[Written around 1979; incomplete—see next page]

THE LIE ALGEBRA OF VECTOR FIELDS ON $\mathbb{R}^n$ SATISFIES POLYNOMIAL IDENTITIES

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William Goldman (student, University of California, Berkeley) has asked (in conversation) whether the Lie algebra of vector fields on the real line generated by two vector fields $f\frac{d}{dx}$ and $g\frac{d}{dx}$ can be free as a Lie algebra on those generators. Surprizingly, the answer is negative, not only on the line, but on any finite-dimensional manifold: we shall exhibit here for each n nontrivial Lie identities (with constant coefficients) holding in the Lie algebra of vector fields on $\mathbb{R}^n$, and hence on any n-manifold. It will be most convenient to first obtain identities in large numbers of variables (§§ 1-3) and then note how these can be made to yield identities in 2 variables (§ 4).

These results are actually algebraic statements about Lie algebras of derivations on a commutative ring. The arguments are combinatoric.

We then further investigate properties of the variety (in the sense of universal algebra) of Lie algebras, $\underline{V}_n$, generated by the Lie algebra of vector fields on $\mathbb{R}^n$, and of some related varieties of Lie algebras.

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2012 Addenda: I was informed by D.Fuchs while working on these notes that the existence of Lie identities for vector fields had also been discovered by B. Lidski, a student of A. Kirillov, and I stopped work on the write-up. (So apologies for their messy incomplete state.) I subsequently heard mention of such work in discussions with other Soviet mathematicians. Lidski's work is apparently unpublished. Some MathSciNet results are given below on this sheet; but I haven't examined them.

In 1999, Askar Dzhumadil'daev wrote me pointing out that assertion (2) of my Theorem 1.2 (on p.3 below) was incorrect. I've crossed out the statement and material depending on it in these notes. (The error in the proof appears on p.6, marked "No!".)

MR0793241
Kirillov, A. A.; Ovsienko, V. Yu.; Udalova, O. D.
Identities in the Lie algebra of vector fields on a straight line.
(Russian)
Akad. Nauk SSSR Inst. Prikl. Mat. Preprint 1984, no. 135, 17 pp.

MR1099432
= translation of above
[with "the real line" for "a straight line" in title]
Selecta Math. Soviet. 10 (1991), no. 1, 7--17.

MR0912412
Molev, A. I.
The algebraic structure of the Lie algebra of vector fields on the line.
(Russian) Mat. Sb. (N.S.) 134(176) (1987), no. 1, 82--92, 142--143;
translation in Math. USSR-Sb. 62 (1989), no. 1, 83--94

MR1064297
Razmyslov, Yu. P.
Finitely generated simple Lie algebras that satisfy the standard Lie identity of degree 5. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1990, no. 3, 37--41, 111; translation in
Moscow Univ. Math. Bull. 45 (1990), no. 3, 39--41

MR0949432
Razmislov, Yu. P.
Constructive reconstruction of a smooth affine algebraic variety from its Lie algebra of vector fields, and simple Lie algebras satisfying a standard identity of degree 5. (Ukrainian. Russian summary)
VÄ«snik KiÃ¯v. UnÄ«v. Ser. Mat. Mekh. No. 27 (1985), 94--96, 128.

1. Definitions. The general C^∞ vector field on $\mathbb{R}^n$ has the form $\sum^n f_i \frac{\partial}{\partial x_i}$ where the f_i are C^∞ functions, and is a derivation on $C^\infty(\mathbb{R}^n)$. We shall prove our results for derivations on an arbitrary commutative ring. We shall find it convenient to consider not only Lie polynomials, but arbitrary polynomials in a family of derivations, i.e. sums and differences of their compositions, which in general will represent higher order differential operators.

Formally, we define a Lie polynomial in variables $U_1, \dots, U_r, \dots$ as an element of the free Lie algebra over $\mathbb{Z}$ in $U_1, \dots, U_n$, $L[U_1, \dots, U_r, \dots]$, and a polynomial as an element of the free associative algebra $\mathbb{Z} < U_1, \dots, U_r, \dots >$. We shall regard $L[U_1, \dots]$ as embedded in $\mathbb{Z} < U_1, \dots >$ (with Lie brackets interpreted as commutators) so that Lie polynomials are in particular polynomials.

Given a Lie polynomial P, homogeneous of degree 1 in one of its variables, V, it is not hard to show that using the Jacobi identity, P can be reduced to a linear combination of monomials of the form $[U_{i_1}, [U_{i_2}, \dots, [U_{i_r}, V] \dots]]$ $(U_{i_j} \neq V)$. So in particular a multilinear Lie polynomial in $U_1, \dots, U_r$ can be written as a linear combination of the monomials $[U_{\pi(1)}, [U_{\pi(2)}, \dots, [U_{\pi(r-1)}, U_r] \dots]]$ where π ranges over the permutation group S_{n-1}. Conversely, these monomials are linearly independent, since when they are expanded as elements of $\mathbb{Z} < U_1, \dots, U_r >$ each has a unique term ending in U_r, namely $U_{\pi(1)} U_{\pi(2)} \cdots U_{\pi(r-1)} U_r$, and these are distinct for distinct π. Hence up to scalars, there is a unique nonzero Lie polynomial multilinear in $U_1, \dots, U_n$ and alternating in the first $r-1$ of these, namely

$$T_r(U_1, \dots, U_{r-1}; U_r) = \sum_{\pi \in S_{r-1}} (-1)^\pi [U_{\pi(1)}, [U_{\pi(2)}, \dots, [U_{\pi(r-1)}, U_r] \dots]].$$

(Incidentally, for $r > 2$, there is no nonzero multilinear Lie polynomial alternating in all r variables. For we see that up to scalars this would have to equal $\sum_{\pi \in S_r} (-1)^\pi [U_{\pi(1)}, \dots, [U_{\pi(r-1)}, U_{\pi(r)}] \dots]$. But by the Jacobi identity applied to $U_{\pi(r-2)}$, $U_{\pi(r-1)}$ and $U_{\pi(r)}$ this is 0.)

2. Our main result is

Theorem 2.1. Let R be a commutative ring, and $D_1, \ldots, D_n\colon R \to R$ derivations. Let $U_1, U_2, \ldots$ denote left R-linear combinations of $D_1, \ldots, D_n$:

$$U_i = \sum_{m \leq n} f_{im} D_m \quad (f_{im} \in R).$$

Then

(1) $T_{n^2+2n+2}(U_1, \ldots; U_{n^2+2n+2}) = 0.$

~~(2) If $D_1, \ldots, D_n$ commute with one another, then $T_{n^2+5}(U_1, \ldots; U_{n^2+5}) = 0.$~~ incorrect!

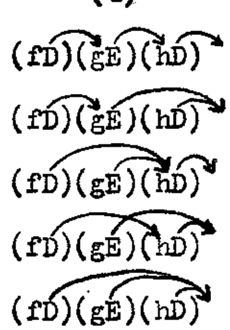
(~~Remark: for n=1, where commutativity is automatic, (1) is a better re~~sult ~~than (2). But as we shall see, the proof of (2) actually would permit us to write $n^2 + 3 + \inf(2, n)$ where we have written n^2+5.~~)

To prove this, let us consider how a composition of operators of the form fD ($f \in R$, $D\colon R \to R$ a derivation) may be "expanded out". If $D\colon R \to R$ is a derivation, $a \in R$ an element, and $X\colon R \to R$ any map, we find that $D\cdot(aX) = (Da)X + a(DX)$ (where Da represents an element of R, but DX a composition of operators.) From this, it is easy to see by induction that a composition of operators of the form fD can be expanded by enumerating all ways in which the derivations can be assigned to ring elements to their right, in a sense which should be clear from the following example: Let $f, g, h \in R$, and $D, E\colon R \to R$ be derivations. One expands (3) by enumerating cases as in (4), and summing the resulting expressions as in (5).

