Notes on orderable and right-orderable groups
(addenda to ''Ordering coproducts of groups and semigroups'')

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These pages discuss some points that I decided not to include in [**1**], because they represented too much of a digression from the subject of that paper, and/or were of limited interest, and/or because my results were not very definitive. §1 concerns a type of condition that occurs frequently in the theory of ordered groups; namely, where one takes arbitrary finite families of elements $h_1, \ldots, h_m$, and considers the values of semigroup expressions in $\{h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m}\}$, as $\varepsilon_1, \ldots, \varepsilon_m$ run over all combinations of +1's and –1's. Some basic results about these conditions are developed from a purely combinatorial point of view. In §2 I note a plausible approach to B. H. Neumann's question of whether every orderable group is embeddable in a divisible orderable group, and then show that it does not work; this may be helpful to someone else studying that question. In §3 we note some unexpected necessary (and possibly sufficient) conditions for a family of right-orderable overgroups G_i of a right-orderable group S to be embeddable in a common right-orderable group by maps agreeing on S. These conditions are always satisfied if the amalgamated subgroup is cyclic, and in §4 some progress is made toward proving the orderability of the coproduct in this case. Our results allow us to prove that every right orderable group is embeddable in a divisible right orderable group.

The four sections are essentially independent of one another and of [**1**], except that §§3 and 4 assume the main result of the (essentially self-contained) last section of [**1**], and occasional comments throughout these pages refer to material in other sections and in [**1**].

1. Combinatorial Observations on $(\varepsilon_1, \ldots, \varepsilon_m)$-Conditions

It is a standard result that a group H is orderable if and only if for every family of elements $h_1, \ldots, h_m \in H - \{1\}$, one can find $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$ such that no nonvacuous product of conjugates in H of $h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m}$ is equal to the identity [**3**, Theorem 1.3.2(a)] [**6**, Theorem II.1.2]. Indeed, given an ordering on H, if we take ε_i to be $+1$ whenever $h_i > 1$, and to be -1 whenever $h_i < 1$, the desired property is clear, while conversely, if the indicated property holds, then by the Compactness Theorem of Model Theory one can get a ''global'' map $\varepsilon\colon H - \{1\} \to \{\pm 1\}$ with the same property, and it is easy to verify that $\{h \mid \varepsilon(h) = +1\}$ will be the positive cone of an ordering of H.

Suppose H is a group which is *not* orderable. By the above result, there exist nonidentity elements $h_1, \ldots, h_m$ such that for every choice of $\varepsilon_1, \ldots, \varepsilon_m$, some nonvacuous product of conjugates of $h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m}$ equals the identity. It follows that any homomorphism of H into an orderable group must have *at least one* of $h_1, \ldots, h_m$ in its kernel. This makes it appear that the study of orderable homomorphic images of H should involve a system of branching choices, first choosing which element of each such family in the original group to assume equal to the identity, then making similar choices for

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families of the same sort in the group gotten by annihilating these elements, and so on.

But something is wrong there, for every group H has a universal orderable homomorphic image; that is, there is a *least* normal subgroup N such that H/N is orderable [**1**]. This must mean that for every family $h_1, \ldots, h_m$ as above, there is some h_i such that, no matter what series of choices one makes in the branching procedure just outlined, one is forced to annihilate h_i.

We shall show below that the identification of such elements involves no nontrivial group theory, but just the combinatoric properties of the types of choices we are considering. We shall then note that the same combinatoric results are applicable to several similar situations in the theory of orderable groups.

General Assumptions and Definitions. *Below, A will be a set, and X some set of* partial *functions from A to $\{\pm 1\}$, each defined on a finite subset of A. For any subset $Y \subseteq X$, the ''support'' of Y will mean the union of the domains of its members, and will be written* $\mathrm{supp}(Y)$. *We shall say that a* (*total!*) *function $p: A \to \{-1, 0, +1\}$ ''escapes'' X if each member of X is either undefined wherever p is nonzero, or is defined and disagrees with p at at least one such point. Given functions $p, q: A \to \{-1, 0, +1\}$, we shall say that q is a refinement of p if it agrees with p wherever p is nonzero.*

(In our motivating context, A might be the set of all nonidentity elements of the group H, or a finite subset $\{h_1, \ldots, h_m\}$. Suppose in that context we are given a family of equations $\xi = 1$ holding in H, where each expression ξ is a product of terms $h^{-1} a^{\pm 1} h$ $(h \in H,\, a \in A)$ such that if a given member of A occurs at all, it always occurs with the same exponent, $+1$ or -1. Let us associate to each of these equations the partial function $A \to \{\pm 1\}$ which takes a member of A appearing in the expression ξ to $+1$ or -1 according to the exponent with which it appears, and is undefined on those members of A which do not appear in ξ; and let us call the resulting set of partial functions X. A function $p: A \to \{-1, 0, +1\}$ will represent an intended assignment to elements of A of the properties of being < 1, $= 1$, or > 1 respectively under a possible ordering of a homomorphic image of H. The statement that p ''escapes'' X thus means that none of the given equations $\xi = 1$ shows that assignment to be impossible by representing 1 as a product ξ of conjugates of elements, all of which are assigned to be ≥ 1, and at least one of which is assigned to be > 1.)

