ON GENERIC RELATIVE GLOBAL DIMENSION 0

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Let K be a ring and R a K-ring, that is, a ring given with a homomorphism $K \to R$. Then R has (left) relative global dimension 0 (we shall henceforth suppress the "left") if and only if every short exact sequence of left R-modules

(1) $$0 \to A \to B \to C \to 0$$

which splits as a sequence of K-modules can also be split as a sequence of R-modules. (Cf. [1], [2], [3].) We shall see below that most of the known cases where this is so have a stronger property, characterized in Lemma 1.1. Some further results and examples are then obtained.

1. All rings will be associative with 1. Recall that if R is a K-ring, there exist an (R,R)-bimodule $\Omega_K(R)$ and a K-derivation $d: R \to \Omega_K(R)$, which is universal among K-derivations from R into (R,R)-bimodules. This bimodule can be obtained as the kernel of the "multiplication" map:

(2) $$0 \to \Omega_K(R) \to R \otimes_K R \to R \to 0$$

and the derivation is given by

(3) $$d(r) = r\otimes 1 - 1\otimes r \quad (r \in R)$$ (cf. [5], [6] and references given there).

If M is any R-bimodule, and $m \in M$ a K-centralizing element, then

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$r \mapsto rm - mr$ is called the <u>inner</u> K-<u>derivation</u> of R into M induced by m. Note that the K-derivation d defined in (3) is inner as a derivation $R \mapsto R \otimes_K R$, but not necessarily as a derivation $R \to \Omega_K(R)$. In fact, if it happens to be inner as a derivation into $\Omega_K(R)$, then by its universal property, all K-derivations from R into (R,R)-bimodules are inner.

If $\underline{\underline{A}}$ is an abelian category, a left R-<u>module object</u> in $\underline{\underline{A}}$ will mean an object A of $\underline{\underline{A}}$ given with a homomorphism $R \to \underline{\underline{A}}(A,A)$. Defining morphisms in the obvious way, one notes that these objects form an abelian category, with kernel and cokernel coinciding with these operations on $\underline{\underline{A}}$.

<u>Lemma 1.1.</u> Let R be a K-ring. Then the following conditions are equivalent:

(4) For every abelian category $\underline{\underline{A}}$, every short exact sequence of R-module objects of $\underline{\underline{A}}$ that splits as a short exact sequence of K-module objects also splits as a short exact sequence of R-module objects.

(4') For every ring T, the ring $R \otimes_{\mathbb{Z}} T$ has relative global dimension 0 over $K \otimes_{\mathbb{Z}} T$.

(4") The ring $R \otimes_{\mathbb{Z}} R^{op}$ has relative global dimension 0 over $K \otimes_{\mathbb{Z}} R^{op}$.

(4"') The ring $R \otimes_{\mathbb{Z}} R^{op}$ has relative global dimension 0 over $K \otimes_{\mathbb{Z}} K^{op}$.

(5) The epimorphism of R-bimodules, $R \otimes_K R \to R \to 0$, induced by multiplication in R, splits.

(5') There exists an element $z \in R \otimes_K R$ which centralizes R, and has image 1 under the above map $R \otimes_K R \to R$.

(6) The universal derivation $d: R \to \Omega_K(R)$ is inner.

<u>Proof.</u> Let us first, without assuming any of these conditions, consider a short exact sequence

$$(7) \qquad 0 \to A \to B \to C \to 0$$

of R-module objects in an abelian category $\underline{\underline{A}}$, which splits as a sequence of K-module objects. As a K-module object in $\underline{\underline{A}}$, and in particular as an object of $\underline{\underline{A}}$, we may make the identification

$$B = A \oplus C. \tag{8}$$

Now let the R-module structures of A and C be given by homomorphisms $r \mapsto r_A$, $r \mapsto r_C$ of R into $\underline{\underline{A}}(A,A)$ and $\underline{\underline{A}}(C,C)$ respectively. If we write

$$\underline{\underline{A}}(B,B) = \underline{\underline{A}}(A\oplus C,\ A\oplus C) = \begin{pmatrix} \underline{\underline{A}}(A,A) & \underline{\underline{A}}(C,A) \\ \underline{\underline{A}}(A,C) & \underline{\underline{A}}(C,C) \end{pmatrix} \tag{9}$$

then from the fact that the maps of (7) are R-module-object homomorphisms, we see that the R-module structure of B is given by a map $R \to \underline{\underline{A}}(B,B)$ of the form

$$r \mapsto \begin{pmatrix} r_A & \delta(r) \\ 0 & r_C \end{pmatrix}. \tag{10}$$

As noted in [5], [6], the necessary and sufficient condition for (10) to be a ring homomorphism is that $\delta(R)$ be a <u>derivation</u> $R \to \underline{\underline{A}}(C,A)$. (Here $\underline{\underline{A}}(C,A)$ is an (R,R)-bimodule via its natural structure of $(\underline{\underline{A}}(A,A),\ \underline{\underline{A}}(C,C))$-bimodule.) Further, because (8) is a decomposition as K-module object, we have $\delta|K = 0$, i.e. δ is a K-derivation.

