A DERIVATION ON A FREE ALGEBRA, WHOSE KERNEL IS A NONFREE SUBALGEBRA

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Recall that in |1| §§2.6-2.7 (especially Propositions 2.6.3 and 2.7.1) the general solution to the equation

(1) $$pq = rs$$

in a class of rings including the free algebras was studied. The first of the solutions found there which is nontrivial enough for our present purposes is

(2) $$p = xyz + x + z, \qquad q = yx + 1$$
$$r = xy + 1, \qquad s = zyx + x + z.$$

In the context of |1|, x, y and z could have been any nonunit elements, but let us here take them to be independent generators of a free algebra $F = k\langle x, y, z\rangle$, over an arbitrary field k. Let

(3) $$G = k\langle p, q, r, s\rangle \subseteq F$$

denote the subalgebra of F generated by the elements (2). A little thoughtful experimentation yields a nonzero derivation on F having G in its kernel*, namely the k-derivation d given by

(4) $$d(x) = xyx + x, \qquad d(y) = -yxy - y, \qquad d(z) = -x.$$

We shall show below that the kernel of d is precisely G, and also that as a k-algebra, G is presented by the four generators p, q, r, s and the single relation (1), from which we will deduce that G is not a free algebra. Thus d has the property asserted in the title.

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*The reader might try discovering this derivation for him/herself, by examining the consequences of the relations $d(p) = 0$ etc. for the elements $d(x)$, $d(y)$, $d(z) \in F$. Here the relations $p = rz+x$, $s = zq + x$ are useful.

To determine Ker d, let us define, for any product A of x, y and z, the <u>distinguished factorization</u> of A to be the factorization obtained by "breaking" A between every pair of successive factors of the form xx or yy, and also between z and any adjacent letter. (For instance, the distinguished factorization of $xyxyyxyzz$ is $(xyxy)(yxy)(z)(z)$.) We see that this will be a factorization of A into factors of the forms

(5) $\quad (xy)^i \ (i \geq 1), \qquad (yx)^i \ (i \geq 1),$

(6) $\quad (xy)^i x \ (i \geq 0), \qquad (yx)^i y \ (i \geq 0),$

(7) $\quad z,$

and it will be the unique factorization into such factors satisfying

(8) Whenever two factors from among the types (5), (6) occur in succession, the break between them occurs between letters xx or yy.

It is easy to check that a product A of x, y and z will be the (unique!) highest-degree* term in some product of the elements p, q, r, s if and only if it satisfies the two conditions

(9) In the distinguished factorization of A, no factor of the form (6) appears.

(10) Every occurrence of z in A is either immediately preceeded or immediately followed by a y.

Let us note the effect of d on the elements (5)-(7): Making use of the relations $d(xy) = d(r-1) = 0$, $d(yx) = d(q-1) = 0$ we easily get:

(11)
$$d((xy)^i) = 0, \quad d((yx)^i) = 0,$$
$$d((xy)^i x) = (xy)^{i+1}x + (xy)^i x, \quad d((yx)^i y) = -(yx)^{i+1}y - (yx)^i y, \quad dz = -x.$$

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*"Degree" will always mean total degree in x, y and z.

Hence if A is a product of x, y and z, of degree n, and we take its distinguished factorization and apply d, we see that every term of d(A) has degree either n or n+2, and that each term of degree n+2 occurring may be obtained by going through the distinguished factorization of A, selecting a factor of one of the forms (6), and increasing the index i in this factor by 1. In particular, in such a term, not every factor (6) in the distinguished factorization can have i=0.

We can now prove

Lemma 1. Ker d = G.

Proof. We already know that $G \subseteq$ Ker d. To get the reverse inclusion, suppose $a \in$ Ker d. Let deg(a) = n. We shall show that every monomial A of degree n occurring with nonzero coefficient in A will satisfy (9) and (10), hence will equal the leading term of an element of G. By an obvious induction on degree, the desired result follows.

Suppose first that not every monomial A of degree n occurring in a satisfies (9). Let i_0 be the largest value of the index i for factors of the form (6) occurring in such monomials. Then when we evaluate d(A), if we look at monomials of degree n+2 whose distinguished factorization involves a factor (6) with $i = i_0 + 1$, each one that appears will come from a unique monomial A, hence there can be no cancellation among such terms. But such terms will indeed arise in applying d to a, so $d(a) \neq 0$, contradiction.