Lemma 1. *Under the above General Assumptions,*

(i) *Every function $A \to \{-1, 0, +1\}$ which escapes X can be refined to a such a function which is maximal with respect to refinement.*

(ii) *Any two functions which escape X and are maximal for that property with respect to refinement have the same zero-set.*

Proof. (i) is easily shown using Zorn's Lemma. To show (ii), it will suffice to show that if p and q are functions escaping X, then the function r defined to agree with p wherever p is nonzero, and with q elsewhere, also escapes X. To see this, consider any $x \in X$. If p is nonzero on some point of the domain of x, then since p escapes X, p must disagree with x at some point where p is nonzero, hence r disagrees with x at the same point. In the contrary case, the domain of x lies wholly in the set where r is defined to equal q, hence the fact that q escapes x implies the same for r. ∎

Thus, if we define the ''core'' of X as the set of elements of A at which every function escaping X is zero, we can say that every function escaping X can be refined to one nonzero everywhere except on the core of X. In particular, the statement that X cannot be escaped by an everywhere nonzero function is equivalent to saying that X has nonempty core.

This confirms what we observed must be true at the beginning of this section, but does not tell us how to recognize any elements of the core of X. For this we use

LEMMA 2. *Suppose, under our general hypotheses, that no everywhere nonzero function escapes* X. *Then*

(i) X *has a finite subset* X' *which cannot be escaped by any everywhere nonzero function.*

(ii) *If* Y *is a finite subset of* X *which cannot be escaped by an everywhere nonzero function, and is minimal for this property, then the core of* Y *is all of* $\mathrm{supp}(Y)$, *which is thus contained in the core of* X.

Proof. (i) If every finite subset of X could be escaped by an everywhere nonzero function, then by the Compactness Theorem, one could find an everywhere nonzero function which escaped all of X.

(ii) That the core of Y is contained in $\mathrm{supp}(Y)$ can be seen by considering the function p with value 0 on $\mathrm{supp}(Y)$ and $+1$ everywhere else. Suppose it is properly smaller. Then we can find some function p which escapes Y but which is nonzero at some point of the domain of some element of Y. Now if the non-zero-set of p meets the domains of all elements of Y, then because p escapes Y, it must disagree with each element of Y at some point of this set, hence any refinement of p would have the same property, which would allow us to find an everywhere nonzero function escaping Y. Hence the subset of elements of Y whose domains are contained in the zero-set of p is nonempty; let us call it Z. By our minimality hypothesis on Y, Z can be escaped by some everywhere nonzero function q. Let r be the function which agrees with p where this is nonzero, and with q elsewhere. It is easy to verify that this will escape Y, contradicting our choice of Y, and completing the proof of (ii). ▮

Thus if the core of X is nonempty, we get a concrete characterization of *some* elements of this core. To see how to get the whole core, let C_0 denote the set of elements of the sort just characterized; i.e., the union of the supports of all subsets $Y \subseteq X$ minimal for the property that they cannot be escaped by everywhere nonzero functions. Let $A_1 = A - C_0$, and let X_1 denote the set of partial functions gotten by restricting members of X to A_1. We now repeat this process, letting C_1 denote the union of the supports of subsets $Y \subseteq X_1$ minimal for the property that they cannot be escaped by any everywhere nonzero function on A_1, letting $A_2 = A_1 - C_1$, and so on.

LEMMA 3. *The core of* X *is the union of the sets* C_i $(i = 0, 1, 2, \ldots)$ *constructed above.*

Proof. The core of X is easily seen to contain this union; to get the reverse inclusion, let h be a member of the core of X. Then we know that no function nonzero on h can escape X, so by the same reasoning used to get (i) of the preceding Lemma, there is a finite subset $X' \subseteq X$ such that no function nonzero on h escapes X'. We shall now show that if $h \notin C_0 \cup \ldots \cup C_n$, then $\mathrm{supp}(X')$ must have cardinality $> n+1$. Since X' is finite, it will follow that h must belong to $C_0 \cup \ldots \cup C_n$ for some n.

Note first that our choice of X' implies in particular that no everywhere nonzero function can escape X'. Let Y_0 denote a subset of X' minimal for this property. By definition, $\mathrm{supp}(Y_0) \subseteq C_0$. Now if $h \notin C_0$, then when we pass to $A_1 = A - C_0$, h lies in this set, and the set X_1' of restrictions to A_1 of the elements of X' still cannot be escaped by any function on A_1 which has nonzero value at h. Note that $\mathrm{supp}(X_1')$ is strictly smaller than $\mathrm{supp}(X')$, because of the deletion of Y_0. We can now iterate this process for n steps. Looking at the strictly decreasing chain of nonempty sets $\mathrm{supp}(X') \supset \mathrm{supp}(X_1') \supset \ldots \supset \mathrm{supp}(X_n')$, we conclude that $\mathrm{supp}(X')$ must have cardinality at least $n+1$, as claimed. ▮

Specializing to our original ordered-group context, we note that the universal orderable homomorphic image of a group H can be constructed as follows. Let X denote the set of partial functions $H \to \{\pm 1\}$

arising from all equations $\xi = 1$ of the sort discussed in the parenthetical paragraph after our ‘‘General Assumptions and Definitions’’. By the above process, one can ‘‘find’’ the core of X. Divide H by the normal subgroup generated by this core, and on the resulting group repeat the process, constructing a new set X etc.. It is not hard to show that at the ωth step one gets a trivial core, i.e., that the resulting group is orderable. The kernel of the map from H to this group must thus be the least normal subgroup yielding an orderable quotient group. (I believe that a careful examination of the properties of this construction will actually show that the core obtained at the first step is a normal subgroup with orderable quotient.)