Now (7) will split over R if and only if we can perturb the given $\underline{\underline{A}}$-splitting (8) by an $\underline{\underline{A}}$-automorphism of B of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $(x \in \underline{\underline{A}}(C,A))$ so that the image of C becomes an R-submodule object. It is straightforward to verify that this means that δ must be the inner derivation $R \to \underline{\underline{A}}(C,A)$ induced by x. Since $d: R \to \Omega_K(R)$ is the <u>universal</u> K-derivation of R into an (R,R)-bimodule, a sufficient condition for this to happen is that d be inner. This establishes the key implication (6) $\Rightarrow$ (4).

As to the other implications, we have (4) $\Rightarrow$ (4') $\Rightarrow$ (4") by first taking $\underline{\underline{A}}$ to be the category of left T-modules (or if one prefers, right T^{op}-modules), then specializing to $T = R^{op}$. Skipping (4"') for the moment, we see that (4") $\Rightarrow$ (5) because the R-bimodule epimorphism

(11) $$R \otimes_K R \to R \to 0$$

is split as a (K,R)-bimodule map by $r \mapsto 1 \otimes r$. We get (5') from (5) by taking for z the image in $R \otimes R$ of $1 \in R$, under an (R,R)-splitting. To get (6) recall that as a derivation into $R \otimes_K R$, d is inner, induced by $1 \otimes 1$. Hence given z as in (5'), d will also be induced by $1 \otimes 1 - z$. But this lies in the kernel of (11), which is $\Omega_K(R)$.

The equivalence of condition (4"') to the others will be easiest to see after we have noted some trivial properties of the above equivalent conditions, in Lemma 1.3 below, so we defer it to the end of this section. Till then "the equivalent conditions of Lemma 1.1" will mean the other conditions. |

Definition 1.2. If the equivalent conditions of the above Lemma are satisfied, the K-ring R will be said to have generic relative global dimension 0.

It might appear that in the proof of Theorem 1.1, the universal derivation $d: R \to \Omega_K(R)$ served as a "magic" tool for getting from concrete conditions to conditions involving arbitrary abelian categories. Actually, however, the direct proof of (5') $\Rightarrow$ (4) is simpler. If in (5') we write $z = \sum a_i \otimes b_i$, and we are given an exact sequence $0 \to A \to B \to C \to 0$ with a K-splitting described by a K-linear projection π of B onto the image of A, then one can easily verify that $\sum a_i \pi b_i$ is an R-linear projection of B onto the image of A. Condition (5') is also generally the easiest to verify in explicit cases; we shall see in the next section that the three main "classical" results on relative global dimension 0 were all proved by displaying, in some guise, an element $z = \sum a_i \otimes b_i$ as in (5'). But perhaps universal derivations have heuristic power: I discovered Lemma 1.1 by looking at the property of having ordinary relative global dimension 0 from the point of view described in the first two paragraphs of the proof.

We now note some trivial properties of the concept we have defined. All follow easily from characterization (4) and/or (4'). To get (16) one first notes that it is true for the tensor product over $\mathbb{Z}$, by (4'), and one then uses the surjectivity of the ring homomorphisms $S \otimes_{\mathbb{Z}} T \to S \otimes_K T$ $(S = R, K)$. To get (17) one uses opposite categories $\underline{\underline{A}}$ and $\underline{\underline{A}}^{op}$.

Lemma 1.3. If $K \to L \to R$ are ring homomorphisms, then

(12) If R has generic relative global dimension 0 over L, and L has generic relative global dimension 0 over K, then R has generic relative global dimension 0 over K.

(13) If R has generic relative global dimension 0 over K, then R has generic relative global dimension 0 over L.

If $K \to R_1$, $K \to R_2$ are ring homomorphisms, then

(14) $R_1 \times R_2$ has generic relative global dimension 0 over K if and only if R_1 and R_2 each have generic relative global dimension 0 over K.

(15) If $K \to R$ is surjective, then R has generic relative global dimension 0 over K.

(16) If R has generic relative global dimension 0 over K, where R and K are algebras over a commutative ring k, then for any k-algebra T, $R \otimes_k T$ has generic relative global dimension 0 over $K \otimes_k T$.

(17) If R has generic relative global dimension 0 over K, then the opposite ring R^{op} has generic relative global dimension 0 over K^{op}. ‖

We can now get the equivalence of condition (4''') with the property of generic relative global dimension 0 as defined by the other conditions of Lemma 1.1. Assuming R has generic relative global dimension 0 over K, look at the induced ring homomorphisms

(18) $$K \otimes_{\mathbb{Z}} K^{op} \to K \otimes_{\mathbb{Z}} R^{op} \to R \otimes_{\mathbb{Z}} R^{op}.$$

Applying (16), (17) and (12) we conclude that $R \otimes_{\mathbb{Z}} R^{op}$ has generic relative global dimension 0 (and hence ordinary relative global dimension 0) over $K \otimes_{\mathbb{Z}} K^{op}$, proving (4'''). Conversely, if (4''') holds, then applying to the composition (18) the observation (13) (or rather, the corresponding observation for ordinary relative global dimension 0) we get (4''), completing the proof.

We remark that the analogs of (12)-(15) all hold for ordinary relative global dimension 0, by the same arguments. In the next section we shall see examples of K-rings R of relative global dimension 0 but not generic relative global dimension 0. In view of (4') this means that the analog of (16) is not true for ordinary relative global dimension.

