each remaining connected component is ≤ 1, and the sum of the reciprocals of the values at the points deleted is small (some function expressible in terms of the graph).

It was emphasized that students should think about all problems, so that we can discuss their difficulties and possible approaches in class.

A solution was shown to (7), and I discussed the concept of a ''compactness proof'' which is the idea behind it.

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24 February, 1988

37. (Paul Erdős offers \$3,000 to the solver of this open question.) If A is a set of positive integers such that $\Sigma_{a\in A}\, a^{-1}=\infty$, then must A contain arbitrarily long arithmetic progressions? (I.e., is it true that for every positive integer N there exist positive integers x and y such that $x,\ x+y,\ \ldots,\ x+(N-1)y$ all lie in A?)

The above question is very difficult, but here are a few easier questions that can help one understand it better.

(a) Show that the above question has an affirmative answer if and only if for every positive integer N there exists a positive constant C such that every set of positive integers A with $\Sigma_{a\in A}\, a^{-1}\geq C$ contains an N-term arithmetic progression.

(b) Can one prove statement (a) with ''set'' replaced by ''finite set''?

(c) Can there exist a set A which for some integer N contains no N-term arithmetic progression, but such that for every integer m not in A, $A\cup\{m\}$ contains such a progression?

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26 February, 1988

38. (From a Soviet mathematical Olympiad.) 50 numbers are written in a circle, each of which is equal to $+1$ or -1. It is required to determine the product of all these numbers. By one question, one may learn the product of three numbers standing in succession. What is the smallest number of questions that must be asked?

39. Let n be a positive integer. Determine the sum of the first n terms of the binomial expansion of $(2-1)^{-n}$.

40. Recall that a *semigroup* means a set with an associative binary operation. In particular, any group can be looked at as a semigroup, and hence one can speak of *subsemigroups* of a group; i.e. subsets closed under the group composition, but not necessarily closed under taking inverses, or containing the identity element.

(a) Show that every subsemigroup of the additive group $\mathbf{Z}$ of the integers is finitely generated.

(b) Show that *not* every subsemigroup of the group $\mathbf{Z}\times\mathbf{Z}$ of pairs of integers under componentwise operations is finitely generated.

(c) A subset S of $\mathbf{Z}\times\mathbf{Z}$ is called *convex* if for any $u, v\in S$ and $\lambda\in[0,1]$ such that $w=\lambda u+(1-\lambda)v$ lies in $\mathbf{Z}\times\mathbf{Z}$, one also has $w\in S$. Is every convex subsemigroup of $\mathbf{Z}\times\mathbf{Z}$ finitely generated?

It was decided that I should cut back on the number of problems I give out, so that we can focus better on problems already given.

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29 February, 1988

It was pointed out that group-theoretic expressions $f(x,y)$ and $g(x,y)$ are "equivalent" in the sense of (17) if and only if they induce the same element of $(S_3)^{S_3\times S_3}$; hence the number of equivalence classes is $\leq 6^{36}$.

Solutions to parts (a) and (b) of (33) were sketched.

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2 March, 1988

I sketched one way of proving an affirmative answer to (19), and gave a hint to a more elementary approach.

Christine Silverio sketched the solution to (24)(a), (b).

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4 March, 1988

(32)(a) and (b) were proved, and (c) discussed. Another part was added:

32. (continued) (d) Try to explain the occurrence of Fibonacci numbers in the decimal expansion of 1/89, and the expansion of other rational numbers to other bases.

7 March, 1988

41. Must every finite string S of consecutive integers have some term which is relatively prime to all the other terms?

42. (a) Suppose S is a closed set of real numbers, and c a positive real number. Determine what implications hold among the following statements:
(i) S meets (has nonempty intersection with) the closed interval $[a, a+c]$ for all real numbers a.
(ii) S meets the half-open interval $[a, a+c)$ for all real numbers a.
(iii) S meets the half-open interval $(a, a+c]$ for all real numbers a.
(iv) S meets the open interval $(a, a+c)$ for all real numbers a.

(This means that for every possible pairwise implication among these conditions, you must try either to prove that the implication holds, or show by example that it does not.)

(b) Consider the above conditions for varying values of c. Suppose one wishes to "classify" sets S according to which conditions hold for which values of c. Determine the different possible behaviors (combinations of conditions satisfied and not satisfied) that can occur.