There are several other constructions in the theory of ordered groups to which the above results can be applied. If we take the characterization of orderable groups recalled at the beginning of this section, and delete the words ‘‘conjugates in H of’’, the resulting condition characterizes *right-orderable* groups [**3**, Theorem 7.1.1(4)]. The characterization of dominion subgroups in the category of orderable groups given in [**1**, §7] is also of this form; in fact, the present results were motivated by that case. (The analogs of our minimal sets Y which cannot be escaped by everywhere nonzero functions are the ‘‘minimal T-traps’’ of [**1**].)

A more difficult case to translate into our formalism is the criterion [**3**, Theorem 1.4.1] = [**6**, Theorem II.3.1] for an infrainvariant subgroup T of an orderable group H (i.e., a subgroup whose set of conjugates within H is totally ordered by inclusion) to be *relatively convex* in H (i.e., convex with respect to some group ordering on H). That criterion is that for each family of distinct elements $h_1, \dots, h_m \in H - T$, there exist $\varepsilon_1, \dots, \varepsilon_m \in \{\pm 1\}$ such that those products of the $h_i^{\varepsilon_i}$ which do not lie in T form a semigroup. To apply our approach we should ask, for each such family of h_i and ε_i, whether there are *two* products of conjugates of the $h_i^{\varepsilon_i}$ each of which lies outside of T but whose product lies in T; and construct a set X consisting of those partial functions $h_m \mapsto \varepsilon_m$ corresponding to the cases where this happens. The Theorem cited shows that T is relatively convex if and only if this set X has empty core. However, when this condition does not hold, it is not clear what one can say about elements of the core of X; in particular, it is *not* evident that some of them must belong to every relatively convex subgroup containing T, because such a subgroup might instead contain some of the pairs of elements outside T with product in T that were involved in the formulation of the condition.

Nevertheless, one can deduce from [**3**, Theorem 1.4.3] = [**6**, Theorem II.3.2] that every infrainvariant subgroup T of H has a ‘‘relatively convex closure’’, and we would expect that our approach should somehow show this. We can, in fact, prove a slight variant of the criterion cited, for which our method works as in the preceding cases:

LEMMA 4. *Let T be an infrainvariant subgroup of an orderable group H. Then the following conditions are equivalent:*

(i) *T is relatively convex.*

(ii) *For every family of elements $h_1, \dots, h_m \in H - T$, there exist $\varepsilon_1, \dots, \varepsilon_m \in \{\pm 1\}$ such that whenever h_i and h_j are conjugate in H, one has $\varepsilon_i = \varepsilon_j$, and such that no nontrivial product of the terms $h_i^{\varepsilon_i}$ belongs to T.*

Proof. Assuming (i), choose an ordering of H with respect to which T is convex, and for every family as in (ii), set ε_i to be $+1$ if $h_i \in H - T$ is positive, -1 if h_i is negative. The conclusion of (ii) is immediate.

Conversely, if (ii) holds, the Compactness Theorem gives us a function $\varepsilon\colon H - T \to \{\pm 1\}$ with the properties indicated. Let us extend ε to a function $\varepsilon'\colon H \to \{-1, 0, +1\}$ by letting $\varepsilon'(h)$ be $+1$ if ε has value $+1$ on some conjugate of h, -1 if ε has value -1 on some conjugate of h, and 0 otherwise. The set on which this function is 0 is the intersection of all conjugates of T, a normal subgroup N of H. It is not hard to check that our original function ε is invariant under right and left translation by elements of T, and to deduce that ε' is constant on cosets of N, and thus induces a function $\varepsilon'': H/N \to \{-1, 0, +1\}$ which is 0 only at the identity. I claim that the set P on which this function is positive forms the positive cone of an ordering on H/N. Indeed, it is immediate that P is conjugacy-invariant and that P^{-1}, $\{1\}$, P partition the group. To show that P is closed under multiplication is to show that if ε'' is $+1$ at two elements, it is $+1$ at their product. This reduces to the same question for ε', and using the infrainvariance of T we can reduce (by a conjugation) to the case where neither element lies in T, and thus to the same question for ε; but this case is immediate.

Since H is orderable, we can lift this ordering of H/N to an ordering of H. We see that this will have the property that elements of $H - T$ are positive or negative according as the original function ε has value $+1$ or -1 on them. We can now show convexity of T under this ordering: if $1 < h < t \in T$ but $h \notin T$, then ε must be positive on both h and $h^{-1}t$, contradicting the assumption that the product of these elements lies in T. ∎

An examination of the proof of the above Lemma shows that condition (ii) can, if one wishes, be weakened by replacing the words ‘‘nontrivial product’’ by ‘‘product of 2 or 3’’. Two-term products are used in relating the set on which our global ε equals $+1$ to the set on which it equals -1, and showing it invariant under right and left translation by elements of T; three-term products are used in showing that the set where it is $+1$ is closed under multiplication.