I don't know whether the analog of (17) is true for ordinary relative global dimension 0. It <u>is</u> true for absolute global dimension 0 by the characterization of rings R with this property ("completely reducible rings") as the semisimple Artin rings, which are finite direct products of matrix rings over division rings. But I don't know of any more elementary argument, and the corresponding statement for global dimension 1 is false [8]. The analog of (14) for (absolute) global dimension 0 is still clearly true; (15) has no analog and that of (16) is of course false; (12) and (13) have analogs with relative global dimension <u>over</u> K replaced by global dimension. Let us write these down, relabeling "L" as "K":

(19) If R has relative global dimension 0 over K, and K has global dimension 0, then R has global dimension 0. If R has global dimension 0 then it has relative global dimension 0 over (any) K.

2. Examples. The following are the five large classes of examples of K-rings of relative global dimension 0 that I know of. We shall see that all but the last have *generic* relative global dimension 0.

Example 2.1. Group rings. Maschke's classical theorem says that a group algebra of a finite group G, over a field K in which $[G:1]$ is invertible, is semisimple, i.e. of global dimension 0. Group theorists are familiar with the stronger result that if k is any commutative ring, and $H \subseteq G$ are groups such that $[G:H]$ is finite and invertible in k, then kG has relative global dimension 0 over kH. Still more generally, suppose k is any ring, G a group, and R a k-ring containing a family of elements r_g $(g \in G)$ such that

(20) each r_g is invertible in R,

(21) $k\, r_g = r_g\, k$ $(g \in G)$,

(22) $r_{g_1} r_{g_2} \in r_{g_1 g_2} k$ $(g_1, g_2 \in G)$,

(23) $R = \oplus_G\, r_g k$.

(It follows from (20) and (22) that $1 \in r_1 k$.) Now let H be any subgroup of G such that $[G:H]$ is finite and invertible in K. We claim that R has generic relative global dimension 0 over the subring $K = \oplus_H\, r_h k$.

Indeed, let $S \subseteq G$ be a set of left H-coset representatives, and let us imitate the key step in the proof of Maschke's Theorem by setting

(24) $$z = [G:H]^{-1} \textstyle\sum_S\, r_s \otimes r_s^{-1} \in R \otimes_K R.$$

Clearly the natural map $R \otimes_K R \to R$ carries z to 1. To show that z centralizes R, it suffices to show that $\sum_S r_s \otimes r_s^{-1}$ commutes with every $\alpha \in k$ and every r_g $(g \in G)$.

Now note that for $\alpha \in k$ and $g \in G$, (20) and (21) give $r_g^{-1} \alpha\, r_g \in k \subseteq K$. Hence $\alpha \sum r_s \otimes r_s^{-1} = \sum r_s (r_s^{-1} \alpha\, r_s) \otimes r_s^{-1} = \sum r_s \otimes (r_s^{-1} \alpha\, r_s)\, r_s^{-1} = (\sum r_s \otimes r_s^{-1})\, \alpha$.

Second, let $\sigma : G \to S$ be the natural projection. Then from (22) we see that for $g_1, g_2 \in G$, we have $r^{-1}_{\sigma(g_1g_2)} r_{g_1} r_{g_2} \in r_{\sigma(g_1g_2)^{-1}g_1g_2} k \subseteq K$. So for any $g \in G$ we have $r_g \sum_S r_s \otimes r_s^{-1} = \sum_S r_{\sigma(gs)}(r^{-1}_{\sigma(gs)} r_g r_s) \otimes r_s^{-1}$ $= \sum_S r_{\sigma(gs)} \otimes (r^{-1}_{\sigma(gs)} r_g r_s) r_s^{-1} = (\sum_S r_{\sigma(gs)} \otimes r^{-1}_{\sigma(gs)}) r_g = (\sum_S r_s \otimes r_s^{-1}) r_g$, as required.

Example 2.2. Matrix rings. Probably the first case where relative global dimension 0 was proved under that name was Hochschild's observation in [2] that for K any ring, the $n \times n$ matrix ring $M_n(K)$ had relative global dimension 0 over the subring K of scalar matrices. The proof he gave is equivalent to noting that the element $z = \sum e_{i1} \otimes e_{1i} \in R \otimes_K R$ satisfies (5').

Example 2.3. Green's criterion. Let $\varphi : K \to R$ be a ring homomorphism. We noted in (15) that if φ is surjective then R has (generic) relative global dimension 0 over K. If a family $e_1, \ldots, e_n$ of orthogonal idempotent elements of R summing to 1 is given, then E. L. Green [3] has given a strong generalization of this fact. For each $i, j \leq n$ let $K_{ij} = \{\alpha \in K \mid e_i \varphi(\alpha) = \varphi(\alpha) e_j\}$, i.e. the set of elements of K carrying $e_j R$ into $e_i R$ and the complement of $e_j R$ into the complement of $e_i R$. One finds that $K_{hi} K_{ij} \subseteq K_{hj}$, whence one can put these additive subgroups of K together to get a "matrix ring" $T_{e_1,\ldots,e_n}(K) = ((K_{ij}))$, and there is a natural ring homomorphism $T_{e_1,\ldots,e_n}(K) \to R$, defined by $\alpha \mapsto e_i\varphi(\alpha) = \varphi(\alpha) e_j$ $(\alpha \in K_{ij})$. Green shows that R has relative global dimension 0 over K if this map is surjective, and applies this result to the study of an important class of algebras in [4]. His proof reduces, from our point of view, to a verification that $\sum e_i \otimes e_i \in R \otimes_K R$ satisfies (5'), so here too we in fact have generic relative global dimension 0.