(3)

(fD)(gE)(hD)

(4)



(5)

f(Dg)(Eh) D
+ f(Dg) h ED
+ f g (DEh) D
+ f g (Dh) ED
+ f g h DED

Let us call an expression as in (3) a "crude monomial", the six expressions in (4) the "diagrams" obtained from (3), and the six summands in (5) the "values" of these diagrams. We shall also say that in, for instance, the third diagram of (4), the term (hD) "receives" the derivations from the terms (fD) and (gE), while the derivation from (hD) "exits".

We shall prove the Theorem by showing that when $T_r(U_1, \dots; U_r)$ is expanded, for appropriate values of r, the diagrams arising may be paired off, so that the values of paired diagrams cancel. Let us write a typical crude monomial occurring in the expansion of $T_r(U_1,\dots; U_r)$

(6) $$(f_1 D_{\mu_1})\dots(f_r D_{\mu_r})$$

(ignoring the original indexing of the "f_{ij}'s" in the statement of the Theorem). Let us call the factor $f_i D_{\mu_i}$ in (6) that comes from U_r the "fixed term", and the factors arising from $U_1, \dots, U_{r-1}$ the "movable terms".* As $T_r(U_1, \dots; U_r)$ is alternating in $U_1,\dots, U_{r-1}$, each crude monomial in the expansion belongs to a system of $(r-1)!$ such monomials, having the same fixed term in the same position, but differing in the arrangement of the movable terms, and having the same coefficient but alternating signs.

Each crude monomial (6) yields in turn $r!$ diagrams. Suppose now that θ is one of these diagrams, and that it has the property

(7) θ contains two movable terms $f_i D_{\mu_i}$ and $f_j D_{\mu_j}$ $(i \neq j)$ such that

(i) $\mu_i = \mu_j$ (i.e., the derivations they contribute are the same) and

(ii) f_i receives the same composition of derivations as f_j (possibly the empty composition) in the diagram θ.

Among all pairs (i,j) satisfying (i) and (ii) in the given diagram θ let us choose the one that minimizes i, and minimizes j for that value of i.

*Thus, we are really thinking of (6) as a product with a distinguished factor, the fixed term, and of T_r as decomposed into a linear combination of such products.

(We could use any other fixed rule for selecting a unique pair from every nonempty set of pairs.) Now define $\alpha\theta$ to be the diagram obtained from θ by interchanging the movable terms $f_i D_{\mu_i}$ and $f_j D_{\mu_j}$, but leaving all other terms, and all arrows, in the same place.

One sees that $\alpha\theta$ again satisfies (7), and that $\alpha\alpha\theta = \theta$. Further the values of θ and $\alpha\theta$ will be equal, and by the alternating condition they will occur in the expansion of $T_r(U_1,\ldots; U_r)$ with the same coefficient but opposite sign. Hence in this expansion, the values of all diagrams satisfying (7) will cancel, and such diagrams can henceforth be disregarded.

Now for what values of r will there exist any diagrams not satisfying (7)? If θ is such a diagram, note that it can involve at most n movable terms $f_i D_{\mu_i}$ that do not receive any derivations (one for each possible value of D_{μ_i}), and at most n^2 movable terms that receive exactly one derivation (n possible values of D_{μ_i} and n of the derivation received.) Also,the number of movable terms receiving more than one derivation can be no greater than the number which receive no derivations, since the total number of derivations received is at most the number of movable terms. (At least one derivation exits, but one is contributed by the fixed term.) So the number of movable terms is at most $n + n^2 + n$, so the total number of terms is at most $n + n^2 + n + 1$. So if we take $r = n^2 + 2n + 2$, the values of all diagrams will cancel, establishing (1).

To get (2), consider again the expansion of $T_r(U_1,\ldots; U_r)$ for arbitrary r, and suppose we have discarded all diagrams satisfying (7). Suppose now that a diagram θ arising from the crude monomial (6) satisfies

(8) θ contains a movable term $f_i D_{\mu_i}$ such that <u>either</u>

(i) $f_{i+1} D_{\mu_{i+1}}$ is again a movable term, but this term does not receive the derivation of $f_i D_{\mu_i}$, <u>or</u>

(ii) $f_{i+1} D_{\mu_{i+1}}$ is the fixed term, and this is followed by another movable term $f_{i+2} D_{\mu_{i+2}}$ (i.e. $i \leq r-2$), but neither of these terms receives the derivation from $f_i D_{\mu_i}$, nor does $f_{i+2} D_{\mu_{i+2}}$ receive the derivation from the fixed term $f_{i+1} D_{\mu_{i+1}}$.

In this case let i be the least index (or any other consistently chosen one) for which (i) or (ii) holds, and let us define $\beta\theta$ to be the diagram obtained by transposing the movable term $f_i D_{\mu_i}$ with $f_{i+1} D_{\mu_{i+1}}$ in case (i), or with $f_{i+2} D_{\mu_{i+2}}$ in case (ii), but this time letting these terms "carry with them" the heads or tails of any associated arrows. We see that $\beta\theta$ will again be a "well-formed" diagram (i.e., arrows go from left to right, which would not be true if (i) respectively (ii) failed), which will occur in the expansion of $T_r(U_1, \ldots; U_r)$ with the same coefficient as θ but opposite sign, and will not satisfy (7) but will satisfy (8), with $\beta\beta\theta = \theta$. ← No! Further, <u>if</u> $D_1, \ldots, D_n$ commute, then θ and $\beta\theta$ will have the same value, and hence will cancel in $T_r(U_1 \ldots; U_r)$. (It is only in the case where the derivations from the two transposed terms, or from one of them and the fixed term if this lies between them, are received by a common term, or both exit, that commutativity is even needed.)

So we now ask when there can exist a diagram θ satisfying neither (7) nor (8). In such a θ, the derivation coming from any term other than the fixed term, the term immediately preceding the fixed term, and the last term must be received by the immediately following term. Hence there can be at most two movable terms receiving no derivations: the first and the one immediately following the fixed term. Also, at most one derivation, namely that of the fixed term or the immediately preceding movable term, but not both, can be received by a term other than that immediately following it, and which receives a derivation from elsewhere. So there can be at most one term receiving more than one derivation. Finally, as before, because θ does not satisfy (7) there can be at most n^2 movable terms receiving exactly one derivation. Hence the total number movable terms is at most $2 + n^2 + 1$. Throwing in the fixed term, and one more movable term, we see that for $r \geq n^2 + 5$, all diagrams satisfy (7) or (8), so $T_{n^2+5}(U_1,\ldots; U_{n^2+5}) = 0$, proving (2) and completing the proof of the Theorem. (Note that when we said there could be at most 2 terms receiving no derivation, we could have made this $\max(2,n)$, getting the answer $n^2+3 + \max(2,n)$ which combines (2) with the n=1 case of (1), as noted parenthetically after the statement of the Theorem.)

3. Remarks.

3.1. To put our results on derivations in perspective, let us consider n arbitrary k-linear maps $D_1,\ldots, D_n$: $R \to R$, where k is a commutative ring and R a commutative k-algebra. Suppose we form linear combinations of these, $U_i = \sum f_{im} D_m$, but with coefficients $f_i \in k$, rather than coefficients from R as above. Then it is trivial that any polynomial in the U_i's which is multilinear and alternating in more than n of them (for instance, $T_{n+2}(U_1,\ldots; U_{n+2})$) will be identically zero. If in fact the D_m's commute, then so will the U_i's, and we even have $T_2(U_1; U_2) = 0$.

On the other hand, suppose that, as in Theorem 2.1, we allow coefficients f_i from R. These will not in general commute with the D_i; and now the U_i need not satisfy any polynomial relations at all. (One can get an example where $n = 1$, D_1 is an <u>automorphism</u> of R, and the $U_i = f_i D_1$ satisfy no polynomial relations.)

So the intermediate situation of Theorem 2.1, where we get identities, but not for trivial reasons, occurs because the D_m are there not assumed to commute with the coefficient ring, but are instead assumed to act on it by derivations.

