HOMOGENEOUS ELEMENTS AND PRIME IDEALS IN $\mathbb{Z}$-GRADED RINGS

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Throughout this note, R will be a $\mathbb{Z}$-graded ring, that is an associative ring given with a decomposition $R = \oplus_{\mathbb{Z}} R_i$ such that $R_i R_j \subseteq R_{i+j}$. Our main result will be

Proposition 1. If P is a nonzero prime ideal of R, then any 2-sided ideal $I \subseteq R$ properly containing P contains a nonzero homogeneous element.

We small make frequent use of

Definition 2. If $0 \neq r \in R$, then $r_+ \in R$ will denote the nonzero homogeneous component of r of highest degree (the "leading term" of r), and $r_- \in R$ the nonzero homogeneous component of lowest degree.

We define the breadth of r to be the non~~zero~~negative integer $\mathrm{br}(r) = \deg(r_+) - \deg(r_-)$. Note that this is 0 if and only if r is homogeneous. We also make the convention $\mathrm{br}(0) = -\infty$.

During most of the proof of Proposition 1 we shall want to assume R itself is a prime ring. This will be possible because in the contrary case, a stronger result is in fact true:

Lemma 3. If R is not a prime ring, then every prime ideal of R contains nonzero homogeneous elements. (Equivalently, if P is any prime ideal of a $\mathbb{Z}$-graded ring R, and $H(P)$ the largest homogeneous ideal of R contained in P, then $H(P)$ is again a prime ideal of R.)

Proof. If R is not prime, let a and b be nonzero elements such that $a R b = 0$. Then it is easy to deduce that $a_+ R b_+ = 0$, hence any prime ideal of R must contain one of the homogeneous elements a_+, b_+. To deduce the parenthetical assertion, note that $R/H(P)$ will again be a $\mathbb{Z}$-graded ring, and in this ring $P/H(P)$ will be prime and have no homogeneous elements. Hence this ring must be prime, hence $H(P)$ is a prime ideal. ||

(Remark 4. The key idea in the above proof is that a $\mathbb{Z}$-graded ring is prime as a ring if and only if it is prime as a graded ring, i.e. for all nonzero homogeneous a, b there exists homogeneous c with $acb \neq 0$. The same results are true if the grading group $\mathbb{Z}$ is replaced by any right orderable group, or still more generally any semigroup with the u.p. property.)

Lemma 5. Let I be any nonzero two-sided ideal of R, and $u \in I$ a nonzero element of minimal breadth. Then for all $r \in R$, $u\, r\, u_+ = u_+\, r\, u$.

Proof. It will suffice to prove the indicated equation for r homogeneous. In this case, $u\, r\, u_+ - u_+\, r\, u$ will clearly be an element of I of breadth less than $\mathrm{br}(u)$, hence it is zero. ||

Digression: Suppose we call nonzero elements u, v of a ring R parallel if $urv = vru$ for all $r \in R$. (If R is prime, this is equivalent to being associates over the extended centroid.) The conclusion of the above Lemma implies that all homogeneous components of the element u are parallel to u_+. This leads to another special case in which we can get a stronger result than Proposition 1 (though we will not need it below), namely

Corollary 6. Let R be a $\mathbb{Z}$-graded ring in which any two parallel homogeneous elements with same centralizer have the same degree. (E.g. a free associative algebra.) Then any nonzero two-sided ideal of R contains nonzero homogeneous elements. ||

To see how to proceed with the proof of Proposition 1, let us note why the result is true for the special case of a polynomial ring over a field, $k[t]$. Here an element of minimal degree in any ideal turns out to be a generator, and an inclusion of ideals corresponds to a divisibility relation among generators, but a nonzero prime ideal will have irreducible generator, hence any larger ideal must be generated by a unit, hence contain 1, which is homogeneous.