Despite the apparent similarity of the last two examples, neither is a case of the other: In the context of Example 2.2 one finds that $\sum e_{ii} \otimes e_{ii}$ does not centralize R, while in the context of Example 2.3 there is no analog of the elements e_{i1} and e_{1i}. Of course, formally one could encompass all three of the preceding examples in the following setup:

Let $\varphi: K \to R$ be a ring homomorphism, and $a_1, \ldots, a_n, b_1, \ldots, b_n$ elements of R. Then it is immediate to verify that sufficient conditions for the element $z = \sum a_i \otimes b_i \in R \otimes_K R$ to satisfy (5') are that $\sum a_i b_i = 1$, and for every $r \in R$ there should exist n^2 elements $\alpha_{ij} \in K$ such that

$$(25) \qquad r\, a_i = \textstyle\sum_j a_j \varphi(\alpha_{ij}) \quad (i \le n) \quad \text{and} \quad \sum_i \varphi(\alpha_{ij})\, b_i = b_j\, r \quad (j \le n).$$

But it is not clear that a formulation of this sort is particularly useful.

Example 2.4. Epimorphisms. Clearly, a trivial case in which the bimodule map $R \otimes_K R \to R$ splits is when it is an isomorphism. This is known to happen if and only if $\varphi: K \to R$ is an epimorphism in the category of rings, giving another generalization of (15) (since the epimorphisms in this category are a larger class than the surjective homomorphisms.) In this case we can take $z = 1 \otimes 1$ in (5').

(Note that the "general setup" of (25), which encompassed Examples 2.1-2.3, does not cover this case unless φ is surjective. For an example of how $1 \otimes 1$ can be R-centralizing when φ is not surjective, and hence also an example showing that the sufficient condition (25) is not necessary, consider the case where R is generated over K by inverses x^{-1} of elements $x \in K$. Then we compute: $x^{-1}(1 \otimes 1) = x^{-1} \otimes (x\, x^{-1}) = (x^{-1}\, x) \otimes x^{-1} = (1 \otimes 1)\, x^{-1}$. Here we have moved x leftwards across the tensor-sign, rather than moving things rightwards as in previous examples. Cf. [6], [7].)

We remark that epic K-rings R are not the only examples in which condition (5') can be satisfied by a rank 1 tensor $z = a \otimes b$. If R is any algebra over a commutative ring k, with a pair of one-sided inverse elements:

$$ab = 1, \quad ba \neq 1, \tag{26}$$

and we let K denote the subalgebra $k + b\,R\,a$, then one finds that (5') is satisfied with $z = a \otimes b$. However the inclusion $K \subseteq R$ is not an epimorphism; this may be deduced from the fact that $ba \in R$ centralizes K but not all of R.

Finally, we have

Example 2.5. Rings of global dimension 0. As noted in (19), if a ring R has global dimension 0, it will have relative global dimension 0 over any K. But in this case it need not have generic relative global dimension 0. For instance, if K is a field and R a K-algebra then by (16) and (19) a necessary condition for R to have generic relative global dimension 0 over K is that for every extension field L/K, $R \otimes_K L$ still have global dimension 0. (Apply (16) with $k = K$, $T = L$. Since $K \otimes_K L = L$ is again a field, (19) requires that $R \otimes_K L$ also be of global dimension 0.) But if, for instance, we take for R an inseparable or transcendental field extension of K, and set $L = R$, this clearly fails.

Since all the above examples of relative global dimension 0 arise from one of the implications

(27)

R has generic relative global dimension 0 over K. $\Rightarrow$ R has relative global dimension 0 over K.

R has global dimension 0. $\Rightarrow$ R has relative global dimension 0 over K.

and since the two conditions on the left are left-right symmetric (cf (17)), all our examples of relative global dimension 0 also have right relative global dimension 0. As I mentioned earlier, I do not know whether there exist K-rings of left but not right relative global dimension 0.

3. Throughout this section, K will be a ring of global dimension 0.

We have seen that in this case the bottom implication of (27) becomes reversible ((19)) but the top implication does not (Example 2.5). In this section we shall investigate which R will in fact have generic relative global dimension 0 over such K.

Note that if

(28) $$z = \sum_{i=1,\ldots,n} a_i \otimes b_i$$

is an element of $R \otimes_K R$, then for any other finite generating set $a'_1, \ldots, a'_{n'}$ of the right K-submodule $\sum a_i K \subseteq R$, we can get an expression $z = \sum a'_i \otimes b'_i$. Hence as $\sum a_i K$ is a direct sum of simple submodules, we can write any z in the form (28) so that we also have:

(29) Each a_i generates a simple right K-module, and the sum of these K-submodules in R is a direct sum.

For each a_i we can find a minimal idempotent $e_i \in K$ (an idempotent which is not a sum of two nonzero orthogonal idempotents) such that

(30) $$a_i = a_i e_i \quad (i=1,\ldots,n).$$

We then have $a_i \otimes b_i = a_i e_i \otimes b_i = a_i \otimes e_i b_i$. Hence replacing each b_i by $e_i b_i$ we can assume

(31) $\qquad b_i = e_i b_i \quad (i=1,\dots,n).$

<u>Lemma 3.1.</u> Suppose K is a semisimple artin ring, and R a K-ring of generic global dimension 0. Let $z \in R \otimes_K R$ be chosen as in (5'), and written as in (28)-(31). Then the left K-module $\sum_i Kb_i$ is a right ideal in R.