Nicholas Field gave a solution to (34). This suggested

34. (continued) (b) Is the same result true for finite families of points of $\mathbf{C}^2$? (By a "line" in $\mathbf{C}^2$ we understand the solution-set of a nontrivial equation $ax+by+c=0$, where $a, b, c \in \mathbf{C}$ are constants and x, y variables.)

(c) If S is a finite set of points in 3-space, $\mathbf{R}^3$, and appropriate degenerate cases are excluded, must there be a plane which contains exactly three points of S? (The "degenerate case" in the original problem was that in which all the points were collinear. In the 3-dimensional case you are asked to decide yourself what case or cases are to be excluded.)

Peter Marshall solved (32)(d) by showing that $(a^2-a-1)^{-1} = \Sigma_{i\geq 0} f_i\, a^{i+1}$. I showed how this result can be motivated, and how similar considerations yield expressions for $\Sigma_{i\geq 0}\, i\, a^{i+1}$, $\Sigma_{i\geq 0}\, i^2 a^{i+1}$, etc..

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9 March, 1988

43. (Open question, from p.24 of P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, 1980.) A system of *covering congruences* means a family of conditions, $x \equiv a_1 \pmod{n_1}$, $x \equiv a_2 \pmod{n_2}$, ..., $x \equiv a_r \pmod{n_r}$, with all n_i distinct and >1, such that every integer satisfies *at least one* of these congruences. Example:

$$0 \pmod 2,\ \ 0 \pmod 3,\ \ 1 \pmod 4,\ \ 3 \pmod 8,\ \ 7 \pmod{12},\ \ 23 \pmod{24}.$$

Is it true that for every N, there is a system of covering congruences with all $n_i \geq N$?

44. Let f and g be continuous periodic functions on the real line, with period 1.
(a) Show that $(\int_0^1 f(x)dx)(\int_0^1 g(x)dx) = \lim_{n\to\infty} \int_0^1 f(x)g(nx)dx$.
(b) Show that $(\int_0^1 f(x)dx)(\int_0^1 g(x)dx) = \lim_{n\to\infty} \int_0^1 f(nx)g((n+1)x)dx$.
(c) Generalize.

I went over the list of past problems, noting which ones were reasonably easy.

11 March, 1988

45. If G is a group, then for $x, y, z \in G$, let $<x, y, z> = xy^{-1}z$. The set G together with the ternary operation $<\,,\,,\,>$, is called a *heap*. More precisely, a heap is defined to mean a set H together with any ternary operation $<\,,\,,\,>$ satisfying all the identities that are satisfied by the operation $xy^{-1}z$ in every group.

(a) Find as strong a set of identities as you can that are satisfied by the ternary operation $<x, y, z> = xy^{-1}z$ in all groups (where we consider one set of identities to be at least as strong as another if it implies the latter). Can you in fact get a set sufficiently strong to imply that every object $(H, <\,,\,,\,>)$ satisfying your identities arises from a group in the manner described?

If G is a group, let us write G_{heap} for the corresponding heap.

(b) A subheap of a heap is defined to mean a subset closed under the operation $<\,,\,,\,>$. For G a group, can one characterize the subheaps of G_{heap} in terms of the subgroups of G?

(c) A homomorphism of heaps is defined to mean a map of underlying sets, which respects the operations $<\,,\,,\,>$. What is the relation between group homomorphisms $G \to H$ and heap homomorphisms $G_{\text{heap}} \to H_{\text{heap}}$? In particular, if $G_{\text{heap}} \cong H_{\text{heap}}$, must we have $G \cong H$?

32. (continued) (e) Show that the Fibonacci numbers satisfy $f_n^2 = f_{n-1} f_{n+1} \pm 1$.

(f) Show, conversely, that if a, b are nonnegative integers such that $b^2 = a(a+b) \pm 1$, then they are successive Fibonacci numbers.

(8)(a) was solved. The solution suggested

8. (continued) (b) For C a closed convex curve, can one show that among triangles containing C there is at least one such triangle T of minimum area? In this case, must T and $m(T)$ coscribe C?

(c) For C as above, must there be at least two triangles T such that T and $M(T)$ coscribe C? (We noted that pairs of triangles obtained as in (a) and as in (b) need not always be distinct.)