Hence all monomials A of degree n occurring in a satisfy (9). Now suppose there were one such monomial A not satisfying (10). Then we can write A = BzC, where B and C are products of monomials of the forms (5) and (7) (possibly empty), and where the condition "A doesn't satisfy (10)" becomes:

(12) B does not end, nor C begin with y.

Now the expansion of $d(A)$ will include a term $Bd(z)C = -BxC$. It is easy to see from (12) that in the distinguished factorization of BxC, the indicated letter x will be one of the factors, in fact, the unique factor of the form (6). Since a involves no monomial of degree n whose distinguished factorization includes a factor of the form (6), it is easily deduced that a term BxC cannot arise from the application of d to any other monomial in a of degree n. But it cannot arise from the application of d to a monomial of degree $n-2$ either, because every monomial which one gets in applying d to a monomial of smaller degree must have in its distinguished factorization a factor of the form (6) with $i > 0$, which BxC does not. So again, no other term in $d(a)$ can cancel BxC, contradicting the assumption $d(a) = 0$.

So we have proved that every monomial of degree n occurring in a satisfies (9) and (10); and the Lemma follows.$\|$

<u>Lemma 2.</u> G is presented in terms of the generators p, q, r, s by the single relation (1).

<u>Proof.</u> In the k-algebra G_0 presented in terms of abstract generators p, q, r, s by the relation (1), it is clear that every element can be written as a k-linear combination of products of p, q, r, s in which

(13) the sequence pq never appears.

Now let $f: G_0 \to G$ be the natural surjective homomorphism. For every product U of p, q, r, s, we can see that $f(U)$ will have a unique term of highest degree, namely the corresponding product of factors xyz, yx, xy, zyx. We shall observe below that if U satisfies (13), then from this highest-degree term of $f(U)$, the above factorization into terms xyz, yx, xy, zyx can be uniquely recovered. Hence for distinct products satisfying (13), the leading terms of their images in G are distinct, so these images are linearly independent, so f is one-one.

If A is the leading term of $f(U)$ (U satisfying (13)) then the promised procedure for factoring A — the correctness of which we leave it to the reader to verify — is as follows. First form the distinguished factorization of A, which we recall will involve only factors of the forms (5) and (7). Break each factor (5) into a product of i factors (xy) or (yx). Then join each factor z to the factor (yx) immediately to its right if there is one there, otherwise to the factor (xy) immediately to its left, which must occur by (10). Thus, a a sequence $(xy)(z)(yx)$ will yield $(xy)(zyx)$, corresponding to rs, rather than $(xyz)(yx)$, representing pq, in accordance with (13). ||

Lemma 3. G is not a 2-fir, and so in particular, not a free k-algebra.

Proof. If G were a 2-fir then the elements p, r being right linearly dependent by (1), would generate a principal right ideal, $pG + rG = uG$ $(u \in G)$. But since the relation (1), which by Lemma 2 defines G, is homogeneous of degree 2 in p, q, r, s it is easy to see that $G/(p,q,r,s)^2 \cong k\langle p,q,r,s\rangle/(p,q,r,s)^2$, and in this ring p and r clearly do not generate a principal right ideal; contradiction. ||

One is, of course, tempted to try to "exponentiate" the derivation d to get a 1-parameter family of automorphisms of F whose fixed subring is G. One finds that one can't do this in F, for in the formal-power-series ring $\hat{F} = k\langle\langle x, y, z\rangle\rangle$ (char $k = 0$), the 1-parameter group of automorphisms induced by d is given by

$$f_t(x) = x\, e^{(1+yx)t} = e^{(1+xy)t}\, x,$$

$$f_t(y) = e^{-(1+yx)t}\, y = y\, e^{-(1+xy)t}$$

$$f_t(z) = z - x\,\frac{e^{(1+yx)}-1}{1+yx},$$

and so does not carry F into itself.

Note, however, that if one could get a derivation d on a free algebra $F = k\langle x_1, \ldots, x_n\rangle$ (char $k = 0$ or "large enough") such that

(15) $(\exists m)$ $\quad d^m(x_1) = \ldots = d^m(x_n) = 0,$

then d could indeed be exponentiated within F. Note also that a sufficient condition for (15) to hold is

(16) $$d(x_i) \in k\langle x_1, \ldots, x_{i-1}\rangle \quad (i = 1, \ldots, n).$$

REFERENCE

1. P. M. Cohn, Free Rings and their Relations, Academic Press, 1971.