<u>Proof.</u> It follows from (29), (30) and the complete reducibility of R as a left K-module that for each $j \leq n$ we can find a right K-linear functional $\theta: R \to K$ such that

(32) $$\theta_j(a_i) = \begin{cases} e_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Now for any $r \in R$, let us apply $\theta_j \otimes id_R: R \otimes_K R \to R$ to the equation saying that z centralizes r, namely

$$\sum a_i \otimes b_i r = \sum r a_i \otimes b_i.$$

Recalling (31) we see that this gives

(33) $$b_j r = \sum_i \theta_j(ra_i)\, b_i \in \sum K b_i.$$

The conclusion of the Lemma clearly follows. ||

Note that since $\sum a_i b_i = 1$, the elements $b_1,\dots,b_n$ have zero common right annihilator. Hence

(34) Under the hypotheses of Lemma 3.1, R has a right ideal with zero right annihilator, which is finitely generated as a left K-module.

Now by (19) we know that R is itself a semisimple artin ring, and by (14) we can even reduce to consideration of the case where R is a matrix ring over a division ring. It is tempting to think that we can deduce from (34) that R will necessarily be finitely generated as a left K-module (and by a symmetric argument, as a right K-module). But in fact, we can only get this conclusion under additional hypotheses. The simplest such case is

Corollary 3.2. If $K \subseteq R$ are division rings and R has generic relative global dimension 0 over K, then R is right and left finite-dimensional over K. In fact

(35) $$[R:K]_{rt} = [R:K]_{left} = n,$$

where n is the number of terms in any expression (28) such that the a_i are right K-linearly independent and the b_i are left k-linearly independent. ||

Corollary 3.3. Suppose K is a field and R a K-algebra of generic relative global dimension 0 over K. Then R is a finite-dimensional semisimple K-algebra. In fact

(36) $$[R:K] \leq n^2$$

where n is the number of terms in any expression (28).

Proof. Left multiply (33) by a_j and sum over j. We get

$$R \subseteq \sum_{i,j} a_j K b_i = \sum a_i b_j K. \ ||$$

Note that in the case $R = M_n(K)$ the upper bound (36) is attained, in contrast to the bound (35) when R is a division ring.

The method of Corollary 3.3 can be used in a slightly more general case; we leave the details of the proof to the interested reader:

Corollary 3.4. Suppose $K \subseteq R$ are algebras over a field k, K is finite-dimensional over k and semisimple, and R has generic relative global dimension 0 over K. Then R is also finite-dimensional over k. ‖

(This is false without the hypothesis that K is semisimple! For instance, if we let R be the 2×2 matrix ring over the free associative algebra $k\langle x_1, \ldots, x_r\rangle$ $(r \geq 1)$, or, for instance, over a division ring rationally generated by this free algebra, and $K \subseteq R$ is the $(r+3)$-dimensional subalgebra having for basis the matrix-units e_{11}, e_{22}, e_{12} and the r elements $e_{12}\, x_i$, then it is well known that $K \to R$ is an epimorphism of rings. Hence R has generic relative global dimension 0 over K, though R is infinite-dimensional.)

The main result of this section is the following precise characterization in the situation of Corollary 3.3:

Theorem 3.5. Let K be a field. Then the K-algebras of generic relative global dimension 0 are precisely the finite direct products of matrix rings

$$(37) \qquad M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r)$$

where each D_i is a finite-dimensional division algebra over K, whose center is a separable extension field of K. ‖

Proof. We already know that any K-algebra of generic relative global dimension 0 will be finite-dimensional and of global dimension 0, i.e. will have the form (37) with each D_i finite-dimensional over K. By (14) it suffices to consider the case $r = 1$, where

$$R = M_n(D).$$

The center C of D will be the center of R. If C is inseparable over K, it is easy to deduce that $R \otimes_K R \to R$ is non-split as a map of $C \otimes_K C$-modules, hence as a map of $R \otimes_K R^{op}$ modules.

On the other hand, if C is separable, then $C \otimes_K C$ is completely reducible, so $C \otimes_K C \to C$ splits over that ring, and so C has generic relative global dimension 0 over K. Furthermore, over C the rings R and R^{op} are finite-dimensional central simple algebras, hence so is $R \otimes_C R^{op}$, so this is completely reducible, so as above we see that R has generic relative global dimension 0 over C. So by (12) R has generic relative global dimension 0 over K. ||

In the more general situation of Corollary 3.4 I don't know whether such a nice criterion can be found, but it would be worth investigating whether some condition such as "center of R separable over its intersection with K" just might work.

Let us now show why we could not deduce in the situation of Lemma 3.1 that R was finite-dimensional as a left K-module, without additional restrictions such as those of Corollaries 3.2-3.4.

Example 3.6. An infinite-dimensional K-ring of generic relative global dimension 0. Let K be a field with a field endomorphism ψ such that $[K:\psi(K)] = \infty$. Let $R = M_n(K)$, but let us make R a K-ring via the map $\varphi(\alpha) = \begin{pmatrix}\alpha & 0\\ 0 & \psi(\alpha)\end{pmatrix}$. We note that $R = \sum_{i,j\leq 2} K e_{ij}$, and that the two summands with $i = 1$ each have left dimension 1 over $\varphi(K)$, while, but the summands with $i = 2$ each have left dimension $[K: \psi(K)] = \infty$. Hence R has infinite left dimension (and by a similar argument infinite right dimension) as a vector space over $\varphi(K)$.

