EPIMORPHISMS OF LIE ALGEBRAS

[addenda are noted on ~~next~~ last page]

by George M. Bergman

abstract

Epimorphisms in the category-theoretic sense, and dominions in the sense introduced by Isbell, are studied in the categories of finite-dimensional and arbitrary Lie algebras over any field K, and finite-dimensional and arbitrary p-Lie algebras over a field K of characteristic p. In the arbitrary Lie case, the finite-dimensional Lie case when char K $\neq$ 0, and the arbitrary and finite-dimensional p-Lie cases, the theory is found to be trivial: all subalgebras are difference-kernels, so the only epimorphisms are the surjective homomorphisms.

Most of the paper studies the characteristic 0 finite-dimensional Lie case, from the point of view of finite-dimensional representation theory. Here nonsurjective epimorphisms do occur. These are related to the concept of "parabolic" subalgebras, but are more general. A further concept, that of "inner dominion", arises naturally in this category, and turns out to be related to the concept of "algebraic subalgebra". Among negative results, it is shown that no nonsurjective homomorphism of Lie algebras with solvable range, or with unimodular image, is an epimorphism.

---

[1]The author was partly supported by an Alfred P. Sloan Research Fellowship.

MOS (1970) classification numbers: Primary: 17B10, 17B99.

Secondary: 16A46, 17B30, 17B35, 17B45, 17B50, 17B65, 18A20.

Keywords: epimorphisms and dominions of (finite-dimensional or arbitrary, p- or ordinary) Lie algebras; annihilator and stabilizer subalgebras of a subspace of a finite-dimensional representation; parabolic subalgebra; algebraic hull.

I am indebted to Linda Rothschild, Joseph Wolf, Gerhard Hochschild and several others for helping me with information and insights in an area with which I was little acquainted.

I shall try to make the category-theoretic concepts used meaningful to the non-categorist. On the other hand, I recall for the non-expert (like myself) in representations of Lie algebras various definitions from that field. So I ask the patience of both kinds of experts among my readers.

1. Definitions and background.

If $\mathbf{C}$ is a category of algebras (in the general sense: groups, rings, modules etc.) and their homomorphisms, B an algebra in $\mathbf{C}$, and $A \subseteq B$ a subalgebra, then Isbell (e.g. [7], cf. also Mazet [10]) says that A *dominates* an element $b \in B$ if for any two homomorphisms in $\mathbf{C}$ from B to a common range object, $f, f' : B \rightrightarrows C$, one has

$$(f|A = f'|A) \Rightarrow (f(b) = f'(b)).$$

(E.g., if $\mathbf{C}$ is the category of all rings, or all semigroups, and $a \in A$ has a two-sided inverse $b \in B$, then A dominates b.) The set of elements of B dominated by A will be a subalgebra of B containing A, called the *dominion* of A in B. Clearly, dominion is a closure operation (cf. [5]) on subalgebras of B. The dominion of A can be written as the intersection of difference-kernels of a set of pairs of maps, $f_i, f_i' : B \rightrightarrows C_i$ $(i \in I)$. If one can form in $\mathbf{C}$ the pushout (universal term to go in the lower right-hand corner, and give commutativity) D of the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, dashed, "h"] \\ B \arrow[r, dashed, "h'"] & D \end{tikzcd} \quad (= B \amalg_A B),$$

then the dominion of A in B can in fact be obtained as the difference-kernel of the single pair $h, h' : B \rightrightarrows D$; alternatively, if we can form in $\mathbf{C}$ the direct product $C = \prod C_i$ for a family (f_i, f'_i, C_i) defining the dominion of A as above, then this dominion is the difference kernel of the single pair of induced maps $f, f': B \rightrightarrows C$.

Recall that a map $g: A \to B$ in an arbitrary category $\underset{\sim}{C}$ is called an epimorphism if for any $f, f': B \rightrightarrows C$ in $\underset{\sim}{C}$ one has the cancellation property:

$$(fg = f'g) \Rightarrow (f = f').$$

Clearly, if $\underset{\sim}{C}$ is a category of algebras, this is equivalent to saying that the dominion (with respect to $\underset{\sim}{C}$) of $f(A) \subseteq B$ is all of B. The class of epimorphisms in a category of algebras always includes the surjective homomorphisms; these may be all, as in the category of groups, or the epimorphisms can form a larger class, as in the category of associative rings where they include all localization-maps. Note that to know which maps are epimorphisms, it suffices to know which inclusions of subalgebras $A \subseteq B$ are epimorphisms, i.e., which subalgebras A have all of B as dominion.

(However, if a subalgebra $A \subseteq B$ has dominion $B_1 \subseteq B$, the inclusion $A \subseteq B_1$ may not be an epimorphism, because there can be maps f, f' defined on B_1 which do not extend to B. Thus the dominion construction can be iterated to give a possibly transfinite descending chain of subalgebras, $B \supseteq B_1 \supseteq B_2 \supseteq \ldots$ containing A, whose eventual constant value D Isbell calls the stable dominion of A in B. This is the largest subalgebra $D \subseteq B$ in which A is included epimorphically.)

Let K be a field. We shall show in Theorem 2.1 that in the category of all Lie algebras over K, the only epimorphisms are the surjective homomorphisms, and more generally, that any Lie subalgebra $A \subseteq B$ is its own dominion.

In other words, if A is a proper Lie subalgebra of B, we can find a pair of homomorphisms of B into a Lie algebra C — equivalently, a pair of representations of B on the same vector-space V — which agree on A but not on all of B, and whose difference-kernel is in fact precisely A.

It is natural then to ask, if B is finite-dimensional, can C (or V) also be taken finite-dimensional? That is, do the above results hold also in the category of finite-dimensional Lie algebras over K? We shall discover that the answer is yes if $\mathrm{char}\, K = p \neq 0$ (Theorem 2.3; and similarly for p-Lie algebras, Theorem 2.2) but no in the characteristic 0 case, which we study in §§3-7.

(Note: Some categorists have adopted the excessively subtle convention of calling a morphism satisfying the above cancellation property an "epi", or "epic morphism", while restricting "epimorphism" to its earlier sense of "surjective homomorphism" — making it a not purely category-theoretic term, since surjectivity cannot be defined in an abstract category. We are not following that convention here, of course.)

2. The trivial cases: characteristic p and/or arbitrary dimension.

To study epimorphisms and dominions of arbitrary Lie algebras over a field K, we shall use their universal enveloping algebras, and an elegant result of L. Silver on epimorphisms and dominions in the category of associative algebras. (Whenever we speak of associative rings or algebras, we shall understand "with 1".)