2. How Not to Embed Orderable Groups in Divisible Orderable Groups.

A classic question of B. H. Neumann is whether every orderable group is embeddable in a divisible orderable group. To get an affirmative answer, it would suffice to show that given an orderable group G, a prime p, and an element y which is not a pth power in G, one can embed G in an orderable group in which y has a pth root. Now in an orderable group, an element which commutes with y must commute with any pth root of y [**3**, Lemma 2.5.1]. Hence, if we let S denote the centralizer of y in G, we may attempt to get the desired extension by adjoining to S a central pth root η of y, noting that the resulting group F is orderable, and forming the group-theoretic coproduct (‘‘free product’’) with amalgamation, $F \amalg_S G$. (It is not hard to verify that the adjunction of a central pth root of y to S is possible, and that the resulting group F is unique up to isomorphism. That this F is orderable may be deduced from [**3**, Theorem 2.2.2] = [**6**, Theorem II.4.1].)

The following observation shows, however, that this coproduct with amalgamation may fail to be orderable.

Lemma 5. *Let G be an orderable group, y an element of G, p a prime such that y is not a pth power in G, S the centralizer of y in G, and F the orderable group gotten by adjoining to S a central pth root η of y.*

Then if G contains an element x such that y and $x^{-1}yx$ are distinct but commute, the group $F \amalg_S G$ is not orderable.

Proof. Observe that $x \notin S$. From this, and the normal form in a coproduct with amalgamation, it is easy to deduce that $x^{-1}\eta x$ does not commute with η in $F \amalg_S G$. But the pth powers of these elements are $x^{-1}yx$ and y, which do commute. This is impossible in an orderable group, by the Lemma cited above. ∎

An example of a group with elements having the above properties is the semidirect product G of a free abelian group on countably many generators $\ldots\ y_{-1}, y_0, y_1, y_2, \ldots$ and an infinite cyclic group $<x>$, acting as a shift on the subscripts of the y's, with y_0 taken to be the y of the Lemma. This G is in fact embeddable in an orderable group in which y has a pth root: the endomorphism of G taking x to itself and y_i to y_i^p is such an embedding. But this group does not have the form $F \amalg_S G$.

Remark: I encountered the above phenomenon after thinking I could prove that an affirmative answer to a question asked in [**1**] would imply an affirmative answer to Neumann's question. This question (I will not refer to points in [**1**] by number, since the numbers may change between the preprint and the published version) asked whether three conditions necessary for a coproduct with amalgamation to be orderable were, together, also sufficient.

The first of these conditions is that S admit some ordering which extends to an ordering of each of the given overgroups. This holds under the hypotheses of Lemma 5, because every ordering of S extends to an ordering of F, as can be seen from the Theorem cited from [**3**] and [**6**] above.

The second condition says that the diagonal image of S in $F \times G$ should be a dominion subgroup in the category of orderable groups. (See [**1**] for discussion of this concept.) Now note that S is a dominion subgroup of G, since it is a centralizer, hence $F \times S$ is a dominion subgroup of $F \times G$. But it was proved in [**1**] that in the category of orderable groups, a dominion subgroup of a dominion subgroup is a dominion subgroup. Hence it will suffice to verify that the diagonal image of S is a dominion in $F \times S$. This image is the inverse image of the diagonal subgroup of $F \times F$ under the inclusion $F \times S \to F \times F$, hence is a dominion, as desired.

The third condition requires that the fixed groups of certain composites of partial endomorphisms of S induced by inner automorphisms of F and G be dominions in F and in G. It appears that in the situation of the above Lemma they are indeed dominions in G. The argument is, roughly, that because F is centrally generated over S, the inner automorphism of S induced by an element of F is equal to the inner automorphism induced by an element of S, which can be looked at as an inner automorphism of G, but with its domain restricted to S. Hence its fixed group is the intersection of the centralizer of an element, which is a dominion in G, with S itself, which is also a centralizer and hence a dominion. One must look a little more closely to deal with composites of such partial maps, but I believe this works out. However, the same fixed groups will not in general be dominions in F. A counterexample, in the situation of the last sentence of the Lemma, is the fixed group of $A_{x^{-1}} A_\eta A_x$. This composite acts as the identity with domain restricted to a certain subgroup of S containing y (namely, $S \cap xSx^{-1}$). Hence its fixed group is that subgroup, which is not a dominion subgroup in F because it contains y but not its pth root η.

3. Compatibility of Right Orderings.

If S is a right orderable group, and we are given a family of right orderable overgroups G_i of S, then a necessary condition for $\coprod_S G_i$ to be right orderable is clearly that S admit a right ordering which extends to every G_i, just as in the case of two-sided orderings. But in the case of right orderings, we can describe what is probably a stronger necessary condition.