To show that R has generic relative global dimension 0 over $\varphi(K)$, we use $z = \sum e_{i1} \otimes e_{1i}$ just as in Hochschild's proof for full matrix rings, Example 2.2 above. In that example, the need to distinguish one index (taken to be 1) seemed an inelegant necessity, but now it is what makes things work: For $\alpha \in K$ we have $\alpha e_{i1} = e_{i1}\varphi(\alpha)$ and $\varphi(\alpha)e_{1i} = e_{1i}\alpha$, (though the corresponding statements for e_{i2} and e_{2i} are false) so $\sum e_{i1} \otimes e_{1i}$ centralizes R, (though $\sum e_{i2} \otimes e_{2i}$ does not.)

4. Remarks.

I am not entirely happy with the term "generic". (Hochschild points out that its use in algebraic geometry suggests "almost everywhere" rather than the "everywhere" that I mean.) But it seemed the best out of a large number of terms that I and others could think of. Among others suggested by condition (4) were "absolute", which unfortunately conflicts with "relative homological dimension", and "categorical", which I didn't use because all homological algebra is categorical. Condition (5) suggests "hereditary", which is obviously out of the question, "universal" which is used by algebraic geometers but which conflicts with "universal construction"; "conservative" and "persistent" which have already been used by P. M. Cohn to describe particular behaviours under extension of scalars by a pure transcendental field extension, and "tenacious" or "incorrigible", which seemed to emphatic for the case at hand. "Strong" would have conflicted with the existing concept of "weak global dimension".

I leave the development of more general "generic relative homological algebra" to those with more homological background than myself. Note that one apparently cannot eliminate the "relative". For instance, no ring R has "generic global dimension 0" in the sense that every short exact sequence of R-module objects in any abelian category splits — consider the category of

$\mathbb{Z}[t]/(t^2)$-modules. On the other hand, it might be of interest to consider (relative or absolute) homological properties of R-module objects in special classes of abelian categories. For instance, is condition (5) equivalent to the same condition with T restricted to be commutative? (For some vaguely analogous considerations, cf. [9].)

One generalization of the concept of relative global dimension 0 (ordinary or generic) which suggests itself is to consider rings and homomorphisms $K \to R \to S$, with the property that every short exact sequence of S-modules that splits over K splits over R. This is comparable to Isbell's generalization of the concept of an epimorphism $K \to R$ to "the dominion $R \subseteq S$ of a morphism $K \to S$" [7].

5. Appendix. ~~An example~~ A "Banach-Tarski paradox" in field theory. If K is a division ring, and R a K-ring of generic relative global dimension 0, as in §3, then on left multiplying (33) by a_j and summing, we get

$$\sum_{i,j} a_j K b_i = R. \tag{38}$$

When I was first attempting to prove R finite-dimensional over K under some hypotheses, I noted that if the a_j are invertible, (38) can be written

$$\sum_{i,j} (a_j K a_j^{-1})\, c_{ij} = R \tag{39}$$

where $c_{ij} = a_j b_i$. Now if R has infinite left dimension over K it will also have infinite left dimension over each $a_j K a_j^{-1}$. This says that for each j, R is a direct sum of infinitely many, but not a sum of finitely many additive subgroups $(a_j^{-1} K a_j)c$ $(c \in R)$. This makes it seem highly unlikely that R could be written as a finite sum (39) either.

But I found an example, which I will now give, showing that this intuition was quite wrong. This is a field E with two subfields, over each of which E is infinite-dimensional, but such that E is the abelian-group sum of these subfields. In fact, one subfield will be the image of the other under an automorphism of E of order 2, so by going to a twisted group ring R over E we can get this automorphism to be inner in R, and get an equation of precisely the form (38). But we will say no more of this, since the commutative example is interesting for itself, while the questions of §3 are now settled.

In the next Lemma infinite-dimensionality is, of course, implicit in the stronger statement (42).

<u>Lemma 5.1.</u> Let k be a field of characteristic $\neq 2$, E an extension field of K such that

(40) $$\mathrm{tr.deg}_k(E) = [E:k]$$

(which happens if and only if $\mathrm{tr.deg.}_k(E) \geq \aleph_0 + |k|$), and τ an automorphism of E over k of order 2. Then there exists an intermediate field $k \subseteq K \subseteq E$ such that

(41) K is purely transcendental over k,

(42) E is transcendental over K (and hence also over τK), but

(43) $E = K + \tau K$.

<u>Proof.</u> Since we are in characteristic $\neq 2$, we have

(44) $$E = E^+ \oplus E^-,$$

where $E^+ = \{ x \in E \mid \tau x = x\}$ and $E^- = \{ x \in E \mid \tau x = -x\}$.

If we set

(45) $$\alpha = \mathrm{tr.deg}_k(E) = [E:k]$$

and make the standard identification of α with the least ordinal of cardinality α, then by (44), (45) we can write

(46) $$E = \sum_{\beta<\alpha} e_\beta k, \quad \text{with} \quad e_\beta \in (E^+ \cup E^-) - \{0\} \ (\beta<\alpha).$$

Note that E will be algebraic over the subfield E^+. It is easy to deduce that

(47) each of E^+, E^- contains a transcendence basis for E over k.