3.2 The method of proof of Theorem 2.1 can be used to give many variant results. For instance, the Lie polynomials T_r ($r = n^2 + 2n + 2$ or $n^2 + 5$) can clearly be replaced by any associative polynomials multilinear in $U_1, \ldots, U_r$ and alternating in $r-1$ of these. (But associative polynomials satisfied by derivations are less interesting than Lie polynomials, because they cannot be interpreted as relations holding identically in any Lie or associative algebra.)

We can also get results allowing more than one "non-alternating" variable. For instance, any multilinear polynomial in $r = n^2 + 2n + 2a$ (respectively $r = n^2 + 2a + 3$) variables and alternating in all but a of these is satisfied by $U_1, \ldots, U_r$ as in (1) (respectively (2)). (This result can be improved slightly when a becomes large compared with n and the limitation on the possible number of terms receiving 0, 2, 3 etc. derivations comes into play.)

Further, the alternating condition itself can be generalized. For instance, for any integer $h > 1$, we may consider multilinear (Lie or associative) polynomials $Q(U_1, \ldots, U_s; U_{s+1}, \ldots, U_{s+a})$ such that the images of Q under the full group of permutations of any h of the variables $U_1, \ldots, U_s$ sum to 0. (The h=2 case is the alternating condition. Note that a polynomial such as T_r which is alternating in all but one variable will satisfy the $h = 3$ condition in all its variables.) We see that if we make the number s large enough so that every diagram must have at least one h-tuple of terms that both give and receive the same combination of derivations, then such a polynomial Q will be identically zero on a family of derivations U_m as in (1). In general, it would be worth studying the question: If Q is a multilinear Lie polynomial in r variables, and A_Q is the linear representation of the symmetric group S_r arising by looking at formal permutations of the variables of Q, what properties of A_Q can insure that A is an identity for derivations as in (1) or (2)?

3.3 I do not know whether the degrees n^2+2n+2 in (1) and n^2+5 in (2) are in fact the least values for which the identities T_r hold, under the stated hypotheses. As we shall see in § , if a polynomial identity $P = 0$ is satisfied under the hypotheses of (1) or (2) this can indeed be verified simply by formally expanding the expression $P(U_1, \dots, U_r)$ with $U_i = \sum f_{im} D_m$, as above. But our analysis of the expansion of $T_r(U_1, \dots ; U_r)$ was quite limited: We only considered the cancellation of certain types of pairs of terms related by simple transpositions of the U_i. Equal terms can also arise in the expansions of monomials related by other members of S_{r-1}, and I simply do not know whether, for some values of r smaller than those given in (1) and (2), the coefficients of every such term cancel.

A comparison with the theory of polynomial identities for associative rings suggests pessimism. There one is interested in the identities satisfied by the ring of d x d matrices over a commutative ring R. One defines the "standard polynomial" $S_r(X_1, \dots, X_r) = \sum (-1)^{\pi} X_{\pi(1)} \cdots X_{\pi(r)}$. It is easy to verify that a d x d matrix ring satisfies $S_{d^2}=0$, because modulo its center it is (d^2-1)-dimensional. But the true minimal identity satisfied (for which no easy proof was known until quite recently) is S_{2d} ([ ], cf. [ ], [ ]).

If one wishes to study the above question further, it may be useful to know the expansion of $T_r(U_1,\ldots; U_r)$ as an associative polynomial. From the fact that it is alternating in $U_1,\ldots, U_{r-1}$ it is clear that T_r is a linear combination of the r alternating associative polynomials $S_{(i)r}(U_1,\ldots, U_{r-1}; U_r) =_{\text{def.}} \sum_{\pi\in S_{r-1}} (-1)^{\pi} U_{\pi(1)}\ldots U_{\pi(i-1)} U_r U_{\pi(i)}\ldots U_{\pi(r-1)}$ $(i=1,\ldots, r)$. Writing $T_r = \sum_i a_{i,r} S_{(i)r}$, one gets the recursion formula $a_{i,r} = a_{i-1,r-1} - (-1)^r a_{i,r-1}$. (To see this, note that $T_r = \sum_{j<r} (-1)^{j-1}[U_j, T_{r-1}(\ldots, \hat{U}_j,\ldots; U_r)]$ and consider the coefficient of $U_1\ldots U_i U_r U_{i+1}\ldots U_{r-1}$.) One can deduce that $a_{i,r}$ is the coefficient of t^{i-1} in $(t-1)(t+1)(t-1)\ldots(t+(-1)^{r-1})$. According as r is odd or even, this is a power of t^2-1, or is $t-1$ times such a power. In each case an expression for $a_{i,r}$ as ± 1 times a binomial coefficient results.

For $d\times d$ matrices over a commutative ring it is known not only that the least r for which the standard associative identity S_r holds is $2d$, but that $2d$ is in fact the least degree of any nontrivial identity, and that S_{2d} is, up to scalars, the lone identity of that degree. For our Lie algebras of linear combinations of n commuting derivations, if r is the least value for which T_r is identically 0 (whatever that value is), it would be desirable to know whether it is also the least value for which any Lie identity is satisfied, and whether the only identities of that degree are the linear combinations of those obtained from T_r by permuting variables. (Over the rationals, these span an $r-1$-dimensional space of Lie polynomials, which is an irreducible representation of the permutation group on the r variables.)

3.4. Vector fields. Taking $R = C^{\infty}(\mathbb{R}^n)$ and $D_i = \frac{\partial}{\partial x_i}$, we see that statement (2) of Theorem 2.1 translates to say that the Lie algebra of all C^{∞} vector fields on $\mathbb{R}^n$ satisfies T_{n^2+5}. It is easily deduced using local coordinates that the same is true on any n-manifold. Statement (1) similarly translates to say that given an n-dimensional tangent-space distribution* on

*Variously called a "distribution", "Frobenius distribution", "Chevalley distribution" or "differential system". Joe Wolf suggests "tangent-space distribution" as an easily-understood term which avoids the possible confusion with "Schwartz distribution" that has led to the recent proliferation of terms.

a manifold of any dimension, any vector-fields $U_1, \ldots, U_{n^2+2n+2}$ belonging to the distribution will satisfy T_{n^2+2n+2}. Note, however, that this does not make T_{n^2+2n+2} an identity of any Lie algebra of vector fields since the class of vector-fields belonging to a tangent-space distribution is not, in general, a Lie algebra. If it happens that it is, then Frobenius's Theorem tells us that the manifold can be foliated by integral manifolds of the distribution [§3.5], [p.10]. Taking local coordinates $x_1, \ldots, x_n$ on these integral manifolds in a coherent fashion, we get a basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ for the distribution, which commute, so that the sharper result (2) in fact applies in this case.

It is natural to ask whether there is an algebraic analog to this last observation. We shall give such a result below under an algebraically convenient hypothesis, namely that our given commutative ring is a field. However, it will be easily seen that the proof, a modified version of the proof of Frobenius's Theorem in | |, can be adapted to varied hypotheses (e.g., to vector-fields $D_1, \ldots, D_n$ on a manifold, in the neighborhood of any point where these give linearly independent tangent vectors. E. R. Kolchin informs me that the next result and Proposition 3.7 below are known to workers in differential algebra, but have apparently not appeared in print.)

Let K be a field. Note that the derivations of K into itself form a left K-vector-space. Further, if S is any family of derivations such that the Lie bracket of any two members of S is a left-K-linear combination of members of S, then the left K-span KS is closed under commutator brackets.

Lemma 3.5 Let K be a field, and V a left K-vector-space of derivations on K, which is finite-dimensional and closed under commutator brackets. Then V has a left basis $D_1, \dots, D_n$ consisting of pairwise commuting derivations.