For any group G, let $RO(G)$ denote the set of all right orderings of G. Note that G has a natural action on $RO(G)$: for $x \in RO(G)$ and $g \in G$, we let xg be the ordering defined by setting $u \leq_{xg} v$ if and only if $gu \leq_x gv$; equivalently, if and only if $gug^{-1} \leq_x gvg^{-1}$. (The fixed points of this action are the two-sided orderings.)

Now if our G is an overgroup of a group S, we have a restriction map $RO(G) \to RO(S)$. Let us define a *G-family* of right orderings on S to mean the image, under this restriction map, of a single

G-orbit in $RO(G)$. (Note that a G-family in $RO(S)$ may not be determined by a single member, as a G-orbit in $RO(G)$ is. Every G-family will be invariant under the action of S on $RO(S)$, and more generally, under the action on this set of the normalizer of S in G. But unless this normalizer is all of G, it is probably possible to have overlapping G-families.)

We now note

LEMMA 6. *Let S be a right orderable group, and let us be given a family of right orderable overgroups G_i of S. Then one has the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *among the conditions listed below.*

(i) *The G_i are embeddable in a right orderable group H by maps agreeing on S.* (*Equivalently, the coproduct with amalgamation $\coprod_S G_i$ is right orderable.*)

(ii) *The G_i can be simultaneously embedded, by maps agreeing on S, in a group H which admits a right-invariant* partial *ordering under which every left coset in H of every G_i is totally ordered.* (*Equivalently, the coproduct with amalgamation $\coprod_S G_i$ admits such a partial ordering.*)

(iii) *There exists a nonempty subset $A \subseteq RO(S)$ which, for each i, is a union of G_i-families.*

Proof. The equivalence of the parenthetical reformulation of (i) to the first formulation was proved in [**1**], and the corresponding result for (ii) holds by the same reasoning.

Now given a right ordering of $\coprod_S G_i$ as in the parenthetical version of (i), it is easy to see that the set A of right orderings on S induced by the orderings on the left cosets thereof in the coproduct has the properties required in (iii). (For each i, it is the image of the G_i-families obtained from the orderings of the left cosets of G_i). A little thought shows that this implication ''factors through'' condition (ii). ∎

(Incidentally, the condition in point (ii) above saying that each left coset hG_i is totally ordered could have been stated more symmetrically as saying that each set $hG_i h'$ is totally ordered, in view of the assumed right invariance of the partial ordering; or as saying that each conjugate of each G_i is totally ordered. I just didn't want to load too many ''equivalently''s into the statement.)

The following considerations will help us understand what it means for condition (iii) to hold. Let S and the G_i be as in the hypothesis of the Lemma. For any subset $X \subseteq RO(S)$, let X' denote the set of elements of X which, for every i, belong to some G_i-family which is contained in X. Clearly, any subset of X which is a union of G_i-families for each i is contained in X'. Let us now construct by transfinite induction the chain $X \supseteq X' \supseteq X''... \supseteq X^{(\alpha)} \supseteq ...$. Being decreasing, this must stabilize in a set A, which will indeed be a union of G_i-families for all i, and will be the largest such set contained in X. Condition (iii) above holds if and only if the A obtained in this way from the whole set $RO(S)$ is nonempty.

For every i, every G_i-family is contained in $RO(S)$; from this we can see that $RO(S)'$ consists of those right orderings on S which extend to right orderings on each G_i. Thus, the condition that $RO(S)'$ be nonempty is the condition, noted at the beginning of this section, that there be some right ordering of S which extends to all the G_i. Our condition (iii), which says that the eventual value of the chain of $RO(S)^{(\alpha)}$'s is nonempty, is thus a strengthening of that condition.

Note that if we construct an analogous chain of sets of 2-sided orderings, it stabilizes after one step, because for such orderings the ''G_i-families'' are singletons. So for that situation, there is no distinction between the weaker and the stronger condition.

I have not examined the behavior of the sets $RO(S)^{(\alpha)}$ for explicit families of groups G_i, so I cannot say definitely that this chain can take longer than one step to stabilize; but I expect it can.

QUESTION 7. *Do there exist examples of orderable groups* S *and families of overgroups* G_i *such that the chain of sets* $RO(S)^{(\alpha)}$ *takes arbitrarily long to stabilize? Do there at least exist examples that take more than one step?*

QUESTION 8. *Is either or both of the implications of Lemma 6 reversible?*

If A is a set which, for each i, is a union of G_i-families, as in point (iii) of Lemma 6, there is way of constructing a ''candidate'' for a partial order as in (ii) on $H = \coprod_S G_i$. Choose an arbitrary element $a(1) \in A$, the ordering we shall try to put on the subgroup $S \subseteq H$. For each i, let us write $RO(G_i, A)$ for the set of elements of $RO(G_i)$ whose whole G_i-orbit is mapped into A by projection to $RO(S)$. Thus by assumption on A we may choose for each i an element $a_i(1) \in RO(G_i, A)$ which restricts to $a(1) \in RO(S)$. This $a_i(1)$ is the ordering we want to put on $G_i \subseteq H$. Now for each i and each $g \in G_i$, let $a_i(g) = a_i(1)g \in RO(G_i)$. (Thus, $u \leq_{a_i(g)} v \Leftrightarrow gu \leq_{a_i(1)} gv$.) Restricting to S, we get orderings $a(g) \in A$, which correspond to the orderings we want to put on the cosets gS. Observe that this definition is well-defined, because the only case where an $a(g)$ can be obtained in more than one way is if $g \in S$, and in this case the orderings in question arise by left S-translation from our original ordering $a(1)$.