Let $R \subseteq S$ be associative algebras over a commutative ring K, and, following Silver [11, form the K-module $S \otimes_R S$. This is not a ring; nevertheless I claim that the dominion of R in S is the difference-kernel D of the two K-module maps $s \mapsto s\otimes 1$ and $s \to 1\otimes s$ of S into $S \otimes_R S$. Indeed, we note that given two homomorphisms of associative algebras, $f, f': S \rightrightarrows T$ agreeing on R, the K-submodule $f(S)f'(S) \subseteq T$ will be a homomorphic image of $S \otimes_R S$, so the maps f and f' will agree on all members of D. On the other hand, to construct such a pair of maps whose difference-kernel is exactly D, note that $S \otimes_R S$ is an (S,S)-bimodule and take for T the split extension $S \oplus (S\otimes_R S)$, and for $f, f': S \rightrightarrows R$ the maps $s \to (s,0)$ and $s \to (s, s\otimes 1 - 1\otimes s)$. To see without scrap-paper that the latter map is an algebra homomorphism, note that it may be written $s \mapsto u^{-1}(s,0)\, u$, where $u = (1, 1\otimes 1) \in T$.

Note that if K is a field and R, S are finite-dimensional over K, then this test-object T will also be finite-dimensional, so D is also a difference-kernel in the category of finite-dimensional associative K-algebras. We conclude that when S is finite-dimensional the dominion of R in S is the same in the categories of all associative K-algebras and of finite-dimensional associative K-algebras.

Now let $A \subseteq B$ be Lie algebras over a field K, and $K[A] \subseteq K[B]$ their universal enveloping algebras. We can choose a totally ordered K-basis for B such that an initial segment of its elements form a K-basis for A.

Applying the Poincaré-Birkhoff-Witt Theorem to the universal enveloping algebras $K[A] \subseteq K[B]$, we see that the latter will be free as a right module over the former, on a basis containing 1. By a symmetric argument, $K[B]$ will also be free as a left $K[A]$-module on a basis containing 1. It follows immediately that in $K[B] \otimes_{K[A]} K[B]$, one has $u\otimes 1 = 1\otimes u$ if and only if $u \in K[A]$; so the dominion of $K[A]$ in $K[B]$ is just $K[A]$, and in particular, we get two maps of K-algebras $K[B] \rightrightarrows T$ with difference-kernel $K[A]$. If we now regard these as maps of Lie algebras over K, and compose them with the canonical inclusion $B \to K[B]$, we get two Lie maps $B \rightrightarrows T$ whose difference-kernel is $K[A] \cap B = A$. Hence:

Theorem 2.1. In the category of arbitrary Lie algebras over a field K, every subalgebra A of a Lie algebra B is its own dominion. In particular, a homomorphism of Lie algebras is an epimorphism in this category only if it is surjective. ||

Next, let K be a field of characteristic $p \neq 0$, and consider p-Lie algebras (= restricted Lie algebras = Lie algebras with p^{th} power operation [8], §V.7) over K. For these one again has a universal enveloping algebra construction, $K[B]_p$, with a normal form like that given by the Poincaré-Birkhoff-Witt Theorem, except that the exponent of each basis element of B in an allowed monomial may now not exceed $p-1$ ([8], Theorem 5.12). We again see that for A a p-Lie subalgebra of B, $K[B]_p$ will be free as a right and as a left $K[A]_p$-module on bases containing 1, and that $K[A]_p \cap B = A$, so we can apply the same reasoning as before. But in this case we can say more: if $\dim_K B = n < \infty$, then $\dim_K K[B]_p = p^n < \infty$, so in this case the test-object T used in determining the dominion of $K[A]_p$ in $K[B]_p$ will also be finite-dimensional. Thus we get:

<u>Theorem 2.2.</u> Let K be a field of characteristic $p \neq 0$. Then in the category of p-Lie algebras over K, <u>and also</u> in the category of finite-dimensional p-Lie algebras over K, every subalgebra A of an algebra B is its own dominion. In particular, a homomorphism in either of these categories is an epimorphism only if it is a surjection. ||

There is a variant of the above construction, used in [8] §VI.3, which, though nonfunctorial (it involves arbitrary choices) allows us get the same result for **ordinary** Lie algebras over a field of finite characteristic:

Let K be a field of characteristic $p \neq 0$ and B a finite-dimensional Lie algebra over K. Choose a basis $\{u_1, \ldots, u_n\}$ for B over K. For each i, the infinite family of vector-space endomorphisms of B, $(\mathrm{ad}\, u_i)^p$, $(\mathrm{ad}\, u_i)^{p^2}, \ldots$ must be linearly dependent, so we can choose a polynomial of the form $f_i(t) = t^{p^{n(i)}} + \alpha_{n(i)-1} t^{p^{n(i)-1}} + \ldots + \alpha_1 t^p$, such that $f_i(\mathrm{ad}\, u_i) = 0 \in \mathrm{End}_{K\text{-lin}}(B)$. From the properties of p^{th} powers and derivations in characteristic p (see [8] for details) one finds that the elements $z_i = f_i(u_i) \in K[B]$ will all be central, and concludes that $K[B]$ has a K-basis consisting of all monomials

$$z_1^{r_1} \ldots z_n^{r_n} u_1^{s_1} \ldots u_n^{s_n} \quad \text{where } 0 \leq r_i;\ 0 \leq s_i < p^{n(i)}.$$

If we now let $I \subseteq K[B]$ denote the 2-sided ideal generated by the central elements z_i and put $S = K[B]/I$, we can see that S will be finite-dimensional, with K-basis consisting of all monomials $u_1^{s_1} \ldots u_n^{s_n}$ $(0 \leq s_i < p^{n(i)})$, and still contain B as a Lie subalgebra. Now let A be a given Lie subalgebra of B, and assume the basis $\{u_i\}$ of B chosen so that $\{u_1, \ldots, u_m\}$ is a K-basis of A, for some $m \leq n$. Then the monomials $u_1^{r_1} \ldots u_m^{r_m}$ will clearly span a subalgebra $R \subseteq S$, such that $A = R \cap B$, and S is right free as an R-module on a basis containing 1. Reversing the ordering of the u_i, we find that S is also free as a left R-module on such a basis. So once more, R will be a difference-kernel in S as finite-dimensional associative algebras, and so $A = B \cap R$ is a difference-kernel in B as finite-dimensional Lie algebras. Thus we have:

Theorem 2.3. Let K be a field of characteristic $p \neq 0$. Then in the category of finite-dimensional Lie algebras over K (as in the category of all Lie algebras over K: see Theorem 2.1) every subalgebra A of an algebra B is its own dominion. In particular, a homomorphism in this category is an epimorphism only if it is surjective.‖

The next four sections will be concerned with epimorphisms and dominions in the category of finite-dimensional Lie algebras over a field K of characteristic 0. However, many of the results will be formulated and proved without assumption on char K. Thus, a general result about difference-kernel (dominion) subalgebras will apply to arbitrary subalgebras when char $K \neq 0$!

3. General representation-theoretic characterizations of dominions and epimorphisms.

The key idea in the next two results, of comparing the annihilators of A and B in a representation, was shown to me by Linda Rothschild.

The actions of elements of a Lie algebra B on elements of a B-module (representation of B) will be indicated by superscripts.