Now let us fix any element $u \in E$ transcendental over k. We shall construct our subfield $K \subseteq E$ so that

(48) u is transcendental over K,

thus assuring (42). We now proceed to construct a generating set for K as a field extension of k. Let us suppose inductively that for some $\beta < \alpha$ we have chosen elements $x_\gamma\ (\gamma < \beta)$ such that

(49) the elements $u,\ x_\gamma\ (\gamma < \beta)$ are algebraically independent over k.

Because β has smaller cardinality than α, we can now clearly choose an element t_β such that

(50) t_β is not algebraic over $k(u,\ x_\gamma (\gamma < \beta),\ e_\beta)$,

(for e_β see (46)), and by (47) we can impose the further condition

(51) $t_\beta \in \begin{cases} E^- & \text{if } e_\beta \in E^+ \\ E^+ & \text{if } e_\beta \in E^-. \end{cases}$

We now define

(52) $x_\beta = t_\beta + e_\beta$.

From (49) and (50) it follows that the family $(u,\ x_\gamma)_{\gamma \leq \beta + 1}$ is algebraically independent over k. As algebraic independence is preserved on going to unions of transfinite chains, and the $\beta = 0$ case of (49) is given by (48), we get a family $(x_\beta)_{\beta < \alpha}$ such that, if we set

$$K = k(x_\beta\ (\beta < \alpha)),$$

(41) and (48) (hence (42)) hold. We also note that

$$\tau x_\beta = \mp t_\beta \pm e_\beta \quad (\beta < \alpha)$$

where the signs depend on the case of (51) holding for the given β. In any case, we have

$$e_\beta = \frac{1}{2}(x_\beta \pm \tau x_\beta) \in K + \tau K,$$

which by (46) gives (43), completing the proof of the Lemma.||

We could, of course, easily get all kinds of variants of this example if we wished — with $\mathrm{tr.deg.}_K E$ as big as $\mathrm{tr.deg}_k E$; with an automorphism τ of arbitrary order; with K algebraically closed instead of purely transcendental (since going to the algebraic closure does not affect (42)); .... Note that for $[E:k]$ countable the above proof does not require the axiom of choice.

From analysis one can get a "natural" example of a similar sort. Let S denote the Riemann sphere

$$S = \mathbb{C} \cup \{\infty\}.$$

<u>Lemma 5.2.</u> Let E be the field of all meromorphic functions on the punctured plane $S - \{0, \infty\}$, K the subfield of functions meromorphic on the complex plane $C = S - \{\infty\}$, and $\tau: E \to E$ the automorphism $f(z) \mapsto f(z^{-1})$. Then (42) and (43) hold.

<u>Proof</u> (suggested by M. Rosenlicht). (42) is immediate; in fact K is clearly algebraically closed in E but distinct from E. To prove (43) take any $f \in E$, find an annulus

$$A = \{ z \mid r < |z| < r + \varepsilon \}$$

in which f has no poles, and expand f there in a Laurent series. This may

may be written as the sum of a power series in z and a power series in z^{-1}:

(53) $$f = f_1 + f_2 \quad \text{on } A,$$
$$f_1 \text{ defined and analytic on } \{z \mid 0 \leq |z| < r + \varepsilon\}$$
$$f_2 \text{ defined and analytic on } \{z \mid r < |z| \leq \infty\}.$$

By (53) f_1 and $f - f_2$ coincide on A. Since f_1 is analytic for $0 \leq |z| < r+\varepsilon$ and $f - f_2$ is meromorphic for $r < |z| < \infty$ they piece together to give a meromorphic extension of f_1 on $S - \{\infty\}$, hence an element of K. Similarly f_2 and $f - f_1$ piece together to give a meromorphic function on $S - \{0\}$, hence an element of τK. By (53) these functions sum to f on A, hence by analyticity their sum must be f wherever it is defined. So $f \in K + \tau K$. ‖

The above argument can clearly be used to show that if $X, Y \subseteq S$ are closed sets which can be separated by an annulus A, and such that $S - X - Y$ is connected, then any function f meromorphic (respectively holomorphic) on $S - X - Y$ is the sum of a function meromorphic (holomorphic) on $S - X$ and a function meromorphic (holomorphic) on $S - Y$. I would suppose that this should be true even if "separable by an annulus" is weakened to "disjoint" and S replaced by any Riemann surface; does anyone know?

The results of this section are so easily come by that I would imagine they should be known somewhere; I would welcome any references.

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Another class of examples. Suppose K is any ring, $d: K \to K$ a derivation, $R = M_2(K)$, and let us map $K \to R$ by the map $\varphi(x) = \begin{pmatrix} x & dx \\ 0 & x \end{pmatrix}$. One can prove

Lemma A.1. If 2 is invertible in K, then R has generic relative global dimension 0 over $\varphi(K)$, using the element $z = 1/2 \sum_{i,j} e_{ij} \otimes e_{ji} \in R \otimes_{\varphi(K)} R$. On the other hand if K is a commutative integral domain in which 2 is not invertible, and $d \neq 0$, then R does not have generic relative global zero over $\varphi(K)$. (Computations of proof available on request.)‖