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14 March, 1988

46. If f is any polynomial in two variables, show that there exists a positive integer n such that for any n real numbers $x_1, \ldots, x_n$ and any n real numbers $y_1, \ldots, y_n$, one has $\det(f(x_i, y_j)) = 0$.

47. Let n be a positive integer, and suppose that all rational numbers between 0 and 1 which can be written with denominator $\leq n$ are written out, in lowest terms, in ascending order: $p_0/q_0 \leq p_1/q_1 \leq \ldots$ p_s/q_s. (Thus, $p_0/q_0 = 0/1$, $p_1/q_1 = 1/n$, $p_s/q_s = 1/1$.) Prove: $p_{i+1}q_i - p_iq_{i+1} = 1$.

I gave a proof of (18). We talked about (41) and (39).

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16 March, 1988

48. (a) (Open question, Harvey Friedman.) Does their exist a one-to-one polynomial function $f: \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ (where $\mathbf{Q}$ denotes the field of rational numbers)?

A much easier problem is:

(b) Find a one-to-one polynomial function $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$.

The following version requires a little skill with analysis, but is still elementary:

(c) Show that there exists a continuous function $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which carries points with rational coordinates to rational numbers, and is one-to-one on $\mathbf{Q} \times \mathbf{Q}$.

Peter Marshall gave a counterexample to (41), and a proof of (39).

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18 March, 1988

49. (a) (Known problem, mentioned by Filip Machi.) Call a rectangle "nice" if one of its edges has integral length. Suppose R is a rectangle which can be decomposed into finitely many (nonoverlapping) nice subrectangles with edges parallel to the edges of R. Show that R is itself nice.

(b) Suppose we identify opposite sides of a rectangle (the result being a torus), and again consider partitions into rectangles with edges parallel to the sides, but this time allow some of these rectangles to cross from an edge to the opposite edge. Is the same result true?

Myra Snell gave a solution to (23)(a). (32)(f) was solved. I suggested an approach to (8)(b).

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21 March, 1988

40. (continued) Replace the last sentence of part (c) with the following:

The smallest convex subset of $\mathbf{Z} \times \mathbf{Z}$ containing S is called the *convex hull* of S; let us write it $C(S)$. Show that if $S \subseteq \mathbf{Z} \times \mathbf{Z}$ is a subsemigroup, then $C(S)$ is also a subsemigroup, and that S is a finitely generated semigroup if and only if $C(S)$ is.

50. Is there a nonconstant polynomial $f(x, y)$ such that the curve $y = x^{1/2} + x^{1/3}$ satisfies $f(x, y) = 0$? If so, can f be chosen that for every x, the set of solutions of $f(x, y) = 0$ is *exactly* the set of sums of a square root of x and the cube root of x?

More generally, can one get such an f for each equation $y = x^{1/m} + x^{1/n}$?

I showed the solutions to (23)(b) and (36)(a).

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23 March, 1988

51. (a) Show that if C is a closed convex curve, then a square can be inscribed in C.

(b) (Open question: [SUPPG], Problem (J)) Is the same true of every simple closed curve C? (A square will be said to be "inscribed" in a not-necessarily convex closed curve if all its vertices lie on the curve. The curve may cross the edges of the square.)

(c) In each case, if C contains in its interior a square of side 1, can one show the existence of an inscribed square of side greater than some fixed positive constant?

25 March, 1988

32. (continued) A sequence of real numbers $(r_0, \ldots, r_n, \ldots)$ is said to satisfy a *linear recursion relation* if there exist real numbers $c_0, \ldots, c_d$, not all zero, such that for all positive n,

$$c_0 r_n + \ldots + c_d r_{n+d} = 0.$$

Thus, the Fibonacci numbers by definition satisfy the linear recursion relation with constants $c_0 = 1$, $c_1 = 1$, $c_2 = -1$.

(g) Show that the sequence of *squares* of Fibonacci numbers $(f_1)^2, (f_2)^2, \ldots, (f_n)^2, \ldots$ satisfies a linear recursion relation.

(h) Is the same true for other positive integer powers of the Fibonacci sequence?

(i) Is the same true for the sequence of inverses of the Fibonacci numbers?

(j) If a, b are positive integers, find a linear recursion relation satisfied by the Fibonacci numbers f_a, f_{a+b}, $f_{a+2b}, \ldots$.