Theorem 3.1. Let $A \subseteq B$ be finite-dimensional Lie algebras over a field K. Then the following conditions are equivalent:

(i) A is the difference kernel of a pair of homomorphisms of B into a finite-dimensional Lie algebra C; equivalently, there exists two representations, $b \mapsto b_i$ $(i=1,2)$ of B on a finite-dimensional vector-space V, such that $A = \{b \in B \mid \forall v \in V,\ v^{b_1} = v^{b_2}\}$.

(ii) There exists a B-module W finite-dimensional over K, and a direct-sum decomposition as vector-space, $W = X \oplus Y$, such that $A = \{b \in B \mid X^b \subseteq X,\ Y^b \subseteq Y\}$.

(iii) There exists a B-module Z finite-dimensional over K, and an element $z \in Z$, such that $A = \{b \in B \mid z^b = 0\}$.

Proof. The two statements of (i) are equivalent because a finite-dimensional Lie algebra has a finite-dimensional faithful module (Ado's and Iwasawa's Theorems.)

(i) $\Rightarrow$ (ii). Given V and two representations as in (i), let $W = V \oplus V$, let $X \subseteq W$ be the diagonal subspace $\{(v,v) \mid v \in V\}$, and $Y \subseteq W$ the subspace $V \oplus 0$. Clearly, $W = X \oplus Y$. If we let B act on W by $(u, v)^b = (u^{b1}, v^{b2})$, then B, and so in particular A, respects Y, while the set of elements of B respecting X is clearly precisely A, giving the desired description of A.

(ii) $\Rightarrow$ (iii). Given $W = X \oplus Y$ as in (ii), let $Z = \mathrm{End}_K W$, and let B act on Z by the adjoint of its action on W. Let $z \in Z$ be the projection of W onto X along Y. Then the desired description of A is immediate.

(iii) $\Rightarrow$ (i). Given Z and z as in (iii), define a Lie algebra structure on $C = B \oplus Z$ by $[(b, s), (b', t)] = ([b, b'], s^{b'} - t^b)$. The two maps of B into C given by $b \mapsto (b, 0)$ and $b \mapsto (b, z^b)$ will be Lie homomorphisms, and they have difference-kernel A, as desired. ∎

If B is a finite-dimensional Lie algebra, it is easy to see that the class of subalgebras $A \subseteq B$ characterized by the above theorem, the difference-kernels, is closed under finite intersections; so by finite-dimensionality of B it is closed under arbitrary intersections, and thus forms a closure system ([6], §2.1). The closure of an arbitrary subalgebra under this system will clearly be its <u>dominion</u> in B (with respect to the category of finite-dimensional Lie algebras). We shall here (experimentally, risking a confusing terminology) call the difference-kernel subalgebras of B <u>the dominions</u> in B (or the dominion-subalgebras of B). Thus, the <u>dominion of a subalgebra</u> $A \subseteq B$ is the least dominion in B containing A. (An alternative would be to drop Isbell's term "dominion" entirely, and speak of "difference-kernel subalgebras" and "difference-kernel hulls".)

From the characterizations of dominions in our category given by Theorem 3.1, we can now write down conditions for a subalgebra A of B to have B for dominion:

Corollary 3.2. Let $A \subseteq B$ be finite-dimensional Lie algebras over a field K. Then the following conditions are equivalent.

(i) The inclusion $A \subseteq B$ is an epimorphism in the category of finite-dimensional Lie algebras over K. Equivalently, any A-module structure on a finite-dimensional K-vector-space V, which can be extended to a B-module structure can be extended uniquely.

(ii) If a B-module W has a direct-sum decomposition $X \oplus Y$ as an A-module, then X and Y are also B-submodules of W.

(iii) In every finite-dimensional representation Z of B, the annihilators of A and B in Z are the same. ▌

Note that in condition (iii), we may restrict attention to subdirectly irreducible B-modules Z.

4. Examples, counterexamples and remarks.

Proposition 4.1. Let K be a field of characteristic 0, let $B = sl(2,K) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \mid \alpha, \beta, \gamma \in K\}$, and let A be the subalgebra $\{\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}\}$. Then the inclusion $A \subseteq B$ is an epimorphism of finite-dimensional Lie algebras.

Proof. By [8], Theorem III.8, all representations of B are completely reducible. The irreducible representations Z are described in [8], Theorem III.12. The only one for which $\mathrm{Ann}_Z A \neq 0$ is the one corresponding to $m=0$ in that Theorem, and there [illegible] $\mathrm{Ann}_Z A = \mathrm{Ann}_Z B = Z$. So condition (iii) of the above Corollary is satisfied. Alternatively, condition (i) can be deduced directly from the development preceding Theorem III.12 in [8]. ▌

We can use this example to get others. Let R be any finite-dimensional associative algebra over a field K of characteristic 0, and V a K-subspace of R, containing 1. The 2×2 matrix ring $S = \begin{pmatrix} R & R \\ R & R \end{pmatrix}$ contains a Lie subalgebra $A = \{\begin{pmatrix} \alpha & v \\ 0 & -\alpha \end{pmatrix} \mid \alpha \in K,\ v \in V\}$, which in turn contains the algebra $\{\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} \mid \alpha, \beta \in K\}$ of the preceding Proposition. Thus, the dominion of A in S will contain $sl(2,K)$, and hence will contain the Lie subalgebra $B \subseteq S$ generated by $sl(2,K)$ and $\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$; and we see that by the same arguments, the inclusion $A \subseteq B$ will be an epimorphism. In general, the form of B is not clear. Note, however, that for all $r, r' \in R$ one has:

$$[\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}$$

$$\tfrac{1}{2}[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}] = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$$

$$[\begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \begin{pmatrix} 0 & r' \\ 0 & 0 \end{pmatrix}] = \begin{pmatrix} rr' & 0 \\ 0 & -r'r \end{pmatrix}.$$

It is easy to see from these relations that when R is commutative, and generated as an associative algebra by V, then B will be precisely $sl(2,R)$; thus the inclusion $A \subseteq sl(2,R)$ is an epimorphism. In this situation B can have large dimensionality while A is small. (The same trick is used in the study of epimorphisms of associative algebras in [7, p.268]; cf. also [1, Example 10.2].) <$\notin$ preprint of [1]>

For an explicit example, note that for any positive integer n, the direct product algebra K^{n+1} is generated by the unit $(1,\ldots,1)$ and the element $(0, 1,\ldots,n)$. Take for V the subspace of K^{n+1} spanned by these two elements. Note that $sl(2, K^{n+1})$ can be written $sl(2,K)^{n+1}$ as a Lie algebra. Applying the preceding observations with $R = K^{n+1}$ we get:

<u>Corollary 4.2.</u> Let K be a field of characteristic 0, n a positive integer, and A the Lie subalgebra of $sl(2,K)^{n+1}$

$$A = \{((\begin{smallmatrix}\alpha & \beta \\ 0 & -\alpha\end{smallmatrix}), (\begin{smallmatrix}\alpha & \beta+\gamma \\ 0 & -\alpha\end{smallmatrix}), \ldots, (\begin{smallmatrix}\alpha & \beta+n\gamma \\ 0 & -\alpha\end{smallmatrix})) \mid \alpha, \beta, \gamma \in K\}.$$

Then the inclusion of the 3-dimensional subalgebra A in the $3(n+1)$-dimensional semisimple Lie algebra $sl(2,K)^{n+1}$ is an epimorphism of finite-dimensional Lie algebras. ‖

One can construct many more such examples. Note that if we take $R = k[t]/(t^{n+1})$, then $sl(2,R)$, again $3(n+1)$-dimensional, now has $3n$-dimensional nil radical, and again has a 3-dimensional epimorphic subalgebra.