We now work out from our ordering on each of these cosets as we did from the ordering $\alpha(1)$ on S. For each $g \in G_i$ and each index $i' \neq i$, we extend the ordering $a(g) \in A$ to an ordering $a_{i'}(g) \in RO(G_{i'}, A)$. This will correspond to the ordering we want to put on the coset $gG_{i'} \subseteq H$. Of course, if g_1 and g_2 lie in the same left coset of S in G_i, then they lie in the same left coset of $G_{i'}$ in H, so we want to make our choices of ordering on the coset $g_1 G_{i'} = g_2 G_{i'}$ agree; we can do this by first choosing $a_{i'}(g)$ for just one representative g of each left coset of S in G_i, and then defining it for other elements of such cosets by letting $a_{i'}(gs) = a_{i'}(g)s$ for $s \in S$. Now for each $g' \in G_{i'}$ we again define $a_{i'}(gg') = a_{i'}(g)g'$, and, restricting to S, get orderings $a(gg') \in A$, for which we again choose extensions $a_{i''}(gg') \in RO(G_{i''}, A)$ for each $i'' \neq i'$. This process may be continued in the obvious manner, to get a right ordering on every left coset in H of every G_i.

This gives us a system of ''intended order-relations'' between certain pairs of elements of H, namely, pairs whose members lie in the same left coset of one of the G_i's; such that what one has on each such coset is a G_i-right-invariant total ordering. We now wish to make this relation H-right-invariant by decreeing that if h, h' are two elements between which we have an intended order-relation, then for any $h'' \in H$, the elements hh'', $h'h''$ have the same order-relation; and we also want to take the transitive closure. The resulting relation is, a priori, a preorder. If it can be shown to be antisymmetric, we will have the implication (iii)$\Rightarrow$(ii)! Whether this can be made to work, in general, I leave for the interested reader to explore.

I do not know how one might approach the question of whether (ii)$\Rightarrow$(i).

4. RIGHT ORDERING COPRODUCTS OVER $<x>$.

There is one class of cases of the problem considered in the preceding section, of right ordering a coproduct $\coprod_S G_i$ (where the G_i are right orderable and S is a common subgroup), which looks particularly hopeful: that in which $S = <x>$, the infinite cyclic group. On the one hand, it is known that the coproduct of any family of *locally indicable* groups with amalgamation of a copy of $<x>$ is locally indicable ([**5**, Theorem 9], cf. [**4**, Proposition 3.1]), and the locally indicable groups form a subclass of the right orderable groups, which for a long time contained all known examples of the latter (cf. [**2**]), so the corresponding result for the larger class of right orderable groups is plausible. On the other hand, observe that $\langle x\rangle$ has just two right orderings. Now if G is any right orderable overgroup of $\langle x\rangle$, we can see from the fact that the set of right orderings of G is closed under taking opposites that both of the right orderings of $\langle x\rangle$ extend to right orderings of G. Hence for any family of right orderable overgroups G_i of $\langle x\rangle$, the set consisting of the two right orderings of $\langle x\rangle$ is, in the language of the preceding section, a union of G_i-families for all i; so condition (iii) of Lemma 6 is always satisfied.

We shall use the result of the last section of [**1**], that a coproduct $\coprod_S G_i$ is right orderable if and only if the G_i can be embedded in some common right orderable group by homomorphisms agreeing on S. Recall also that a group G is right-orderable if and only if it can be represented as a group of order-automorphisms of a totally ordered set, equivalently, if and only if it can be represented as a such a group whose nonidentity elements act without fixed points. Hence our problem is equivalent to that of showing the existence of an ordered set on which all the G_i can act without fixed points, by actions agreeing on $\langle x\rangle$. (Our reason for choosing fixed-point-free actions rather than general actions is that there is less complication in classifying possible actions of $\langle x\rangle$ of the former sort than of the latter.) To put the question another way: Does there exist a fixed-point-free ordered $\langle x\rangle$-set A whose structure extends to a structure of fixed-point-free ordered G_i-set for each i? For brevity, let us call a nonempty totally ordered set on which a group G acts by fixed-point-free order-automorphisms a *G-chain*, and for G an overgroup of $\langle x\rangle$ let us call an $\langle x\rangle$-chain whose structure can be extended to a structure of G-chain *G-admissible*.

By standard arguments we can restrict attention to the case where we have only finitely many G_i and each is finitely generated, and in this case the A we desire, if it exists, can be taken countable. Hence we want to study the structures of countable $\langle x\rangle$-chains, and try to understand what properties of such objects are related to their G-admissibility for various finitely generated right-orderable overgroups $G \supseteq \langle x\rangle$.