Proof. Let $E_1, \dots, E_n$ be an arbitrary K-basis of V. From the fact that these derivations are K-linearly independent operators on K, it follows that we can find $f_1, \dots, f_n \in K$ such that the matrix $((E_i f_j))$ is nonsingular. Let $((c_{ij}))$ be its inverse, so that

(9) $$\textstyle\sum_i c_{hi} E_i f_j = \delta_{hj} \quad (h, j \leq n).$$

Applying the invertible matrix $((c_{ij}))$ to the n-tuple of derivations (E_i), we get a new basis for V, consisting of the derivations

(10) $$D_h = \sum c_{hi} E_i.$$

By (9) and (10) we see that

(11) $$D_h f_j = \delta_{hj}.$$

It follows immediately that all the compositions $D_h D_i$ annihilate all f_j, hence so will all the commutators $[D_h, D_i]$. But these commutators belong to V, hence are linear combinations of the D's, hence by (11) they are determined by their values on the f_j's. Hence they must be zero, as claimed. ‖

Using the theory of localization of commutative rings, we can now get:

Corollary 3.6. Let R be a commutative ring without nilpotent elements, and L a left R-module of derivations on R, which has dimension $\leq n$ in the sense that $\Lambda^{n+1} L = 0$. Then

(12) $T_{n^2+2n+2} = 0$ identically on L, and

(13) If L is closed under commutator brackets, then $T_{n^2+5} = 0$ identically on L.

Proof. Suppose r is an integer such that T_r is not identically 0 on L. Then we can find a nonzero element $a \in R$ of the form $a = T_r(U_1, \ldots; U_r) f$ ($U_i \in L$, $f \in R$). Since a is not nilpotent, the multiplicative semigroup it generates does not contain 0, and we can find a maximal multiplicative semigroup $S \subseteq R - \{0\}$ which contains a. From the fact that R contains no nilpotent elements, and the maximality of S, it follows that $S^{-1}R$ will be a field K. Every derivation $D \in L$ induces a unique derivation $\bar{D}$ on K, and it is easy to deduce that this process respects commutator brackets and the R-module structure of L. Hence $\Lambda^{n+1} \bar{L} = 0$, and if we let V denote the K-vector-space $K\bar{L}$, we have $\Lambda^{n+1} V = 0$, i.e. $\dim_K V \leq n$. Note also that

(14) $$T_r(\bar{U}_1, \ldots; \bar{U}_r) \bar{f} = \bar{a} \neq 0$$

because we made S invertible in K.

We can now apply Theorem 2.1 (1) to (14) to conclude that $r < n^2+2n+2$. If L is closed under commutator brackets, so that V is as well, we can apply Lemma 3.5 and Theorem 2.1 (2) to conclude that $r < n^2+5$. Thus we get (12) and (13) respectively. ||

(I do not know whether the above result remains true if R is allowed to have nilpotent elements. If we let N denote the nil radical of R, and if R has no additive 2-torsion, then all derivations on R take N into N, and we can apply the above Corollary to R/N to conclude that T_r (for the indicated values of r) will be N-valued. By the same localization approach used above, the general question can be reduced to the question for a local ring with nilpotent maximal ideal.)

The proof of Frobenius's Theorem in | | had to be simplified to get our proof of Lemma 3.5, since the original proof involved solving differential equations, which cannot be done within an arbitrary field with derivations. But that more complicated proof had an (unstated) bonus: In the constructed basis $D_1, \ldots, D_n$, the first term D_1 could be chosen arbitrarily. To get the analogous algebraic statement we must allow ourselves to extend the field K and the derivations on it. The sort of extension we want is, of course, one that preserves the Lie relations on these derivations. This condition could be formulated directly, but we shall take a shortcut. We use the present version of Proposition 3.5 to choose an arbitrary commuting basis $d_1, \ldots, d_n$ of V; then say that we wish to look at extension fields K' of K given with extensions of $d_1, \ldots, d_n$ which continue to commute. (It is not hard to verify that this is equivalent to extending the action of V so as to preserve commutator brackets and K-linear relations.)

This, in fact, puts us into the framework of <u>differential</u> <u>algebra</u>, which studies fields K (or sometimes general commutative rings R) given with families $d_1, \ldots, d_n$ of pairwise commuting derivations | |. We shall state the next result in the language of differential algebra. (However, except in an appendix, § , where we prove a lemma used in the next result, we shall not use the language of differential algebra in the rest of this paper.)

Proposition 3.7. Let $(K; d_1, \ldots, d_n)$ be a differential field, with $d_1, \ldots, d_n$ left linearly independent over K. Let $D_1, \ldots, D_m$ be $m \le n$ linearly independent and pairwise commuting left K-linear combinations of $d_1, \ldots, d_n$. Then over some extension $(K'; d_1, \ldots, d_n)$ of the differential field $(K; d_1, \ldots, d_n)$, the given family can be extended to a commuting basis $D_1, \ldots, D_n$ of the K'-linear span of $d_1, \ldots, d_n$.

Proof. Let $D_i = \sum g_{ij} d_j$. The D_i are linearly independent, hence after a possible reordering of $d_1, \ldots, d_n$, we can assume the $m \times m$ matrix $((g_{ij}))$ $(i, j \le m)$ is nonsingular; i.e. that $D_1, \ldots, D_m, d_{m+1}, \ldots, d_n$ form a basis for the span of $d_1, \ldots, d_n$.

Now let us form K' by adjoining to K solutions $f_1, \ldots, f_n$ to the system of linear partial differential equations $D_i f_j = \delta_{ij}$ $(i \le m, \ j \le n)$. By a lemma in differential algebra which we prove in an appendix, § below, such an extension is possible, and moreover may be performed so that the elements $d_i f_j$ $(m < i \le n, \ 1 \le j \le n)$ are algebraically independent over K.

This means that when we apply the n derivations $D_1, \ldots, D_m, d_{n+1}, \ldots, d_n$ to the n elements $f_1, \ldots, f_n$, we get a matrix with block form $\binom{I\ 0}{*\ *}$, where the $*$'s consist of algebraically independent entries. In particular, this matrix will be invertible. As in the proof of Proposition 3.5, we now apply the inverse of this matrix to our system of n derivations, and conclude that the resulting system forms a pairwise commuting basis of the space of derivations we are considering. But the block form of the matrix insures that the first m terms of this basis will remain $D_1, \ldots, D_m$. ||

3.8. Suppose now that we are interested in investigating whether a given polynomial $P(U_1, \ldots, U_r)$ is satisfied identically by linear combinations of n commuting derivations on commutative rings R. For reasons that will be made precise in , we can assume without loss of generality that R is a field. We can also certainly assume without loss of generality that $U_1 \neq 0$. But by the above Proposition we can now go to an extension field of

R, over which the vector space spanned by the given n commuting derivations is also spanned by a system $D_1, \ldots, D_n$ with $U_1 = D_1$. It follows that in determining whether $P = 0$ is an identity, we may without loss of generality take $U_1 = D_1$. Of course, in many arguments, like those of Theorem 2.1, there is no advantage to such a reduction; but it can be convenient when performing hand computations for low values of r, as we shall do in parts of the next section.

4. Identities in two variables, and some precise results on the case n=1.

4.1 Subalgebras of free Lie algebras. It is known that every Lie subalgebra of a free Lie algebra over a field k is again free, and that the free Lie algebra on two generators, $L_k[x,y]$ contains subalgebras free on countably many generators, $L_k[V_1(x,y), V_2(x,y), \ldots]$ ( ). It follows immediately that if a Lie algebra $\underline{A}$ satisfies a nontrivial identity in r variables, $f(U_1,\ldots, U_r) = 0$, then it also satisfies the nontrivial identity in two variables, $f(V_1(x,y),\ldots, V_r(x,y)) = 0$.

The next Lemma exhibits some explicit free subalgebras of free Lie algebras, thus allowing us to write down explicit 2-variable identities in our Lie algebras of derivations. It also turns out to be a very convenient aid to computations in free Lie algebras.