J. Wolf has observed to me that the key to Proposition 4.1 seems to be the fact that the subalgebra of $sl(2,K)$ in question is <u>parabolic.</u> We recall that an algebraic subgroup G of a linear algebraic group H is said to be parabolic if the homogeneous space H/G is complete as an algebraic variety. /ref.?/ This is equivalent to a condition on the Lie algebras $A \subseteq B$ associated with $G \subseteq H$, namely that when one extends scalars to the algebraic closure $\bar{K}$ of K, then $A_{\bar{K}}$ contains a maximal solvable subalgebra of $B_{\bar{K}}$ (cf. [3], Cor. 16.13). Hence a subalgebra A of an arbitrary Lie algebra B (not necessarily coming from an algebraic group and subgroup) is also called <u>parabolic</u> when this condition holds.

However, not all epimorphisms of Lie algebras correspond to parabolic subalgebras. The A of the preceding Corollary is not parabolic.

I suspect that $A \subseteq B$ being epimorphic corresponds (in some similar sense) to homogeneous spaces H/G which have no nonconstant polynomial functions; a weaker condition than completeness. For a motivation, note that the condition that H/G have no nonconstant polynomial functions is equivalent to saying that all G-invariant polynomial functions on H are constant, i.e., H-invariant, and compare this with Corollary 3.2 (iii). I do not know enough of the relevant theory to pursue these ideas further. I leave it to those who do to check whether (in characteristic 0) parabolicity does indeed imply epimorphism, and whether anything can be made of "no nonconstant polynomial functions on H/G" as a general criterion for epimorphism. Yes.

I also wonder whether parabolic subalgebras $A \subseteq B$ might not be characterizable as those subalgebras that are epimorphic in all intermediate subalgebras B_0 $(A \subseteq B_0 \subseteq B)$; perhaps K should here be algebraically closed.

We know from Theorem 2.3 that the inclusion $\{\begin{pmatrix}\alpha & \beta \\ 0 & -\alpha\end{pmatrix}\} \subseteq sl(2,K)$ is <u>not</u> an epimorphism when char $K \neq 0$. We can also show this by a simple explicit example. To make the example work uniformly in characteristic 2 and odd prime characteristic, let us use a slightly nonstandard definition of $sl(2,K)$. If K is any commutative ring, $sl(2,K)$ will denote the Lie algebra free as a K-module, on a basis $\{c,d,e\}$, with Lie operation given by:

$$[c,d] = c, \quad [c,e] = d, \quad [d,e] = e.$$

If $1/2 \in K$ (in particular, when K is a field of characteristic 0 or an odd prime) this describes, as expected, the algebra of trace-zero 2x2 matrices over K, via:

$$c = \frac{1}{2}\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}, \quad d = \frac{1}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}, \quad e = \begin{pmatrix}0 & -1 \\ 0 & 0\end{pmatrix}.$$

For $1/2 \notin K$ this is not so. In particular, for K a field of characteristic 2, the trace-zero matrix algebra is nilpotent, while the algebra we have agreed to call $sl(2,K)$ is simple, as in other characteristics. This is the point of the definition.

Our example is based on the representation of the next Lemma. Note that if p is an odd prime, then for any integer i one has $\binom{i+p}{2} \equiv \binom{i}{2}$ (mod p), while modulo 2, $\binom{i+4}{2} \equiv \binom{i}{2}$. Here $\binom{i}{2}$ denotes the binomial coefficient $i(i-1)/2$.

Lemma 4.3. Let K be a commutative ring, and r a positive integer such that for all integers i, $\binom{i+r}{2} = \binom{i}{2}$ in K. Then for any elements $\alpha, \beta \in K$, the Lie algebra $sl(2,K)$ (as defined above!) has an action on the free K-module on r generators x_i $(i \in Z/rZ)$ given by:

$$x_i{}^c = x_{i+1}$$

$$x_i{}^d = (i + \alpha)\, x_i \qquad\qquad (i \in Z/rZ)$$

$$x_i{}^e = (\tbinom{i}{2} + i\alpha + \beta)\, x_{i-1}$$

Proof. By choice of r, the last formula makes sense. The computation is immediate.‖

Let us call this $sl(2,K)$-module $V(\alpha,\beta;\, K,\, r)$.

Now take any K, r, α,β as above, and consider $V(\alpha,\beta;\, K,\, r)$ and $V(\alpha,\, \beta+1,\, K,\, r)$ as two representations of $sl(2,K)$ on the same space $x_0K+\ldots+x_{r-1}K$. The subalgebra of $sl(2,K)$ on which these representations agree is precisely $cK + dK$, so this subalgebra is a dominion in $sl(2,K)$, in contrast to Lemma 4.1.

(This example shows other things. Note that the element $x = \sum x_i$ under any representation of this form satisfies $x^c = x$; so c acts non-nilpotently. On the other hand, if we form the 2-dimensional commutative

K-algebra $K[\varepsilon]$ with $\varepsilon^2 = 0$, and look at the representation $V(\varepsilon,\beta;\ K[\varepsilon],\ r)$ of $sl(2,K[\varepsilon])$, then d acts on x_0 by multiplication by ε, which is nilpotent. Hence if we regard this as a 2r-dimensional representation of $sl(2,K)$, we see that under this representation, d acts non-semisimply. Cf. [8], Theorem VI.2.)

Note that condition (i) of Corollary 3.2 is not equivalent to the weaker condition that a B-module be determined <u>up to isomorphism</u> by its A-module structure. For instance, when char $K = 0$, the subalgebra $dK \subseteq sl(2,K)$ has this property (cf. [8], Theorem II.12), but this inclusion is not an epimorphism (e.g., apply Corollary 3.2 (iii) to the adjoint representation of B.) On the other hand, it seems plausible that in characteristic p, even this weak condition may imply $A = B$.

6. Some further general results.

We shall need the following known fact for the proof of part (c) of the next Theorem:

Lemma 6.1. Let B be a Lie algebra over a field K, and V a B-module. Then there exists an action of B by derivations on the exterior algebra ΛV, extending its action on V. Let X be any subspace of V of dimension $n < \infty$, and write $\Lambda^n X = zK \subseteq \Lambda V$. Then $\{b \in B \mid X^b \subseteq X\} = \{b \in B \mid z^b \in zK\}$.