The next result is an analog of the statement that an ideal of $k[t]$ is generated by any element u of minimal degree. (To see this analogy, consider (1), "ignoring" the homogeneous elements h and u_+, and also the r which may without loss of generality be taken homogenous.)

Lemma 7. Suppose R is prime, I is a nonzero two-sided ideal of R, and u a nonzero element of I of minimal breadth. Then for every nonzero $w \in I$ there exists nonzero $x \in R$, and homogeneous $h \in R$, such that

(1) $\forall r \in R, \quad x\, r\, u = w\, h\, r\, u_+.$

Proof. We shall use induction on $\mathrm{br}(w)$, the case $\mathrm{br}(w) < 0$ being vacuous because w is required to be nonzero.

Given w as above, let us take any homogeneous element $g \in R$ and define

(2) $$w' = w\, g\, u_+ - w_+\, g\, u.$$

Because of the cancellation of the terms $w_+\, g\, u_+$ we see that $\mathrm{br}(w') < \max(\mathrm{br}(w), \mathrm{br}(u))$, which equals $\mathrm{br}(w)$ by choice of u. Now if $w' = 0$ for all choices of g, we get $w_+\, r\, u = w\, r\, u_+$ for all $u \in R$, and we get (1) by taking $x = w_+$, $h = 1$. In the contrary case let us use any g such that $w' \neq 0$. Then applying our inductive hypothesis to w', we can find a nonzero $x' \in R$ and a homogeneous $h' \in R$ such that

(3) $\forall r \in R, \quad x'\, r\, u = w'\, h'\, r\, u_+.$

If we substitute (2) into (3), and apply Lemma 5 to the last term, we get

$$\forall r \in R, \quad x' r u = w g u_+ h' r u_+ - w_+ g u_+ h' r u.$$

If we now put $h = g u_+ h'$ and $x = x' + w_+ g u_+ h'$, this becomes (1). It remains only to verify that $x \neq 0$.

Because R is prime, (3) tells us that $\deg x' = \deg w'h' \leq \deg w' + \deg h'$. By (2), $\deg w' < \deg w + \deg g + \deg u$, so we get $\deg x' < \deg w + \deg g + \deg u + \deg h'$; hence adding to x' the homogeneous element $w_+ g u_+ h'$ cannot send it to zero, i.e. $x \neq 0$. ||

We can now prove Proposition 1. We shall begin with the analog of the observation that in $k[t]$, larger ideals have generators of smaller degrees. This is not true in general for graded rings (consider $(t-1)$ and $(2t-2)$ in $\mathbb{Z}[t]$) but it <u>is</u> when the smaller ideal is prime and without homogeneous elements. (More generally, if it is right or left "homogeneous-prime", i.e. $a R h \subseteq Q$ (resp. $h R a \subseteq Q$) implies $a \in Q$ when h is homogeneous.)

<u>Proof of Proposition 1.</u> We may clearly assume that P itself has no nonzero homogeneous elements, and hence that R is prime.

Let w denote an element of minimal breadth in $P - \{0\}$, and u an element of minimal breadth in $I - P$. Since w_+ is homogeneous, $w_+ \notin P$, so as P is prime, we can find nonzero homogeneous $g \in R$ such that $w_+ g u \notin P$. Hence $w_+ g u - w g u_+$ is an element of $I - P$ of breadth $< \max(\mathrm{br}(w), \mathrm{br}(u))$. But it must have breadth $\geq \mathrm{br}(u)$, by choice of u, hence $\mathrm{br}(u) < \mathrm{br}(w)$.

It follows that u is of minimal breadth in $I - \{0\}$, and we can apply Lemma 7 to this u and w, getting (1). This result, together with the primeness of R, implies that $\mathrm{br}(x) + \mathrm{br}(u) \leq \mathrm{br}(w)$, but it also says $x R u \subseteq P$, hence $x \in P$, hence $\mathrm{br}(x) \geq \mathrm{br}(w)$. It follows that $\mathrm{br}(u) = 0$, i.e. u is homogeneous. ||