Lemma 4.2. Let k be a commutative ring, and X, Y be disjoint sets. Then in the free Lie algebra $L_k[X \cup Y]$, the elements

(15) $$(\mathrm{ad}\ x_1)\ldots(\mathrm{ad}\ x_n)\ y \qquad (n \geq 0;\ x_1,\ldots, x_n \in X;\ \ y \in Y).$$

form a free generating set as Lie algebra for the ideal of $L_k[X \cup Y]$ generated by Y.

Proof. To show that the elements (15) satisfy no Lie relation it will certainly suffice to show that in the free associative algebra $k\langle X \cup Y \rangle$ they satisfy no associative-algebra relation. This in turn will follow if we can find a semigroup-ordering on the free semigroup $\langle X \cup Y \rangle$ such that the leading terms (i.e., maximal nonzero terms) with respect to this order, of the elements (15) occur with coefficient 1 and satisfy no semigroup relation in the semigroup $\langle X \cup Y \rangle$.

To do this, choose any total ordering on $X \cup Y$ such that all elements of Y are greater than all elements of X, and order $\langle X \cup Y \rangle$ by putting $U > V$ if U has greater length than V, or if they have the same length, and in the last place where they differ, the letter occurring in U is greater than the letter occurring in V. (I.e., for words of equal length use lexicographic order reading from the right.) Now in the expansion of (15) in $k < X \cup Y >$, there is a unique term ending with y, namely $x_1 \ldots x_n y$, so this is the leading term of (15). It is clear that in any product of elements of $< X \cup Y >$ of this form, the factors are uniquely recoverable. (Split it after each member of Y occurring.) Hence these elements generate a free subsemigroup, completing the proof of the independence of the terms (15).

Let us call the set of elements as in (15) $< \mathrm{ad}\ X > Y$. The Lie algebra which it generates clearly lies in the ideal of $L_k[X \cup Y]$ generated by Y. Conversely, we see that this Lie subalgebra is closed under bracketting with all elements of X, because $< \mathrm{ad}\ X > Y$ is, and it is closed under bracketting with all elements of Y because it contains these. Hence it is an ideal of $L_k[X \cup Y]$, and as it contains Y, it must be precisely the ideal indicated.‖

4.3. In particular, in the free Lie algebra on two generators $L_k[x, y]$, the ideal generated by y (spanned as an R-module by all Lie monomials other than x) is free as a Lie algebra on $\{ (\mathrm{ad}\ x)^n y \mid n = 0, 1, \ldots \}$. Hence if a Lie algebra $\underline{L}$ satisfies a multilinear identity of degree r, $P(x_1, \ldots, x_r) = 0$, it will satisfy the 2-variable identity $P(y, \ldots, (\mathrm{ad}\ x)^{r-1} y) = 0$ of degree $r(r-1)/2$ in x and r in y, and thus of total degree $r(r+1)/2$. For instance, the Lie algebra of vector fields on the real line satisfies the nontrivial identity $T_5(y, (\mathrm{ad}\ x)y, (\mathrm{ad}\ x)^2 y, (\mathrm{ad}\ x)^3 y; (\mathrm{ad}\ x)^4 y) = 0$, of total degree 15.

For r large there are tricks that will give us 2-variable identities of smaller degree than $r(r+1)/2$. Within the subalgebra of $L_k[x, y]$ freely generated by $\{(\mathrm{ad}\ x)^i y\}$, consider the ideal generated by all elements $(\mathrm{ad}\ x)^i y$ such that $i > 0$. (This is the commutator ideal of $L_k[x, y]$.) Again applying Lemma 4.2, we see that this will have the free generating set $\{(\mathrm{ad}\ y)^i (\mathrm{ad}\ x)^j y \mid$

$i \geq 0,\ j > 0\}$. Note that this has $d-1$ generators of each degree d rather than just one. One can deduce that a multilinear expression of degree r may be transformed into a 2-variable expression whose degree is on the order of $r^{3/2}$. To get a still better rate of growth consider, within the Lie algebra freely generated by $\{(\mathrm{ad}\ x)^i\ y\}$ the ideal generated by y alone. This will have as free generators the elements $(\mathrm{ad}\ (\mathrm{ad}\ x)^{i_1}y)\ldots(\mathrm{ad}\ (\mathrm{ad}\ x)^{i_n}y)\ y\quad (n \geq 0,\ \ i_1,\ldots,i_n \geq 1)$. One can show that the number of these of degree d is precisely f_{d-2}, the $d-2^{nd}$ Fibonacci number. We leave further investigation of this phenomenon to the interested reader.

The identity T_5 for vector fields on the line is of too small a degree for the above tricks to help, but there is another observation we can apply. Recall that the Lie polynomial $T_r(U_1,\ldots,\ U_{r-1};\ U_r)$ $(r > 2)$ is alternating in the first $r-1$ variables, but not in all r. This means that it becomes the zero polynomial if any two of $U_1,\ldots,\ U_{r-1}$ are identified, but not if any two of $U_1,\ldots,\ U_r$ are. It is easily deduced that $T_r(U_1,\ldots,\ U_{r-1};\ U_1)$ is a nontrivial Lie polynomial, but of course it is identically zero on any Lie algebra on which T_r is zero. It follows that vector fields on the line satisfy $T_5(y,\ (\mathrm{ad}\ x)y,\ (\mathrm{ad}\ x)^2y,\ (\mathrm{ad}\ x)^3y;\ y) = 0$, a nontrivial identity of degree only 11.

<u>4.4.</u> So far we have been obtaining Lie polynomials

$$(16)\qquad T_r(\ V_1(x,y),\ \ldots;\ V_r(x,y)\ )$$

which had to be nonzero in $L_k\,[x,y]$ by general arguments. But other polynomials of the form (16) have a <u>chance</u> of being nontrivial, as long as $V_1,\ldots,\ V_{r-1}$ are linearly independent, and $V_r \neq 0$. For $r = 5$ (the case of vector fields on the line) the two simplest polynomials with these properties (modulo interchange of x and y) are

(17) $$T_5(x,\ y,\ [x,y],\ [x,[x,y]];\ x),$$

(18) $$T_5(x,\ y,\ [x,y],\ [y,[x,y]];\ x),$$

both of degree 8. To aid us in determining whether these represent nonzero elements of $L_k[x,\ y]$, let us first note that in expanding an instance of T_5, each term

(19) $$(\mathrm{ad}\ a)(\mathrm{ad}\ b)(\mathrm{ad}\ c)(\mathrm{ad}\ d)\ e$$

can be paired off with the term $-(\mathrm{ad}\ b)(\mathrm{ad}\ a)(\mathrm{ad}\ c)(\mathrm{ad}\ d)\ e$, and the sum of the two reduced to $(\mathrm{ad}\ [a,b])(\mathrm{ad}\ c)(\mathrm{ad}\ d)\ e$. This in turn can be paired off with $-(\mathrm{ad}\ [a,b])(\mathrm{ad}\ d)(\mathrm{ad}\ c)\ e$ to give $(\mathrm{ad}\ [a,b])(\mathrm{ad}\ [c,d])\ e$, or $[[a,b],\ [[c,d],e]]$. In this way, the $(5-1)! = 24$ terms of T_5 are reduced to $24/4 = 6$. Applying this to (17), and writing y_i for $(\mathrm{ad}\ x)^i\ y$, we get

(20) $$T_5(x,\ y,\ [x,y],\ [x,[x,y]];\ x)$$
$$= [[x,y_0],\ [[y_1,y_2],x]] + [[x,y_1],[[y_2,y_0],x]] + [[x,y_2],[[y_0,y_1],x]]$$
$$+[[y_0,y_1],\ [[x,y_2],x]] + [[y_1,y_2],[[x,y_0],x]] + [[y_2,y_0],[[x,y_1],x]].$$

We can now use the fact

(21) $$[x,y_i] = y_{i+1}$$

to simplify (20) to a Lie polynomial in the y_i's alone, which will be nonzero if and only if it is so as a member of $L_k[y_0,\ y_1,\ldots]$. It is easy to see that when we do so, the variable y_4 will appear exactly once, namely in the fourth term of (20), which reduces to $[[y_0,y_1],-y_4]$. Since this is nonzero, it follows that (17) is nonzero, and represents a nontrivial identity in x and y satisfied by vector fields on the line, of degree 5 in x and 3 in y.