Proof. The first assertion is well-known (cf. [8], Prop. 5.4). In the second, "$\subseteq$" is clear. Conversely, for $b \in B$ such that $X^b \not\subseteq X$, take $x \in X$ with $x^b \notin X$. Then $x \wedge z = 0$, so $x \wedge z^b = -x^b \wedge z \neq 0$, so $z^b \notin zK$. ∎

Part (c) of the next Theorem was suggested by Isbell's corresponding result for nilpotent associative algebras, [7], 2.3. Part (d) was shown to me (with a slightly different proof) by Warren Dicks.

Theorem 5.2. Let B be a finite-dimensional Lie algebra over a field K, and $A \subseteq B$ a proper subalgebra. Then the inclusion $A \subseteq B$ is not an epimorphism if either

(a) A lies in a proper ideal $I \triangleleft B$, or

(b) A is reductive in B (i.e., B completely reducible as an A-module) or more generally, if A lies in a proper direct summand of B as A-module; or

(c) A is nilpotent, or more generally, unimodular (acts trivially on its highest nonzero exterior power); or

(d) B is solvable.

Proof. (a): Immediate by looking at the factor-map and zero-map, $B \rightrightarrows B/I$.

(b) Say $B = X \oplus Y$ as A-module, with $A \subseteq X$ and $Y \neq 0$. If the inclusion $A \subseteq B$ were an epimorphism then by Corollary 5.2, X and Y would be B-submodules, i.e., ideals of B, contradicting (a).

(c) Let Z be the n^{th} exterior power of the adjoint representation of B, where $n = \dim_K A$, and $zK \subseteq Z$ the 1-dimensional subspace corresponding to A. By unimodularity, $z^A = 0$. If $A \subseteq B$ were an epimorphism this would imply $z^B = 0$, so by Lemma 5.1 A would be an ideal of B, again contradicting (a).

(d) B will have a nonzero abelian ideal; Let $I \triangleleft B$ be a minimal such.

If $A + I \neq B$, then the map of A into B/I is nonsurjective, and by induction on $\dim B$, we conclude that it is not an epimorphism. It follows that $A \subseteq B$ is not either.

If $A + I = B$, note that $A \cap I$ must be properly contained in I. Because I is an ideal of B, $A \cap I$ is an ideal of A, and because I is abelian, $A \cap I$ is an ideal of I; hence it is an ideal of $A + I = B$. So by the minimality assumption on I, we have $A \cap I = 0$, i.e. $B = A \oplus I$. As the summands are A-submodules of B, the inclusion of A in B is not an epimorphism by (b). ||

Note that part (c) implies that any epimorphism with nilpotent domain is surjective. This is not true, however, of epimorphisms with unimodular domain, since any Lie algebra can be written as a homomorphic image of a unimodular one.

We shall see in the next section that a nilpotent, or indeed a 1-dimensional subalgebra of a finite-dimensional Lie algebra B need not be a dominion. (That is, the dominion of such an A, though it cannot be all of B, can be strictly larger than A.) I don't know whether every subalgebra of a solvable Lie algebra is a dominion, nor whether every semisimple subalgebra of an arbitrary Lie algebra is.

See addendum

If B is a Lie algebra over a field K, and E is a field extension of K, let B_E denote $B \otimes_K E$, made a Lie algebra over E in the natural manner. For any subalgebra $A \subseteq B$, A_E can be identified with a subalgebra of B_E. Given a subalgebra $C \subseteq B_E$ over E, on the other hand, there will exist subalgebras $C_0 \subseteq C_1 \subseteq B$ such that $(C_0)_E \subseteq C \subseteq (C_1)_E$, which are respectively maximal and minimal for these inclusions to hold. Obviously, there are a large number of possible connections between dominions of related subalgebras of B and B_E that one could investigate; and still more relations arise if we consider restriction as well as extension of scalars. Here we shall only consider the simplest case:

Proposition 5.3. Let $A \subseteq B$ be finite-dimensional Lie algebras over a field K, let C denote the dominion of A in B, and let E be an extension-field of K. Then the dominion of A_E in B_E, as Lie algebras over E, is C_E. In particular, the inclusion $A_E \subseteq B_E$ is an epimorphism of finite-dimensional Lie algebras over E if and only if $A \subseteq B$ is an epimorphism over K.

Proof. One direction goes easily: C is a dominion in B, so it satisfies the equivalent conditions of Theorem 3.1. Choose data as in any one of those conditions, e.g., Z and z as in (iii). Extending scalars to E, we get corresponding data Z_E, z for $C_E \subseteq B_E$, so C_E is a dominion in B_E, over E. Thus, since C_E contains A_E, it contains the dominion of A_E.

To get the reverse inclusion, let D denote the dominion of A_E in B_E over E. Again, since D is a dominion it will satisfy the equivalent conditions of Theorem 3.1, this time over the field E, so let us again choose Z, z as in that Theorem. Since $\mathrm{Ann}_{B_E}\, z = D$, we have in particular $z^{A_E} = 0$, so $z^A = 0$.

Now if the field extension E/K is finite-dimensional, we can restrict operators and regard Z as a B-module finite-dimensional over K. Since $z^A = 0$ the definition of C implies $z^C = 0$, hence $C \subseteq D$, hence $C_E \subseteq D$, as desired.

If E/K is not finite-algebraic, we reduce to the finite-algebraic case by a specialization. The method is standard so I won't go through the details. /referen The point is that if some $c \in C$ does not lie in D, we take a specialization $E \to F$ with F finite-algebraic over K, whose domain of definition includes the finitely many structure constants of the B_E-module Z, and which does not annihilate (some coefficient of) the nonzero element $z^c \in Z$. We then apply the preceding argument to the B_F-module induced by Z. Since A annihilates z, but $c \in C$ does not, we have a contradiction. ▌

6. Inner dominions.

Condition (ii) of Theorem 3.1 naturally leads one to consider the simpler condition (ii) of the following theorem. One might even expect them to be equivalent, but we shall see that they are not.

Theorem 6.1. Let $A \subseteq B$ be finite-dimensional Lie algebras over a field K. Then the following conditions are equivalent:

(i) There exists a finite-dimensional B-module Z, and a 1-dimensional K-subspace $zK \subseteq Z$, such that $A = \{b \in B \mid z^b \in zK\}$.

(ii) There exists a finite-dimensional B-module W, and a K-subspace $X \subseteq W$, such that $A = \{b \in B \mid X^b \subseteq X\}$.

(iii) There exists a linear algebraic group H over K (a sub-group-scheme of some $GL(n,K)$), an algebraic subgroup $G \subseteq H$, and a homomorphism f of B into the associated Lie algebra $T(H)$ of H, such that $A = f^{-1}(T(G))$.

Further, every dominion $A \subseteq B$ satisfies the above conditions.

Proof. We first show that any dominion $A \subseteq B$ will satisfy (i). We choose a B-module and element, $z_0 \in Z_0$, satisfying condition (iii) of Theorem 3.1, then let $Z = K \oplus Z_0$, with B acting trivially on K ($K^B = \{0\}$). Then $z = (1, z_0)$ will have the desired property.