(k) Show that a sequence $r_0, \ldots, r_n, \ldots$ satisfies a linear recursion relation if and only if the power series $\Sigma\, r_i t^i$ represents a rational function (a ratio of polynomials with real coefficients, $p(t)/q(t)$).

(l) Show that a sequence of integers satisfies a linear recursion relation with real coefficients if and only it satisfies a linear recursion relation with integer coefficients.

(m) Show that every polynomial sequence satisfies a linear recursion relation and that every geometric progression satisfies a linear recursion relation.

(n) If $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$ are sequences of integers satisfying linear recursion relations, will the sequence $a_{b_1}, \ldots, a_{b_n}, \ldots$ satisfy a linear recursion relation?

27. (continued) (c) Let $P = \{1, 2, \ldots, r\}$ for some r. For $i, j \in P$ define $i \leq_2 j$ to mean $i = j$ or $j - i \geq 2$. Verify that $\leq_2$ is a partial ordering. (You might draw the picture for $r = 8$.) Considering P partially ordered by this relation, determine the ratios $n(s, t)/n_S$, as defined in part (b), for all $s, t \in P$. Do these partially ordered sets satisfy Fredman's conjecture?

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4 April, 1988

52. Does there exist a nonzero polynomial f in three variables, such that the solution-set of $f(x, y, z) = 0$ in 3-dimensional space contains a smooth Möbius strip?

53. If C is an arc in the plane, i.e. a continuous function $[0, 1] \to \mathbf{R}^2$, the *length* of C is defined to be the supremum, over all sequences $1 \leq t_1 \leq t_2 \leq \ldots t_n \leq 1$, of $\Sigma_i\, d(C(t_i), C(t_{i+1}))$; a nonnegative real number or $+\infty$. If a sequence of arcs, C_i $(i = 1, 2, \ldots)$ converges pointwise to an arc C, what can be said about the relationship between the lengths of the C_i's and the length of C?

In connection with (32)(j), Peter Marshall showed (and I gave another proof) that for all integers a, b, c there exist integers A, B, C such that $(\forall n)\ Af_{n+a} + Bf_{n+b} + Cf_{n+c} = 0$. In connection with (27), I showed that the number of total orderings refining the partial ordering on the set $\{1, 2, \ldots, r\}$ described in part (c) is the Fibonacci number f_{r+1}.

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6 April, 1988

54. (a) (Open question, P. Erdős, \$300?) Suppose S is a set of nonnegative integers such that every nonnegative integer n can be written as a sum of two members of S, $n = s + s'$. Prove, or find an S which is a counterexample: For every positive integer r there exists a positive integer n which can be so written in at least r distinct ways: $n = s_1 + s_1' = \ldots = s_r + s_r'$ (where $s_1 < \ldots < s_r$ and $s_1' > \ldots > s_r'$ are members of S).

(b) If S is a counterexample to the statement in part (a), what can be said about the number of elements in S between 1 and n, as a function of n?

(c) Find a set S satisfying the hypothesis of (a), such that the number of elements in S between 1 and n is as small as you can make it as a function of n, for n large.

(d) Find a set S not satisfying the conclusion of (a), such that the number of elements in S between 1 and n is as large as you can make it as a function of n, for n large.

(e) Can you prove anything about whether there exists an S such that every nonnegative integer can be written as a sum of two members of S in one and only one way up to order?

(f) Question (a) can be thought of as conjecturing that not only is there no set S yielding every n as a sum in exactly one way, but that a set yielding every n as a sum in at least one way cannot come close to yielding every n in at most one way. Investigate the reverse question: can a set yielding every n in at most one way come close to yielding every n in at least one way? You will have to decide what to mean by the latter condition. Likewise: can there exist a set which comes close to yielding every n in at most one way and also comes close to yielding every n in at least one way?

(g) What can be said about the existence of a pair of infinite sets of nonnegative integers S, T having or close to having the property that every $n \geq 0$ can be written $s+t$ ($s \in S$, $t \in T$) in exactly one way?

I gave some hints on (15) and a solution to (48)(b), and reviewed what we had seen on (24)(a), (b).