When we expand (18) we get an expression like (20), except that y_2 is everywhere replaced by $[y_0,y_1]$. In this case, we find that application of (21) leads to exactly two occurrences of y_3. Namely, in the expansion of the fourth term and the sixth term, we get summands $-[[y_0,y_1], [y_0,y_3]]$ and $-[[[y_0,y_1],y_0],y_3]$ respectively. To check whether these cancel, we may expand them as linear combinations of $(\mathrm{ad}\ y_3)(\mathrm{ad}\ y_0)(\mathrm{ad}\ y_0)\ y_1$ and $(\mathrm{ad}\ y_0)(\mathrm{ad}\ y_3)(\mathrm{ad}\ y_0)\ y_1$, which we can see by Lemma 4.2 to be independent. The second of these elements occurs only in the first summand, and there it has coefficient +1. Hence (18) is also nontrivial. Note that it has degree 4 in x and 4 in y.

If we interchange x and y in (17) we get an identity of degree 3 in x and 5 in y. If we do this to (18), the resulting identity again has degree 4 in x and 4 in y, and it is not evident whether it is equivalent to (18). In an appendix, § below, we shall get our hands dirty with computations and determine all 2-variable identities of degree ≤ 8 satisfied for $n = 1$. We shall find that (except in characteristic 2) there are precisely three such linearly independent identities, one each of degrees (3,5), (4,4), and (5,3) in x and y respectively. Thus, the consequences of T_5 determined above give all the 2-variable identities of minimal degree. We shall also sketch proofs that T_5 itself is the essentially unique multilinear identity of degree ≤ 5, and note some results on the case of finite characteristic. (§ requires § , but the two of these do not depend on any intervening material.)

We remark that though 5 and 8 are the least degrees of multilinear, respectively 2-variable Lie polynomials satisfied identically for $n = 1$, there are associative identities of lower degrees. The multilinear identity $S_3(x_1, x_2, x_3) = \sum (-1)^{\pi} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}$ and the 2-variable identity $x^2y^2 - 3(xy)^2 + 2\, xy^2x + 2\, yx^2y - 3(yx)^2 + y^2x^2$ are easily established.

continuation of above?:

4.5 Bases for free Lie algebras. Suppose $L_k[X]$ is a free Lie algebra, and $u \in L_k[X]$ some monomial other than one of the variables. Let y be one of the variables occurring in u. Then by Lemma 4.2 we can express u as a Lie polynomial in the elements of the set $< ad(X-\{y\})> y$, and we see that it will have smaller degree in this set of variables than in X. This idea, slightly refined, can be used to construct a k-module basis for $L_k[X]$:

Lemma 4.6. Let k be a commutative ring, and $L_k[X]$ a free Lie algebra, where X is a totally ordered set. For any $y \in X$, write $X_{<y} = \{x \in X \mid x < y\}$, and let us order the set of monomials $< \mathrm{ad}\, X_{<y} > y$ by putting $(\mathrm{ad}\, x_1)\ldots(\mathrm{ad}\, x_n)\, y < (\mathrm{ad}\, x_1')\ldots(\mathrm{ad}\, x'_{n'})\, y$ if $n < n'$, or if $n = n'$ and $x_i < x_i'$ for the largest i such that $x_i \neq x_i'$. Then a basis for $L_k[X]$ as a free k-module is given by the set of "acceptable monomials" defined inductively as follows:

(i) For any $y \in X$, the only acceptable monomial involving no indeterminate but y is y itself.

(ii) If u is a monomial involving more than one indeterminate, let y be the largest indeterminate (under the given ordering) occurring in u. Then u is acceptable if and only if it has the form of an expression in the elements of the (ordered) set $< \mathrm{ad}\,(X_{<y}) > y$, and this expression (of lower degree!) is in fact an acceptable monomial in the free Lie algebra on this ordered set.

Proof. Because the identities defining a Lie algebra are homogeneous in each variable, $L_k[X]$ will be a direct sum of components homogeneous in every indeterminate. If we collect these summands according to the largest indeterminate they involve, we see that $L_k[X]$ is the direct sum over all $y \in X$, of the ideal of $L_k[X_{\leq y}]$ generated by y (where $X_{\leq y} = X_{<y} \cup \{y\}$). And we know by Lemma 4.2 that this ideal is a free Lie algebra on $< \mathrm{ad}(X_{<y}) > y$.

It now follows easily by induction that for all n, the acceptable monomials of degree $\leq n$ in any free Lie algebra on an ordered set form a free basis for the Lie polynomials of degree $\leq n$. ||

4.7. As examples, let us list the forms of all acceptable monomials of degrees ≤ 4. Here x, y etc. will denote elements of the given ordered basis, not necessarily distinct unless so indicated.

(22) degree 1: y

degree 2: $[x, y]$ $(x < y)$

degree 3: $[w, [x,y]]$ $(w, x < y)$

$[y, [x,y]]$ $(x < y)$

degree 4: $[v, [w, [x,y]]]$ $(v, w, x < y)$

$[y, [w, [x,y]]]$ $(w, x < y)$

$[y, [y, [x,y]]]$ $(x < y)$

$[[w,y], [x,y]]$ $(w < x < y)$

Many of these cases can be consolidated by using "$\leq$" signs in our side-conditions, but the above is the form in which Lemma 4.6 gives them to us.

(By choosing different orderings of $< ad(X_{< y}) > y$ in the above Lemma, and more generally different ways of partitioning $X_{\leq y}$, e.g. into $\{y\}$ and $X_{< y}$ rather than $X_{<y}$ and $\{y\}$, we can get other bases for $L_k[X]$. I do not know whether bases so obtained overlap the known constructions, such as that of P. Hall ([ ], §IA4.5. For generalization see [ ]). Inevitably, they agree in degrees ≤ 3. But constructions based on Lemma 4.2 seem oriented toward systems of brackets that maximize "depth" and minimize "branching", while P. Hall systems tend to do the opposite.)

We are now ready for

4.8. Precise determination of identities of minimal degree in two variables satisfied for $n = 1$. Let k be a commutative ring. If R is a commutative k-algebra and $D: R \to R$ one derivation, we will abbreviate Da to a', and more generally $D^i a$ to $a^{(i)}$. The R-module of derivations DR forms a Lie algebra over k; one checks that the bracket operation is

$$[aD, bD] = (ab' - a'b)D. \tag{23}$$

We now wish to study Lie polynomials in x and y annihilated by all assignments

$$x \mapsto f\,D, \quad y \mapsto g\,D \qquad (f, g \in R;\ R, D \text{ as above}) \tag{24}$$

It will suffice to consider Lie polynomials of the form

$$P(x,y) \in L_k[x, y], \text{ homogeneous of degree } r > 0 \text{ in } x \text{ and } s > 0 \text{ in } y. \tag{25}$$

Also, as we noted at the end of §3, we can without loss of generality take $f = 1$ in (24), i.e. simplify (24) to

$$x \mapsto D, \qquad y \mapsto g\,D. \tag{26}$$

Any Lie polyomial $P(x,y)$ can now be evaluated in RD as $u\,D$, where u is a (commutative associative) polynomial in $g, g', \ldots, g^{(i)}, \ldots$. From (23) we get the following formula for Lie brackets of monomials in the $g^{(i)}$:

$$[g^{(i_1)} \ldots g^{(i_p)}D,\ g^{(i_{p+1})} \ldots g^{(i_q)}D] = \sum_{h \leq q} \mp g^{(i_1)} \ldots g^{(i_h+1)} \ldots g^{(i_q)}D \tag{27}$$

where "$\mp$" is $-$ for $1 \leq h \leq p$ and $+$ for $p+1 \leq h \leq q$. When we actually calculate with such polynomials it will be convenient to abbreviate

$$g^{(i_1)} \ldots g^{(i_p)}D \text{ to } (i_1, \ldots, i_p), \tag{28}$$

Also, whenever our context allows us to specify the order-relation among the i_h's, we will write them so that

(29) $$i_1 \leq \cdots \leq i_p \quad (\text{in } (28)).$$

Returning to $L_k[x, y]$, we recall that any Lie polynomial (25) can be written as a Lie polynomial in $y_0, y_1, \ldots,$ where $y_i = (\mathrm{ad}\ x)^i\, y$, and this polynomial will be homogeneous of total degree s and total "weight" r, where y_i is considered to have weight i $(i = 0, 1, \ldots)$.

