The question considered here isn't terribly important, but you may find the counterexample interesting. No present plans of publishing this. Comments welcomed as always.

When $\mathbf{C}^{\mathrm{pt}}$ has products that are not products in $\mathbf{C}$

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If $\mathbf{C}$ is a category having a final object F, then $\mathbf{C}^{\mathrm{pt}}$, the category of ''pointed'' objects of $\mathbf{C}$, has for objects all pairs (ε, X) where X is an object of $\mathbf{C}$ and ε a morphism $F \to X$, and has for morphisms all morphisms of second components which make commuting triangles with F. It is easy to verify that if (ε_i, X_i) $(i \in I)$ are objects of $\mathbf{C}^{\mathrm{pt}}$, and the objects X_i have a product ΠX_i in $\mathbf{C}$, then, writing ε for the morphism $F \to \Pi X_i$ induced by the ε_i, the pair $(\varepsilon, \Pi X_i)$ will be a product of the objects (ε_i, X_i) in $\mathbf{C}^{\mathrm{pt}}$.

Suppose, on the other hand, that we know that the objects (ε_i, X_i) have a product (ε, X) in $\mathbf{C}^{\mathrm{pt}}$. Must X be a product of the X_i in $\mathbf{C}$? From the preceding observation this will be so if and only if the X_i *have* a product in $\mathbf{C}$, but a little reflection produces no reason to expect that such a product must exist. However, a counterexample was unexpectedly difficult to find. I shall develop below a rather curious counterexample, then show how, with hindsight, one can get a less exotic one.

We begin with the following general construction. Suppose A is an abelian monoid. We define a category $\mathbf{Set}_A$ whose objects are sets S given with maps $p\colon S \to A$ (which for brevity will not be shown in writing these objects), and in which a morphism $f\colon S \to T$ in $\mathbf{Set}_A$ means a finitely-many-to-one set map $f\colon S \to T$ such that

(1) $$\text{for all } t \in T,\ \Sigma_{f(s)=t}\, p(s) = p(t).$$

It is immediate that set-theoretic composition of maps makes this a category.

We shall be interested in objects of $\mathbf{Set}_A$ with finite underlying sets. The full subcategory of $\mathbf{Set}_A$ consisting of these objects is not in general connected; its connected components are the full subcategories $\mathbf{Set}_{A,\,a}$ $(a \in A)$ composed of all objects S satisfying $\Sigma_S\, p(s) = a$. Each such component has a final object, F_a, a one-element set such that p takes the one element to a.

Now take for A the group Z_2. If s is an element of an object S of $\mathbf{Set}_{Z_2}$, we shall call $p(s)$ the *parity* of s, and call s *even* or *odd* according as $p(s)$ is 0 or 1. Let $\mathbf{C}$ be the full subcategory of $\mathbf{Set}_{Z_2,\,0}$ consisting of those objects all of whose members have the same parity. In other words, an object of $\mathbf{C}$ consists either of an arbitrary finite number of even elements, *or* of an even number of odd elements.

We see that $\mathbf{C}$ has a final object F_0, consisting of a single even element. This admits morphisms only to objects of $\mathbf{C}$ which have even elements; by choice of $\mathbf{C}$ such objects consist entirely of even elements. It is easy to deduce that $\mathbf{C}^{\mathrm{pt}}$ is isomorphic to the category of pointed finite sets, which of course has finite products. We claim, however, that these do not constitute products of these same objects

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25/5/10: I've updated this in two trivial ways: using the word ''monoid'', which at the time I insisted on calling ''semigroup with neutral element'', and updating my e-mail address at the end.

in $\mathbf{C}$. For example, let X be the object of $\mathbf{C}^{\mathrm{pt}}$ consisting of two even elements x and y, with x the basepoint (the image of the unique element of F_0 under the map ε). The product in $\mathbf{C}^{\mathrm{pt}}$ of two copies of X is their set-theoretic product W, with all four elements even, and (x, x) as basepoint. Now let Y be an object of $\mathbf{C}$ consisting of four odd points, s, t, u, v, and define morphisms $f, g\colon Y \to X$, letting the first take s and t to x, and take u and v to y, while the second takes s and u to x, and takes t and v to y. If W with its two projections to X were the product of two copies of X in $\mathbf{C}$, we would have a morphism $Y \to W$ carrying each of s, t, u, v to a distinct element of W; but this map would not satisfy (1). So we have obtained the desired counterexample.

(Does anyone know whether the construction $\mathbf{Set}_{Z_2}$ has been used anywhere else?)

The key properties that made this work are that the full subcategory of $\mathbf{C}$ consisting of objects *admitting* pointed structures has finite products, but that outside of this category, there is an object Y such that the set-valued functor $\mathbf{C}(Y, -)$ does not respect those products. With this observation in mind one can find a simpler example. Let $\mathbf{C}$ be the category obtained from $\mathbf{Set}$ by adjoining one additional object Y, setting $\mathbf{C}(Y, Y) = \{\mathrm{Id}_Y\}$ and $\mathbf{C}(X, Y) = \varnothing$, for $X \in \mathrm{Ob}(\mathbf{Set})$, but defining $\mathbf{C}(Y, X)$ to be the set of all nonempty subsets of X (or if we prefer, the set of all one- and two-element subsets of X), and letting composition with any $f \in \mathbf{Set}(X, X')$ carry $X_0 \in \mathbf{C}(Y, X)$ to the image-set $f(X_0) \in \mathbf{C}(Y, X')$. It is straightforward to verify that this makes $\mathbf{C}$ a category, which has for final object the final object of $\mathbf{Set}$. The full subcategory of $\mathbf{C}$ consisting of elements admitting pointed structures, i.e., the nonempty objects of $\mathbf{Set} \subseteq \mathbf{C}$, has small products; but $\mathbf{C}(Y, -)$ does not respect these products.

Incidentally, returning for a moment to the definition of the categories $\mathbf{Set}_A$, let us note (though this is irrelevant to our examples) that one can, if one likes, weaken the condition that all morphisms be finitely-many-to-one on underlying sets. On first sight, it might appear that we could replace it by the condition that for each $t \in T$, we have $p(s) = 0$ for all but finitely many elements $s \in f^{-1}(t)$, since this allows us to make sense of (1). However, this condition is not respected by composition of set-maps. A condition that does respect composition, and generalizes the conjunction of ‘‘finitely-many-to-one’’ and (1), is

(2) For all $t \in T$, there exists a way of partitioning $f^{-1}(t)$ into finite subsets, such that for all but one of these subsets, the sum of $p(s)$ over the elements of the subset is 0, while the sum over the one remaining subset equals $p(t)$.

I have no idea what if anything this construction may be good for, however.

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