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8 April, 1988

32. (continued) (o) Determine the complex numbers α such that the geometric progression $(1, \alpha, \alpha^2, \ldots)$ satisfies the same linear recursion relation as the Fibonacci sequence. Show that the vector space of all sequences satisfying that linear recursion relation has, on the one hand, a basis consisting of those geometric progressions, and on the other hand, a basis consisting of the two vectors $(f_0, f_1, \ldots, f_n, \ldots)$ and $(f_1, f_2, \ldots, f_{n+1}, \ldots)$. Give explicit expressions for these two bases in terms of one another. Translate these into formulas expressing Fibonacci numbers and powers of the above values of α in terms of one another.

(p) Investigate to what extent the results obtained in (o) can be extended to characteristic p. After getting such general results as you can, say what you can concretely in the cases $p = 2, 3, 5, 7, 11$.

(q) Let τ denote the largest of the values of α determined in (o). Let us consider expressions for numbers in "base τ", i.e., finite strings of 0's and 1's, with a decimal point somewhere among them, interpreted so that the n+1st position to the left of the decimal point has value τ^n, and the nth position to the right of the decimal point has value τ^{-n}. Show that 100.01 is the base-τ expression for an integer (which?). Can all positive integers be represented in base τ? What value does 10.1 represent? What can you say about the set of real numbers representable in base τ? Are representations in general unique?

(r) Investigate how one could do arithmetic in base τ.

I went through the solution of (24)(c), (d), and began discussion of (9).

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11 April, 1988

55. Let $F_{a,b}$ denote the set of all finite "words" in two symbols $a, b,$ and $F_{a,b,x}$ the set of all such "words" in three symbols $a, b, x.$ (Examples: $abbbabaa \in F_{a,b} \subseteq F_{a,b,x}$; $xbax \in F_{a,b,x}$. We shall find it convenient to consider the "empty word" to belong to these sets, and denote it 1. Formally, these sets are the free semigroups on 2 and 3 generators respectively, but we don't need to know this.)

Given $f = f(a,b,x),\ g = g(a,b,x) \in F_{a,b,x},$ let $S_{f,g}$ denote $\{s \in F_{a,b} \mid f(a,b,s) = g(a,b,s)\},$ where by $f(a,b,s)$ we mean the result of "substituting" the word s for all occurrences of the letter x in f (and analogously for $g(a,b,s)$). Intuitively, $S_{f,g}$ is the "solution-set" of the equation $f = g$ in $F_{a,b}$. A simple example is

(a) Show that $S_{abx,\,xba} = \{a,\ aba,\ ababa, \dots\}.$

Many of the properties of this example hold more generally. In (b)-(d), we assume elements $f \neq g \in F_{a,b,x}$ given.

(b) Show that if $s, t \in S_{f,g}$ and length $s \leq$ length t, then s right and left divides t (i.e., t both begins and ends with the subword s). In particular, this says that if length $s =$ length t, then $s = t$.

(c) Show that if $S_{f,g}$ is infinite, then there exists an integer N and elements $u, v \in F_{a,b}$ such that $\{s \in S_{f,g} \mid \text{length } s \geq N\} = \{uv^i \mid i = 0, 1, 2, \dots\}.$

(d) Show that if $a^m, a^n \in S_{f,g},$ where $m \neq n,$ then $S_{f,g} = \{a^i \mid i = 0, 1, 2, \dots\}.$

(e) Do there exist f, g such that $1 < \mathrm{card}(S_{f,g}) < \infty$?

(f) Try to find necessary and sufficient conditions on a subset $S \subseteq F_{a,b}$ for it to equal $S_{f,g}$ for some pair of distinct elements $f, g \in F_{a,b,x}$. (Thus, (b)-(d) give necessary conditions.)

Myra Snell gave a solution to (42)(a). I gave further suggestions on how to approach (49) and (15), and further discussion of (9).

13-15 April, 1988

56. (Based on [SUPPG], §4) Call a subset S of the plane *rational* if every pair of points of S is a rational distance apart. Items (a), (b) and (e) are open.

(a) Does there exist a rational set of points dense in the plane?

(b) For every finite set of points $p_1, \dots, p_n$ and every $\varepsilon > 0$, does there exist a rational set of points $q_1, \dots, q_n$ such that $d(p_i, q_i) < \varepsilon$ $(i = 1, \dots, n)$?

(c) For any point p of the plane, let O_p denote the operation of *inversion through* p, i.e., the map of $\mathbf{R}^2 - \{p\}$ into itself which takes a point at distance d from p to the point on the same ray through p having distance d^{-1} from p. Investigate how inversion can be used to study rational sets.