Further, (i) $\Rightarrow$ (ii) is clear, and so is (ii) $\Rightarrow$ (iii) if we take $H = \mathrm{Aut}(W)$, and G the subgroup stabilizing X.

(iii) $\Rightarrow$ (i) is a Lie algebra version of Chevalley's Theorem ([4], exposé 10, Prop.5), which says that given a linear algebraic group H and an algebraic subgroup G, there exists a linear algebraic action of H on a finite-dimensional vector space Z, and a 1-dimensional subspace zK, such that G is the stabilizer of zK in H. Now note that in this situation, Z will become a $T(H)$-module. If char $K = 0$, then by "exponentiation" we can see that $T(G) = \{u \in T(H) \mid z^u \in zK\}$, hence in the situation of condition (iii), we get $A = f^{-1}(T(G)) = \{b \in B \mid z^{f(b)} \in zK\}$, establishing (i). If char $K \neq 0$, then by Proposition 2.3 and the preceding remarks, every subalgebra $A \subseteq B$ satisfies (i)-(iii), and the question is trivial. (The general hypothesis of [4], where Chevalley's Theorem is proved, is that K is algebraically closed, but this is not needed in the simple proof of that result. Note that (ii)$\Rightarrow$ (i) can be seen directly from Lemma 5.1. This is equivalent to a step in the proof of Chevalley's Theorem.)∥

It is clear from condition (ii) that the class of subalgebras A of a Lie algebra B characterized in the above Theorem will be closed under intersections, and so form a closure system. Since the closed subalgebras of this system include the dominions, the closure of an arbitrary subalgebra $A \subseteq B$ will be contained in its dominion, so we shall call it the <u>inner dominion</u> of A in B. The class of subalgebras defined by the above Theorem will be called the inner dominion subalgebras of B.

(In oldfashioned language, the inner dominions in B are the subalgebras that can arise as isotropy subalgebras of points of projective spaces under infinitesmal linear actions of B, while the dominions are those that arise as isotropy subalgebras of points of vector spaces under such actions.)

(Recall that if H is an algebraic Lie group, and $B = T(H)$ the associated Lie algebra, then a subalgebra of B is called algebraic if it can be written $T(G)$ for some algebraic subgroup $G \subseteq H$, and the associated closure operator on subalgebras of B is called "the algebraic hull". The class of algebraic subalgebras, however, does not depend on the Lie algebra

B alone, but also on the representation $B = T(H)$. The concept of inner dominion is a modification of this, which removes this dependence. The idea suggested in §4, that it should be possible to relate the conditions "$A \subseteq B$ epimorphic" and "H/G is a variety with no nonzero constant sections", can probably be carried out in a similar manner.)

That the concepts of inner dominion and dominion are distinct is shown by:

Proposition 6.2. If A is a proper subalgebra of a finite-dimensional Lie algebra B, then the inner dominion of A is also a proper subalgebra. In particular, every maximal subalgebra of B is an inner dominion.

Proof. The normalizer of A in B, $\{b \in B \mid [A, b] \subseteq A\}$, is by construction an inner dominion, and contains A. If it is a proper subalgebra of B, we are done. If not, A is an ideal, hence a dominion (cf. Theorem 5.2 (a)), hence an inner dominion. ▌

On the other hand, from condition (i) of Theorem 6.1 we can see that every inner dominion subalgebra of B contains a dominion subalgebra of B, of codimension ≤ 1, so the two classes are not that far apart. The following observation might be of some use:

Lemma 6.3. Let B be a finite-dimensional Lie algebra, and $A \subseteq B$ an inner dominion subalgebra. Say $A = \{b \in B \mid X^b \subseteq X\}$, for some subspace X of some finite-dimensional B-module W. (E.g., X may be taken 1-dimensional.)

Suppose that the dual A-module, X^*, can be embedded in some B-module U. Then A is in fact a dominion subalgebra.

Sketch of proof. Let Z be the B-module $W \otimes_K U$. The A-submodule $X \otimes X^* \subseteq Z$ can be identified with $\mathrm{End}_K(X)$. The element z corresponding to id_X will be annihilated by A, but not by any other member of B. ▌

Note that X^* can always be written as a homomorphic image as A-modules of the B-module W^*. It becomes an A-submodule of W^* if X is a direct summand in W as A-modules. So the condition of the above lemma generalizes that of Theorem 3.1 (ii).

G. P. Hochschild has shown me two examples of Lie subalgebras which are not inner dominions; these are given (in slightly modified form) below. His motivation of these in terms of algebraic hulls is what suggested condition (iii) of Theorem 6.1.

Recall the notation for $sl(2,K)$ set up in §4.

Lemma 6.4. Let K be a field of characteristic 0. Then the 1-dimensional subalgebra of $sl(2,K)^2$ spanned by the element (c,d) has for inner dominion (and dominion) the 2-dimensional subalgebra spanned by $(c,0)$ and $(0,d)$.

Proof. The 2-dimensional subalgebra named is the centralizer of the given element (or of itself), hence is an annihilator subalgebra of $sl(2,K)^2$ under the adjoint representation, hence is a dominion (Theorem 3.1 (iii)). It will hence suffice to prove that the inner dominion of $(c,d)K$ contains this subalgebra.

Now by the representation theory of $sl(2,K)$ in characteristic 0 ([8], III.8) the action of c in any representation is nilpotent, while that of d is semisimple. Hence on any $sl(2,K)^2$-module W, $(c,0)$ and $(0,d)$ will have commuting, respectively semisimple and nilpotent actions. If the action of (c,d) belongs to the (associative) subalgebra of $\mathrm{End}_K(W)$ taking some subspace $X \subseteq W$ into itself, then its nilpotent and semisimple parts $(c,0)$ and $(0,d)$ must also belong to this subalgebra ([8], Theorem III.16). So these lie in the inner dominion of (c,d) in $sl(2,K)^2$, as claimed. ||

Lemma 6.4. Let K be a field of characteristic 0, properly containing the field $\mathbb{Q}$ of rational numbers; and let α be any element of $K - \mathbb{Q}$. Then the 1-dimensional subalgebra of $sl(2,K)^2$ spanned by the element $(d, \alpha d)$ has for inner dominion (and dominion) the 2-dimensional subalgebra spanned by $(d,0)$ and $(0,d)$.

Proof. For the same reason as before, the 2-dimensional subalgebra named will contain the dominion of the given 1-dimensional subalgebra, and it suffices to prove that it lies in the inner dominion thereof.

This time, we use the fact that in any representation of $sl(2,K)$, d acts semisimply with half-integer eigenvalues. ([8], p.85, Lemma 5, where $H = 2d$.) Since $(d,0)$ and $(0,d)$ commute, any $sl(2,K)^2$-module Z can be decomposed $Z = \oplus Z_{ij}$, where $(d,0)$ acts on Z_{ij} as multiplication by $i/2$, and $(0,d)$ as multiplication by $j/2$.