Example A.2. Let k be a field, $K = k(t)$ a simple transcendental extension of k, and $d: K \to K$ the derivation carrying t to 1. Define $R = M_2(K)$ and φ as above. Then R will have generic relative global dimension zero over $\varphi(K)$ if and only if char $k \neq 2$. Note that the intersection of $\varphi(K)$ and the center of R, i.e. the largest subfield over which both R and $\varphi(K)$ are algebras, will be the kernel of d in K. This is k if char $k = 0$, and $k(t^p)$ if char $k = p \neq 0$. In the former case K and R are infinite-dimensional over this subfield; we shall say no more about that now. In the latter, we note that (center of $\varphi(K)$)$=\varphi(K)$ is inseparable over $k(t^p)$. On p.15 line 10 we wondered whether separability of that extension might be a necessary and sufficient condition for generic relative global dimension 0. We see by the odd-characteristic case of this example that it is not necessary. The characteristic 2 case shows that a correct criterion for generic relative global dimension 0 must be quite subtle, since the cases are superficially parallel! (I would imagine we can get analogous examples not having generic relative global dimension 0 in odd characteristic p, by using an indecomposable $p \times p$ matrix-representation of an inseparable field extension.)

Incidentally, the condition mentioned is certainly sufficient, for by Theorem 3.5 it implies that R has generic relative global dimension 0 over the intersection of K with the center of R; hence by (13) R has generic relative global dimension 0 over K.

Generalizing Green's criterion (Example 2.3).

Lemma A.3. Suppose $\varphi: K \to R$ is a ring homomorphism, $e_1, \ldots, e_n \in R$ are orthogonal idempotents summing to 1, and $T = T_{e_1,\ldots,e_n}(K) = ((K_{ij}))$ is defined as in Example 2.3. Then if R has (generic, respectively ordinary) relative global dimension 0 over T, it has (generic, respectively ordinary) relative global dimension 0 over K as well.

Proof. Even if $T \to R$ is not surjective, the computation upon which Green's result is based shows that the indicated element z of the (R,R)-bimodule $R \otimes_K R$

centralizes the image of T in R. Hence if a short exact sequence of R-modules in an abelian category $\underline{A}$ has a K-splitting, z gives a T-splitting, and this gives an R-splitting for appropriate $\underline{\underline{A}}$ (depending on our hypothesis on R and T.) ||

This Lemma does not follow from Example 2.3 by (12), because Example 2.3 does not work by proving T to have generic relative global dimension 0 over K — indeed, T is in general not a K-ring; we merely have a diagram $\genfrac{}{}{0pt}{}{K}{T}\!\rightrightarrows R$.

Let us show that the converse of the Lemma is false. Let k be a field of characteristic 0, $K = k[x]$ a polynomial ring in one indeterminate, and $d\colon K \to K$ the derivation taking x to 1. Take $R = M_2(K)$ and $\varphi\colon K \to R$ as in Lemma A.1. Then by that Lemma, R has generic relative global dimension 0 over K. Now if we take for the system of idempotents in Green's construction $e_{11}, e_{22} \in R$, we find that $T = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \subseteq R$. T has global dimension 0, so gl.dim.(R,T) = gl.dim.$(R) = 1$. Clearly this could not happen if $K \to R$ factored through $T \to R$.

<u>"Transfer dimension" 0</u>

Given any ring homomorphisms $\genfrac{}{}{0pt}{}{K}{L}\!\rightrightarrows R$ and a right R-module C, let us say that C is <u>(K,L)-transfer-projective</u> if every short exact sequence (1) of R-modules that <u>splits over K splits</u> over L as well. We can similarly define other (K,L)-transfer homological concepts. In particular it is natural say that R has "generic global (K,L)-transfer dimension 0" if every short exact sequence (1) of R-module objects in any abelian category which splits over K splits over L; equivalently if all R-module objects in all abelian categories are (K,L)-transfer projective. (I suggested in the abstract a more limited case of this concept on p.17.) Then the proof of Lemma A.3 actually shows that

(54) In the context of Green's constr., R has generic (K,T)-transfer dimension 0.

Other easily verifiable results are

(55) If R has generic (K,L)-transfer dimension 0 and generic (L,M)-transfer dimension 0, then R has generic (K,M)-transfer dimension 0.

(56) If $L \subseteq K \subseteq R$, then R has generic (K,L)-transfer dimension 0.

(57) If $L \subseteq R$ is the dominion of $K \to R$ [7], then R has gen. (K,L)-tr. dim. 0 (cf. 2.4).

(58) If $u \in R$ is invertible and $K \subseteq R$, then R has gen. (K, uKu^{-1})-tr. dim. 0.

If R has generic (K,L)-transfer dimension 0, and we let S denote the subring of R generated by the images of K and L, then for any short exact sequence of R-module objects (1) splitting over K, we get a splitting over L, and it is natural to ask whether we will even get a splitting over S, i.e. whether R will have generic global (K,S)-transfer dimension 0. Consider the following example. Let k be a field, R the upper triangular matrix ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, $K \cong k \times k$ the subalgebra spanned by the idempotents $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, and $L \cong K$ the subalgebra spanned by e_{11} and e_{22}. Then by either (54) using the idempotents e_{11} and e_{22}, or (58) using $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we find that R has generic (K,L)-transfer dimension 0. But K and L generate all of R, which does not have relative global dimension 0 over K.

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D. Sarason tells me that the proof of Lemma 5.2 can indeed be adapted to the case where 0 and ∞ are replaced by disjoint closed subsets of S with connected complements. Whether S can be replaced by a more general Riemann surface he didn't know.