Let us determine explicitly the expansions in RD of monomials in the y_i of low degrees:

(30) degree $s = 1$: $y_i \to (\mathrm{ad}\ D)^i(fD) = f^{(i)}D = (i)$ (by (28)).

(31) degree $s = 2$: $[y_i, y_j] \mapsto -(i+1, j) + (i, j+1)$ (using (27)).

(32) degree $s = 3$: $[y_i, [y_j, y_k]] \longmapsto (i+1, j+1, k) - (i+1, j, k+1) - (i, j+2, k) + (i, j, k+2)$.

In (30) and (31) we see that distinct allowable monomials in the y_i (in the sense of Lemma 4.6; i.e. all cases of (30), and those cases of (31) where $i < j$) are formally linearly independent. (To see this for (31) it is helpful to restrict attention to expressions of a fixed weight r, so that the only Lie monomials in question are $[y_i, y_{r-i}]$ $(0 \leq i < r/2)$ and the commutative monomials are $(i, r-i+1)$ $(0 \leq i < (r/2)+1)$.) It follows that if any nontrivial homogeneous polynomial (25) is identically zero under assignments (24), its degree s in y must be ≥ 3. By symmetry, its degree r in x must also be ≥ 3.

We now turn to (32), and consider all acceptable monomials $[y_i, [y_j, y_k]]$ of given weight $i+j+k = r$. The acceptability condition means that $j < k$, $i \leq k$. (Cf. (22)). We note that among all such monomials, the only one that will have a term $(0,0,s+2)$ in its expansion is $[y_0, [y_0, y_s]]$. (See (32). Note that in applying (32), we must rearrange each term appearing on the right-hand side so as to satisfy (29). Thus, if we did <u>not</u> restrict ourselves to acceptable monomials, $[y_0, [y_s, y_0]]$ would also have a term $(0,0,s+2)$.) Hence, any polynomial P whose evaluation under (26) is identically zero cannot involve a term $[y_0, [y_0, y_s]]$. Let us now enumerate the remaining acceptable monomials for small values of r:

(33) weight r=3: $[y_0, [y_1, y_2]]$, $[y_1, [y_0, y_2]]$.

(34) weight r=4: $[y_0, [y_1, y_3]]$, $[y_1, [y_0, y_3]]$, $[y_1, [y_1, y_2]]$, $[y_2, [y_0, y_2]]$.

(35) weight r=5: $[y_0, [y_1, y_4]]$, $[y_1, [y_0, y_4]]$, $[y_0, [y_2, y_3]]$, $[y_1, [y_1, y_3]]$, $[y_2, [y_0, y_3]]$, $[y_2, [y_1, y_2]]$.

From this point, it is just a few minutes of scratch-work to determine precisely the identities satisfied in these degrees and weights. Let me outline how the calculation proceeds for $r = 5$:

Write out the expansions of the six acceptable monomials, using (32), and permuting each symbol to get (29). Suppose some linear combination of these expansions is 0. Let $\alpha, \beta \in k$ be the coefficients of the first and third of summands in this combination. Looking successively at the coefficients of (016), (115), (124), and (133) we find that the coefficients of the 2d, 4th, 6th and 5th summands are $-\alpha$, α, β and α respectively. Looking at the coefficient of (025) we now get $\beta = -2\alpha$. Under these conditions, the coefficients of the remaining terms (034) and (223) also cancel. We conclude that there is, up to scalar multiples, a <u>unique</u> <u>identity</u> of degree 5 in x and 3 in y, given by summing the terms of (35) with coefficients 1, -1, -2, 1, 1, -2.

In the corresponding calculation on (34), we calculate that all four monomials must have the same coefficient α, which must satisfy $2\alpha = 0$. So if k is an integral domain of characteristic $\neq 2$, there are no identities of this degree, but if char $k = 2$ there is, up to scalars, a unique identity given by the sum of all terms of (34).

In (33) one finds no relations.

For $s = 4$ I will briefly record the results. We have the formulae

(36) $[y_h, [y_i, [y_j, y_k]]] \mapsto -(h+1,i+1,j+1,k)+(h+1,i+1,j,k+1)+(h+1,i,j+2,k)$
$-(h+1,i,j,k+2)+(h,i+2,j+1,k)-(h,i+2,j,k+1)-(h,i,j+3,k)-(h,i,j+2,k+1)$
$+(h,i,j+1,k+2)+(h,i,j,k+3)$.

(37) $[[y_i, y_k], [y_j, y_k]] \mapsto +(i+1,j+2,k,k)-(i+2,j+1,k,k)+(i+2,j,k,k+1)$
$-(i,j+2,k,k+1)+(i,j+1,k,k+2)-(i+1,j,k,k+2)$.

We see that the acceptable monomial $[y_0, [y_0, [y_0, y_r]]]$ is the only one having $(0,0,0,r+3)$ in its expansion, and hence (like $[y_0, [y_0, y_r]]$ for $s=3$) cannot appear in any identity.

For $r = s = 4$, we find that the remaining acceptable monomials are

(38) $[y_0, [y_0, [y_1, y_3]]]$, $[y_0, [y_1, [y_0, y_3]]]$, $[y_1, [y_0, [y_0, y_3]]]$,
$[y_0, [y_1, [y_1, y_2]]]$, $[y_1, [y_0, [y_1, y_2]]]$, $[y_1, [y_1, [y_0, y_2]]]$, $[y_2, [y_0, [y_0, y_2]]]$.

(There are none of the form (37).) On expanding these by (36), we find that if 2 is not a zero-divisor, there is a unique linear relation among them, with coefficients 1, -3, 2, 1, -2, 0, 1 respectively, while in characteristic 2 we have both this relation and one with coefficients 1, 0, 1, 0, 1, 1, 0.

Outline of main results in the remainder of "The Lie algebra of vector fields..." (in preparation).

If $P(U_1, \ldots, U_r)$ is a nontrivial Lie polynomial in r variables, then we verify that $P(U, (\mathrm{ad}\ V)U, \ldots, (\mathrm{ad}\ V)^{r-1}U)$ is a nontrivial Lie polynomial in two variables. Hence any Lie algebra satisfying a nontrivial Lie identity satisfies one in two variables.

For $n=1$ (the case of vector fields on $\mathbb{R}$) hand calculations show that there are no Lie identities in two variables of total degree < 8, but that there are 3 linearly independent identities of that degree. (The general argument above gives an identity of degree 15 in two variables. Incidentally, there is an <u>associative</u> identity of degree 4.)

Let $\underline{V}_n$ denote the <u>variety</u> (in the sense of universal algebra) of Lie algebras defined by the identities holding for all $U_1, \ldots, U_r, \ldots$ as in Theorem 2.1 (2). (Equivalently, by vector-fields on $\mathbb{R}^n$.)

Then the Lie algebra sl_{n+1} belongs to $\underline{V}_n$. It is easy to deduce that the variety generated by the union of the chain $\underline{V}_1 \subseteq \underline{V}_2 \subseteq \ldots$ is the variety of all Lie algebras. Hence the chain is not eventually constant. (But I cannot prove that successive terms are always distinct.)