Now suppose $(d, \alpha d)$ sends some 1-dimensional subspace $zK \subseteq Z$ into itself. Write $z = \sum z_{ij}$, where $z_{ij} \in Z_{ij}$. Then $z^{(d,\alpha d)} = \sum \frac{1}{2}(i + \alpha j) z_{ij}$. For this to lie in zK, all the coefficients $\frac{1}{2}(i + \alpha j)$ associated with nonzero components z_{ij} of z must be equal in K. But since $\alpha \notin \mathbb{Q}$, no two such terms are equal, so there can be at most one nonzero component z_{ij}. So z is an eigenvector for both $(d,0)$ and $(0,d)$, so these take zK into itself. Thus, they lie in the inner dominion of $(d, \alpha d)K \subseteq sl(2,K)^2$, as claimed. ‖

Note that the inclusions of these 1-dimensional subalgebras in their dominions are not epimorphisms, since these dominions are abelian. Hence the dominion construction in this category must indeed be iterated more than once to get Isbell's "stable dominion" (see parenthetical discussion on p.3).

We see that in characteristic 0, we have the following distinct classes of subspaces of a general finite-dimensional Lie algebra B:

$$\{\text{ideals}\} \subseteq \{\text{dominions}\} \subseteq \{\text{inner dominions}\} \subsetneq \{\text{subalgebras}\},$$

while the last three fall together in finite characteristic.

An interesting contrast to the above two examples is given by:

<u>Lemma 6.5.</u> Let B be a Lie algebra over a field K, and let $P = pK$ be the 1-dimensional Lie algebra over K. Then for any $b_0 \in B$, the 1-dimensional subalgebra $\langle p, b_0\rangle \subseteq P \times B$ is a dominion. In particular, the subalgebras of $P \times sl(2,K)^2$ spanned by (p,c,d), and by $(p,c,\alpha c)$ (any $\alpha \in K$) respectively, are dominions.

<u>Proof.</u> $(p,b_0)K \subseteq P \times B$ is the graph of the Lie algebra homomorphism $P \to B$ taking p to b_0; and the graph of any homomorphism $f: A \to B$ is a difference-kernel, namely of the two maps $A \times B \to B$ given by $(a,b) \to f(a)$ and $(a,b) \to b$. ||

The analog of Proposition 5.3 holds for inner dominions, with essentially the identical proof. (It is most convenient to work in terms of condition (ii) of Theorem 6.1).

<u>Theorem 6.6.</u> Let $A \subseteq B$ be finite-dimensional Lie algebras over a field K, let C denote the inner dominion of A in B, and let E be an extension-field of K. Then the inner dominion of A_E in B_E, as Lie algebras over E, is C_E. ||

7. Pre-finite-dimensional algebras *

In this last section, we look at our results on epimorphisms of finite-dimensional Lie algebras from another point of view.

Let us call a Lie algebra A over a field K pre-finite-dimensional* if it can be written as a subdirect product of finite-dimensional Lie algebras; equivalently, if the intersection of all ideals of finite codimension in A is $\{0\}$. (Equivalently, if it is embeddable in a pro-finite-dimensional Lie algebra as usually defined: an inverse limit of finite-dimensional ones.) We make the same definitions for modules etc..

Then it is not hard to see that if $A \subseteq B$ are finite-dimensional Lie algebras, the dominion of A in B will be the same in the categories of finite-dimensional and pre-finite-dimensional Lie algebras, though we know that for char $K = 0$, this is not the same as dominion in the category of all Lie algebras over K.

Thus, the study of epimorphisms and dominions in the category of pre-finite-dimensional K-algebras generalizes the finite-dimensional theory presented above. Note that for pre-finite-dimensional Lie algebras $A \subseteq B$, the dominion of A in B can be computed as the intersection of all difference-kernels of pairs of maps of B into finite-dimensional Lie algebras C, which agree on A.

I can now state what I really had in mind in Corollary 4.2: In the category of pre-finite-dimensional Lie algebras over a field K of characteristic 0, the inclusion of the 3-dimensional subalgebra $\{\begin{pmatrix}\alpha & \beta+\gamma t \\ 0 & -\alpha\end{pmatrix} \mid \alpha,\beta,\gamma \in K\}$ in the infinite-dimensional Lie algebra $sl(2,K[t])$ is an epimorphism.

*Standard term: residually finite-dimensional

Likewise, the problem of determining "the general form of B" mentioned on p.9 can now be formulated more naturally. Let R be the free associative algebra on n indeterminates, $K<x_1,\ldots,x_n>$ and let V denote the K-subspace of R spanned by $1, x_1, \ldots, x_n$. Then as pre-finite-dimensional Lie algebras, the $n+2$-dimensional algebra of matrices $\begin{pmatrix}\alpha & v\\ 0 & -\alpha\end{pmatrix}$ $(\alpha \in K,\ v \in V)$ will be epimorphically included in the Lie algebra B of matrices over R generated by $sl(2,K)$ and the $\begin{pmatrix}0 & x_i\\ 0 & 0\end{pmatrix}$ $(i \leq n)$. The problem, then, is to describe this B. (B will be pre-finite-dimensional because the Lie algebra of all 2x2 matrices over R is, because R is, as an associative algebra. Note also: R has an involution $a \mapsto \bar{a}$ changing the signs of the generators (and by definition of involution, reversing the order of multiplication.) One gets from this an involution on 2x2 matrices over R: $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \mapsto \begin{pmatrix}\bar{d} & \bar{b}\\ \bar{c} & \bar{a}\end{pmatrix}$. B will be contained in the Lie algebra of anti-symmetric elements under this involution; possibly this might equal B.)

We observed in §2 that epimorphisms and dominions in the category of finite-dimensional associative K-algebras were the same as in the category of arbitrary associative K-algebras. However, epimorphisms in the category of pre-finite-dimensional associative K-algebras need not be! The trouble is that though $S \otimes (S \otimes_R S)$ will be finite-dimensional whenever S is, it may not be pre-finite-dimensional if S is — the tensor product $S \otimes_R S$ may have elements which are killed whenever we kill an ideal of finite codimension in S.

Indeed, that this does occur can be deduced from the existence of nonsurjective epimorphisms in the category of finite-dimensional Lie algebras! By [8], Exercise VI.7, 8 (p.206), the universal enveloping algebra $K[B]$ of a finite-dimensional Lie algebra B over a field K of characteristic 0 is pre-finite-dimensional as an associative algebra. Now let $A \subseteq B$ be a nonsurjective epimorphic inclusion of finite-dimensional Lie algebras. In the category of all associative K-algebras, the inclusion $K[A] \subseteq K[B]$ is not an epimorphism; in fact, the proof of Theorem 2.1 shows that $K[A]$ is a dominion in $K[B]$. But any two maps f and f' of $K[B]$ into a finite-dimensional associative K-algebra C which agree on $K[A]$ can be looked at as Lie maps of B into C agreeing on A, hence they will be equal, by the assumption on $A \subseteq B$. It follows that the inclusion $K[A] \subseteq K[B]$ <u>is</u> an epimorphism of pre-finite-dimensional Lie algebras. (Thus we see that epimorphisms and dominions of finite-dimensional associative K-algebras agree with those of both arbitrary and pre-finite-dimensional associative K-algebras, though the latter do not agree with each other!)