If A is an $\langle x\rangle$-chain, and a and b elements of A, it is not hard to verify that b lies in the convex closure of the $\langle x\rangle$-orbit of a if and only if a lies in the convex closure of the $\langle x\rangle$-orbit of b. (This is true with $\langle x\rangle$ replaced by any group.) Thus, the convex closures of orbits of points form the equivalence classes in an equivalence relation on A; let us call these classes *spans*. Let us call an element a *increasing* if $ax > a$, *decreasing* if $ax < a$, and call this property the *polarity* of a. It is not hard to verify that polarity is constant on spans, so we may speak of increasing and decreasing spans. Finally, let us define a *block* to mean a maximal convex set all of whose members have the same polarity. Thus, each block is a union of spans, and the blocks also partition A. Two elements belong to the same block if and only if they have the same polarity and no element between them has different polarity.

In considering the structure of $\langle x\rangle$-chains, one may look at the structures of the spans, the way the spans are put together to form blocks, and the way blocks are put together to form the whole structure. We shall show that the first two aspects of the structure present no obstacles to our task. This will allow us to get a partial positive result, and to see what the remaining problem is.

We first observe

(1) If M is a span in a countable $\langle x\rangle$-chain, and as an ordered set has no jumps, then it is isomorphic as an $\langle x\rangle$-chain to the ordered set of rational numbers, with x operating by addition of 1 or by subtraction of 1, according to the polarity of M.

Indeed, assume without loss of generality that M is increasing. Let c be an element of the order-completion of M which does not belong to M itself. Then M is the union of the intervals $(cx^i, cx^{i+1}) \cap M$, i ranging over $\mathbf{Z}$, and x will take each of these intervals bijectively to the next. Each of these intervals is a countable totally ordered set without jumps and without greatest or least element, hence is isomorphic as a totally ordered set to the rational numbers. Thus the structure of the

$<x>$-chain M is determined up to isomorphism. Since the hypotheses of (1) are satisfied by the specific $<x>$-chain mentioned in the conclusion, every $<x>$-chain satisfying those hypotheses is isomorphic thereto.

Now note that if A is an $<x>$-chain, then $A\times\mathbf{Q}$ can be made an $<x>$-chain by letting x act on the first component only. This $<x>$-chain will clearly satisfy the hypothesis of (1), and hence its conclusion. If A is G-admissible for some group G, then G can also be made to act on $A\times\mathbf{Q}$, again by letting it act trivially on second components. Hence we have constructed from A a new G-admissible $<x>$-chain, whose spans have the form described in (1), while the way the spans fit together into blocks and the blocks to form the whole $<x>$-chain is unchanged.

Next, suppose A is any $<x>$-chain, and let A' denote an ultrapower of A with respect to a nonprincipal ultrafilter F on the nonnegative integers. Note that if A is without jumps as a totally ordered set, then so is A'. I claim further that the totally ordered set of spans of A' has no jumps. For suppose $a<b$ are elements of A' belonging to different spans. Let these be the images in the ultraproduct of sequences $a(i)$, $b(i)$ of elements of A. We may assume without loss of generality that a is an increasing element of A' (equivalently, that $a(i)$ is an increasing element of A for ''almost all'' i, in the sense determined by F), since if the reverse is true, we may replace x by x^{-1} in the arguments to follow. Now it will either be true that for ''almost all i'', $a(i)$ and $b(i)$ belong to distinct spans in A, or that for almost all i they belong to the same span. In the first case, consider the sequence $c(i)=a(i)x^i$. For almost all i, $c(i)$ is in a distinct span from $b(i)$; moreover for each integer n, we have $c(i)>a(i)x^n$ for all but finitely many i. Hence the element $c\in A'$ determined by this sequence is in a span above that of a and below that of b. On the other hand, if $a(i)$ and $b(i)$ belong to the same span for almost all i, then for each such i, let $n(i)$ denote the least integer such that $a(i)x^{n(i)}>b(i)$. These integers must be unbounded above on every set large with respect to F. It is easily deduce that the sequence $c(i)=a(i)x^{[n(i)/2]}$ determines an element $c\in A'$ in a span between those of a and b.

The method used in the first of the above two cases also clearly shows that a block of A' has no greatest span, and a symmetric argument shows that it has no least span.

Of course, even if A is countable, A' will not be. But given a finite family of finitely generated groups G_i acting on A, we can easily find a countable subset $A''\subseteq A'$ closed under the induced actions of these groups, and still having the properties that there are no jumps in the order structure of A'', and no jumps or greatest or least member in the ordered set of spans of any block of A''. Thus, each span will look like a copy of the rationals with x acting by adding 1, and the set of spans in each block will have the order-structure of the rationals.

We thus see that for every finitely generated right-orderable overgroup G of $<x>$ we can find G-chains A whose spans and blocks as $<x>$-chains are of a ''standardized'' sort. If we could also get the structure of the ordered set of blocks to have a standard form (as a countable totally ordered set with a specification of a ''polarity'' for each element), then all finitely generated overgroups of $<x>$ would act on a common $<x>$-chain, as desired.