(d) Show that there exist infinite rational sets which are neither concyclic nor collinear.

(e) Does every rational set of 6 points have a subset of either 3 collinear or 4 concyclic points?

(f) What would an affirmative answer to (e) imply about the structure of larger rational sets?

(g) Given a finite rational set S, one can look at the set $R(S)$ of points p of the plane such that $S \cup \{p\}$ is again a rational set. Conceivably, this might be dense for all rational sets S. Examine what relation this statement would have to (a), (b) and (e).

I gave a solution to (49)(a), with several variants.

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20 April, 1988

I showed how to adapt one of the solutions to (49)(a) given last time to show that in the situation of (49)(b), either one side of the rectangle must be integral, or both sides must be *rational*, with relatively prime denominators, and gave examples showing that the latter case can indeed occur, with any two relatively prime integers as the denominators.

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24 April, 1988

57. (a) For n a positive integer, let $\mathrm{Mag}(n)$ denote the set of $n \times n$ matrices A of nonnegative integers with the property that all rows of A have the same sum, and that all columns of A have the same sum. (''Mag'' stands for ''magic square''. A common definition of magic square also imposes the restriction that no entry occur more than once, but the above definition makes $\mathrm{Mag}(n)$ closed under addition.) Let $\mathrm{Per}(n)$ denote the set of $n \times n$ *permutation matrices*: matrices with exactly one 1 in each row and exactly one 1 in each column, and all other entries 0. Prove the

Theorem of Birkhoff and von Neumann: Every member of $\mathrm{Mag}(n)$ is a sum of permutation matrices (equivalently, $\mathrm{Mag}(n)$ is generated as an additive semigroup by $\mathrm{Per}(n)$).

I suggest you show first that this is equivalent to the statement that the permutation matrices are precisely the *minimal* nonzero members of $\mathrm{Mag}(n)$, in an appropriate sense.

(b) More generally, let $\mathrm{Mag}(m, n)$ denote the set of $m \times n$ matrices A of nonnegative integers with the property that all rows of A have the same sum, and all columns of A have the same sum. (While in a magic *square*, the common sum of the rows and the common sum of the columns are the same, for these rectangles they will have the ratio $n:m$.) What can be said about the additive generators of $\mathrm{Mag}(m, n)$?

(I do not know the answer. It might be useful to start by studying special classes of cases, such as $n = m+1$ or $n = 2m$.)

Generalizing in another direction, we might consider 3-dimensional ''matrices'', i.e. arrays (a_{ijk}). Let us define $\mathrm{Mag}^{3,2}(n)$ to be the set of such $n \times n \times n$ arrays with the property that $\Sigma_{j,k}\, a_{i_0,j,k}$ is the same for all i_0, $\Sigma_{i,k}\, a_{i,j_0,k}$ is the same for all j_0, and $\Sigma_{i,j}\, a_{i,j,k_0}$ is the same for all k_0. Likewise, let us define $\mathrm{Mag}^{3,1}(n)$ to be the set of $n \times n \times n$ arrays such that $\Sigma_k\, a_{i_0,j_0,k}$ is the same for all i_0 and j_0, $\Sigma_j\, a_{i_0,j,k_0}$ is the same for all i_0 and k_0, and $\Sigma_i\, a_{i,j_0,k_0}$ is the same for all j_0 and k_0.

(c) Investigate the additive generators of $\mathrm{Mag}^{3,2}(n)$ and/or $\mathrm{Mag}^{3,1}(n)$.

(Generally, we would write $\mathrm{Mag}^{d,e}(n)$ for the set of d-dimensional $n \times \ldots \times n$ matrices A such that parallel e-dimensional submatrices of A have the same sum. Still more generally one could combine this idea with that of question (b) above, and study $\mathrm{Mag}^{d,e}(n_1, \ldots, n_d)$, defined to consist of all $n_1 \times \ldots \times n_d$ matrices with this property; but for simplicity I will not ask about this.)

58. (Open question: [SUPPG] part of problem (E)) Suppose P is a simple closed polygon in the plane, in general *not* convex, and x and y are two points in the interior of P. Is it possible to draw a ray (vector) out of x which, possibly after a finite number of reflections off the sides of P, but without striking any vertex, will eventually reach y? (In the language of [SUPPG]: ''Is every polygonal plane region illuminable from each of its points?'')