In the above situation, we can deduce that the $K[B]$-bimodule $K[B] \otimes_{K[A]} K[B]$ and the coproduct over $K[A]$ ("free product algebra with amalgamation of $K[A]$") $K[B] \coprod_{K[A]} K[B]$ both fail to be pre-finite-dimensional. The universal pre-finite-dimensional image of each of them will be $K[B]$.

<INSERT on p.22, after first paragraph:>

To the three equivalent conditions of Theorem 6.1, one may add a fourth:

(ii$\frac{1}{2}$): There exists a finite-dimensional associative K-algebra S, a subalgebra $R \subseteq S$, and a Lie algebra homomorphism f of B into S (made a Lie algebra by commutator brackets) such that $A = f^{-1}(R)$.

The implications (ii) $\Rightarrow$ (ii$\frac{1}{2}$) $\Rightarrow$ (iii) are immediate, the latter via "group of units". It is also easy to show that in (ii$\frac{1}{2}$) and (iii), the homomorphism f can be taken to be an embedding.

REFERENCES

1. G. M. Bergman, Coproducts and some universal ring constructions, (to appear).

2. G. M. Bergman, and W. Dicks, On universal derivations, (to appear).

3. A. Borel, Groupes linéaires algébriques, Ann. Math. 64 (1956) 20-82.

4. Séminaire Claude Chevalley, 1956-1958, Classification des groupes de Lie algébriques, v.I, Secretariat Mathematique, Paris, 1958.

5. P. M. Cohn, On the embedding of rings into skew fields, Proc. London Math. Soc. (3) 11 (1961) 511-530.

6. P. M. Cohn, Universal Algebra, Harper and Row, 1965.

7. J. R. Isbell, Epimorphisms and dominions, IV, J. London Math. Soc. (2) 1 (1969) 265-273.

8. N. Jacobson, Lie Algebras, Interscience Tracts in Pure and Applied Mathematics, No. 10.

9. J. T. Knight, On epimorphisms of non-commutative rings, Proc. Camb. Phil. Soc., 68 (1970) 589-600.

10. P. Mazet, Caractérization des épimorphismes par relations et générateurs, Sem. P. Samuel (2) 1967-1968. CR Ac Sci Paris, Sér A-B 266 (1968) A309-311

11. L. Silver, Noncommutative localization and applications, J. Alg. 7 (1967) 44-76.

Department of Mathematics
University of California
Berkeley, Ca. 94720

A D D E N D A to "EPIMORPHISMS OF LIE ALGEBRAS" ~~(to be incorporated in final version)~~

The following paper has been brought to my attention, of which §§5-8 have considerable overlap with the present work:

G. A. Reid, Epimorphisms and surjectivity, Inventiones math. 9(1970)295-307.

One result which Reid obtains which we did not is that epimorphisms are surjective in the category of (finite-dimensional) real compact Lie algebras; this uses the corresponding result on Lie groups, which is proved using the Peter-Weyl Theorem.

Hochschild has shown me that every subalgebra of a nilpotent Lie algebra is a dominion, using results on extension of representations, and that any semisimple subalgebra of an arbitrary Lie algebra is a dominion.

The question of whether every subalgebra of a solvable Lie algebra is a dominion seems to be quite difficult.

The examples of epimorphisms given in §4 were based either on parabolicity alone (Prop.4.1) or parabolicity combined with "sub-Lie-algebra generated by" (Cor.4.2). Here is an example which also involves inner dominions: Let $B = sl(2,K) \times sl(2,K)$ (K a field containing Q), let α be an element of $k-Q$, and let $A \subseteq B$ be the subalgebra spanned by $(d, \alpha d)$, $(e,0)$, $(0,e)$. Then no subalgebra of A is parabolic in a larger subalgebra of B. (Explicitly, one verifies that any subalgebra $C \subseteq B$ not contained in A has a solvable subalgebra larger than $C \cap A$.) However, the inner dominion of A in B is the subalgebra spanned by $(c,0)$, $(d,0)$, $(0,c)$, $(0,d)$, which is parabolic in B by Prop.4.1, so the dominion of A in B is B, i.e., $A \subseteq B$ is epic.

It would be interesting to know whether all epi's and dom's of Lie alg.s arise by combination of parabolicity, inner dominions, and generation. I.e., if $A \subseteq B$ is a sub-Lie-algebra (closed under generation), which is an inner dominion (i.e., closed under the operation of inner dominion) and such that if $A_0 \subseteq A$ is parabolic in $B_0 \subseteq B$, then $B_0 \subseteq A$ (closed with respect to parabolicity), then (i) does $A \subseteq B$ epimorphic imply $A = B$? (ii) more generally, must A always be a dominion in B?

The corresponding (perhaps equivalent) question for Lie groups is the following: Let H/G be a homogeneous space of a linear algebraic group H, whose function sheaf has no nonconstant global sections; i.e., such that H has no nonconstant polynomial functions invariant under G-translations. If H/G is positive-dimensional, must it have a <u>complete</u> subvariety of positive dimension which is the homogeneous space $H_0/G \cap H_0$ for some subgroup $H_0 \subseteq H$? (I.e., if $G \subsetneq H$ is epimorphic, must there be a case of parabolicity, $G \cap H_0 \subseteq H_0$, which shows it is not "parabolically closed"? Example: Let H/G be the projective plane minus a point, H its full group of linear transformations. Then any projective line not containing the deleted point will give the desired complete subvariety. Hartshorne tells me that a reasonable variety without nonconstant global sections of its function sheaf <u>will</u> contain a complete curve; the problem is whether one can choose this to be <u>homogeneous</u> under some subgroup of H.)

If the answers to (i) and/or (ii) above are affirmative, one would also like to know whether the operations of inner dominion, parabolic partial-extension, and subalgebra-generated can each be performed once on an arbitrary subalgebra to get its dominion (or at least, to get B if it is epimorphic), or whether iterations, of possibly arbitrary length, may be needed.

Finally, what can one say in general about the inner dominion operation? Are the examples of Lemmas 6.4 and $6.4\frac{1}{2}$ somehow canonical? If so, does the operation really break down into two operations: "separating" purely additive and purely multiplicative summands, and separating purely multiplicative summands with incommensurable eigenvalues?

C. Praeger (M.Sc. thesis, Oxford, 1970, unpublished; present address: Austral.Natl.U.) examined epimorphisms in some varieties of algebras, and found that the equivalent of Prop.4.1 above holds for K the field of 3 elements, <u>within the variety</u> of Lie algebras generated by $B = sl(2,K)$.