For any $n \geq 1$, $r > 1$, we find that the Lie algebra L <u>free in</u> $\underline{V}_n$ on r generators is <u>prime</u>. That is, for any ideals $I, J \triangleleft L$, $[I, J] = 0 \Rightarrow I$ or $J = 0$.

For any integer n, let $\underline{V}_{v,n}$ denote the variety of Lie algebras determined by the Lie identities of volume-preserving vector-fields on $\mathbb{R}^n$. Thus, $\underline{V}_{n-1} \subseteq \underline{V}_{v,n} \subseteq \underline{V}_n$. We show that $\underline{V}_{v,n}$ satisfies T_{n^2+4}. Let $\underline{V}_{h,2n}$ denote the variety determined by identities of Hamiltonian vector-fields on $\mathbb{R}^{2n}$. Thus, $\underline{V}_n \subseteq \underline{V}_{h,2n} \subseteq \underline{V}_{v,2n}$. We show that $\underline{V}_{h,2n}$ satisfies T_{2n^2+n+5}. Free Lie algebras on $r > 1$ generators in these varieties (except, of course, for $\underline{V}_{v,1}$) are, like free Lie algebras in $\underline{V}_n$, prime.

. Appendix: a result in differential algebra. The following result was called on in the proof of Proposition 3.7. There we were only concerned with solving the partial differential equations $D_i f_j = \delta_{ij}$, but it is natural to formulate the result for a general differential extension. We also indicate the universal property of the resulting differential ring.

Theorem .1. Let $(K; d_1, \dots, d_n)$ be a differential field, with $d_1, \dots, d_n$ left linearly independent over K. Suppose

(A1) $$D_i = \sum g_{ij} d_j \qquad (g_{ij} \in K, \ i \leq m, \ j \leq n)$$

are $m \leq n$ commuting derivations, so that $(K; D_1, \dots, D_m)$ again becomes a differential field; and that the D_i are linearly independent, in fact, that

(A2) the $m \times m$ matrix $(\!(g_{ij})\!)$ $(i, j \leq m)$ is nonsingular.

Now let

(A3) $(L; D_1, \dots, D_m)$

be an extension of the differential field $(K; D_1, \dots, D_m)$; and suppose that L has a transcendence basis X over K (in the sense of ordinary field theory) such that L is separable over $K(X)$. (Automatic in characteristic 0.)

Let us adjoin to L a family X' of additional indeterminates x_α, where x ranges over X, and α ranges over all $(n-m)$-tuples $(\alpha_{m+1}, \dots, \alpha_n)$ of nonnegative integers, excluding the zero $(n-m)$-tuple.

Then the derivations $d_1, \dots, d_n$ of K can be extended to commuting derivations on the polynomial ring $L[X']$ (and hence on its field of fractions $L(X')$) in a unique manner such that (i) the linear combinations $\sum g_{ij} d_j$, restricted to L, are precisely the given extensions of $D_1, \dots, D_m$ in (A3), and (ii) each $x_\alpha \in X'$ arises by

(A4) $$x_\alpha = d_{m+1}^{\alpha_{m+1}} \cdots d_n^{\alpha_n} x.$$

The resulting differential ring $(L[X']; d_1, \ldots, d_n)$ has the universal property that given any differential ring $(E; d_1, \ldots, d_n)$ extending $(K; d_1, \ldots, d_n)$ and any homomorphism $\phi: L \to E$ respecting the operators $D_i = \sum f_{ij} d_j$, there will exist a unique homomorphism of differential rings $\bar{\varphi}: (L[X']; d_1, \ldots, d_n) \to (E; d_1, \ldots, d_n)$ extending φ.

Proof. From the fact that L is separably algebraic over $K(X)$, it is straightforward to show that there are unique extensions of $d_{m+1}, \ldots, d_n$ to commuting derivations on $L[X']$ satisfying (A4). To extend $d_1, \ldots, d_m$ will be a more delicate process.

Note that by (A1) and (A2) one can express $d_1, \ldots, d_m$ on K in terms of $D_1, \ldots, D_m, d_{m+1}, \ldots, d_n$, by a system of equations

(A5) $$d_i = \sum_{1 \le j \le m} p_{ij} D_j + \sum_{m < j \le n} q_{ij} d_j \quad (i \le m)$$

which is equivalent to (A1). Since we have extensions of $D_1, \ldots, D_m$ to L, and of $d_{m+1}, \ldots, d_n$ to all of $L[X']$, we may use (A5) to extend $d_1, \ldots, d_m$ to derivations $d_i: L \to L[X']$. These will satisfy (i) because (A5) is equivalent to (A1). We extend these to derivations on all of $L[X']$ by prescribing

(A6) $$d_i(x_\alpha) = (d_{m+1}^{\alpha_{m+1}} \cdots d_n^{\alpha_n})(d_i(x)) \quad (x_\alpha \in X', \; i \le m).$$

We claim that this makes d_i $(i \le m)$ commute with each d_j $(m < j \le n)$. Indeed, they already commute on K; comparison with (A4) shows that they now also commute on $X \cup X'$. Since L is separably algebraic over $K(X)$, we see that the derivations $[d_i, d_j]$ will be 0 on all of $L[X']$, as desired.

It remains to prove that $[d_i, d_j] = 0$ for $i, j \le m$. Again we start by noting that this holds on K. To show it on L we recall that $D_1, \ldots, D_m$ commute there by hypothesis. In view of (A1) this says that as operators on L,

$$(A7) \qquad 0 = [D_{i'}, D_{j'}] = [\sum g_{i'i}\, d_i,\ \sum g_{j'j}\, d_j]$$

$$= \sum_{i,j} (g_{i'i}\, d_i(g_{j'j})\, d_j - g_{j'j}\, d_j(g_{i'i})\, d_i) + \sum g_{i'i} g_{j'j} [d_i, d_j].$$

Now if we restrict (A7) to an equation in operators on K, the terms of the last sum are identically 0. This means that the other sum is also 0. But this is just a linear expression in $d_1, \ldots, d_n$ with fixed coefficients from K. Since $d_1, \ldots, d_n$ are linearly independent, these coefficients must all be 0, so (even over L) (A7) reduces to

$$(A8) \qquad \sum_{i,j} g_{i'i}\, g_{j'j}\, [d_i, d_j] = 0. \quad (i', j' \le m).$$

The sum in (A8) is <u>a priori</u> over $i, j \le n$, but the terms with i or $j > m$ we have already shown to be 0, so (A8) reduces to a sum over $i, j \le m$. By (A2) it follows that the $[d_i, d_j]$ are all 0 in (A8), i.e., as functions $L \to L[X']$.

Finally, we note that since d_i and d_j both commute with d_h for $h > m$ on all of $L[X']$, so does $[d_i, d_j]$. Hence for $x_\alpha \in X'$ we see $[d_i, d_j]\, x_\alpha = [d_i, d_j](d_{m+1}^{\alpha_{m+1}} \cdots d_n^{\alpha_n})x = (d_{m+1}^{\alpha_{m+1}} \cdots d_n^{\alpha_n})[d_i, d_j]\, x = 0$, since $x \in L$. So $[d_i, d_j] = 0$ on $L[X']$, as desired. This proves the main assertion of the Theorem.

Since the formulas by which we defined $d_1, \ldots, d_n$ on $L[X']$ were forced on us by conditions (i) and (ii), the universal property is easily verified.||

In the particular case of Proposition 3.7, we take for $(L, D_1, \ldots, D_m)$ the extension of $(K; D_1, \ldots, D_m)$ obtained by adjoining to K an n-tuple of indeterminates, $X = \{f_1, \ldots, f_n\}$ and extending the D_i's by defining

$D_i(f_j) = \delta_{ij}$. Then the hypotheses of the above Theorem are satisfied, and we see that in the extension $L(X') = K(X \cup X')$ the elements $d_i f_j$ $(i > m)$ are indeed independent indeterminates over K.