I have not been able to get such a result, but from the partial results described above we can prove

LEMMA 9. *Let G_1 and G_2 be right orderable overgroups of $<x>$, and suppose that at least one of these admits a right ordering under which all conjugates of x are >1. Then $G_1\amalg G_2$ is right orderable.*

Proof. Standard arguments again allow us to assume G_1 and G_2 finitely generated. Suppose G_1 ordered so that all conjugates of x are >1. Then if we let G_1 act on itself by right translation, the action of the element x will be everywhere increasing; in other words, the resulting G_1-chain will form a single block as an $<x>$-chain, with increasing polarity. Applying the constructions described above, we get a countable G_1-chain A_1 which as an $<x>$-chain consists entirely of one increasing block, and such that this block has the ''standard'' structure described. We can also let G_1 act on the opposite ordered set, and so get a G_1-chain A'_1 consisting of a single *decreasing* block with the standard structure.

On the other hand, our general results tell us that we can get a G_2-chain A_2 whose increasing and decreasing blocks each have the standard structure. We may now let G_1 act on this totally ordered set by giving each increasing block the G_1-chain structure of A_1 and each decreasing block the G_1-chain structure of A'_1. Thus G_1 and G_2 act faithfully on a common totally ordered set by actions agreeing on $<x>$, as required to establish our conclusion. ∎

Note that any *abelian* overgroup of $<x>$ satisfies the cndition used in the above Lemma; in particular, $\mathbf{Q}$ does. This allows us to deduce

COROLLARY 10. *Any right orderable group is embeddable in a divisible right orderable group.* ∎

(Cf. the much stronger result about locally indicable groups, [**4**, Corollary 4.6], that every locally indicable group is embeddable in an equationally complete locally indicable group. We also remark that only a small part of the preceding discussion is needed to get the above Corollary. The core of the proof of the Corollary is: If G is a group of order-automorphisms of a countable ordered set A, then G can also be represented by order automorphisms of $A\times\mathbf{Q}$, lexicographically ordered. But this is order-isomorphic to $\mathbf{Q}$, and the group of order-automorphisms of $\mathbf{Q}$ is divisible; cf. (1).)

Though I do not know how to prove that all finitely generated right orderable overgroups of $<x>$ admit actions on a common $<x>$-chain, let us note what seems to be the natural candidate for the structure of such a chain. Define a *bipartite chain* as a totally ordered set whose elements are partitioned into two subsets, the ''increasing elements'' and the ''decreasing elements'', such that between any two increasing elements there is a decreasing element, and vice versa. We note

(2) Up to isomorphism, there is a unique countable bipartite chain which as an ordered set has no greatest element, no least element, and no jumps.

Indeed, one can construct inductively an isomorphism between any two such chains, by listing the elements of their disjoint union, and choosing at each stage a match for the first element of this list that has not yet been matched, which we require to lie in the appropriate range within the other chain, and to have the same polarity. (There is not an obvious ''most natural'' representative of this isomorphism class of bipartite chains, but one representative is the ordered set of all rational numbers with odd denominator, partitioned according to whether the numerator is even or odd. Another may be gotten by taking any countable $\mathbf{Q}$-vector-subspace of $\mathbf{R}$ containing $\mathbf{Q}$, and partitioning it into rational and irrational elements.)

QUESTION 11. *Does every finitely generated right orderable overgroup G of $<x>$ admit a G-chain A such that, considering A as an $<x>$-chain, the structure of its bipartite chain of blocks is that characterized in* (2)*?*

An affirmative answer would, as we have seen, imply that every coproduct of right orderable groups with amalgamation of a cyclic subgroup is right orderable.

Might the methods we have used so far, of interdigitating multiple copies of A, and forming ultrapowers, be useful in obtaining a G-chain A' of the desired sort from an arbitrary G-chain A?

Unfortunately, the method of interdigitation appears to work only at the finest level, as when we passed from A to $A\times\mathbf{Q}$, putting an array of copies around each element, and at the coarsest level, as in the proof of Lemma 9, where we stacked together copies of the whole $<x>$-chains A_1 and A_1'. If we try to do the same at the intermediate level of spans or blocks, we run into the difficulty that elements of G not centralizing $<x>$ may not carry spans to spans or blocks to blocks. Thus, spans or blocks may get stretched by such elements to cover many spans or blocks, and there is then no evident way of extending the action of such elements to the new copies we seek to ''interdigitate'' among these.

With the technique of ultraproducts there is a different difficulty. An ultraproduct A' of copies of a G-admissible $<x>$-chain A is again a G-admissible $<x>$-chain, but if the set of blocks of A has a least element or a greatest element or a jump, the same will be true of the set of blocks of A'. Indeed, the reason taking ultraproducts could make a difference at the level of *spans* is that the property of belonging to the same span is not an elementary one. (The statement of that property involves an existential quantification over integer exponents of x.) But the property of having increasing or decreasing polarity is elementary, hence so is the property of belonging to the same block, hence so is that of having a jump in the system of blocks.

In any case, the results we have obtained reduce our original problem to the question: How do the properties of a right orderable overgroup G of $<x>$ affect the possible structures of the bipartite chain of *blocks* of a G-admissible chain?

We remark that if we had the desired result on orderability of coproducts with amalgamation of $<x>$, we could deduce the same result with $<x>$ replaced by any subgroup of the rational numbers, using the fact that such groups are direct limits of copies of $<x>$.

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