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ON MONOIDS, 2-FIRS, AND SEMIFIRS

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Abstract. Several authors have studied the question of when the monoid ring DM of a monoid M over a ring D is a right and/or left fir (free ideal ring), a semifir, or a 2-fir (definitions recalled in §1). It is known that for M nontrivial, a necessary condition for any of these properties to hold is that D be a division ring. Under that assumption, necessary and sufficient conditions on M are known for DM to be a right or left fir, and various conditions on M have been proved necessary *or* sufficient for DM to be a 2-fir or semifir.

A sufficient condition for DM to be a semifir is that M be a direct limit of monoids which are free products of free monoids and free groups. Warren Dicks has conjectured that this is also necessary. However F. Cedó has given an example of a monoid M which is not such a direct limit, but satisfies all the known necessary conditions for DM to be a semifir. It is an open question whether for this M, the rings DM are semifirs.

We note here some reformulations of the known necessary conditions for a monoid ring DM to be a 2-fir or a semifir, motivate Cedó's construction and a variant thereof, and recover Cedó's results for both constructions.

Any homomorphism from a monoid M into $\mathbb{Z}$ induces a $\mathbb{Z}$-grading on DM, and we show that for the two monoids just mentioned, the rings DM are "homogeneous semifirs" with respect to all such nontrivial $\mathbb{Z}$-gradings; i.e., have (roughly) the property that every finitely generated homogeneous one-sided ideal is free of unique rank.

If M is a monoid such that DM is an n-fir, and N a "well-behaved" submonoid of M, we prove some properties of the ring DN. Using these, we show that for M a monoid such that DM is a 2-fir, mutual commutativity is an equivalence relation on nonidentity elements of M, and each equivalence class, together with the identity element, is a directed union of infinite cyclic groups or of infinite cyclic monoids.

Several open questions are noted.

1. Definitions, and overview

Rings are here associative and unital.

We recall that a ring R is called a *free ideal ring*, or *fir*, if all left ideals and all right ideals of R are free as R-modules, and free R-modules of distinct ranks are non-isomorphic. The original motivating examples are the free associative algebras $k\langle X\rangle$ over fields k. Many related constructions are known to have the same property, for instance, the completions $k\langle\langle X\rangle\rangle$ of these algebras, group algebras of free groups, and more generally, monoid algebras of coproducts (a term I prefer to "free product") of free groups and free monoids, and the generalizations of these examples with a division ring D in place of the field k. The firs

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also include the classical principal ideal domains, and various noncommutative generalizations of these, e.g., Ore polynomial rings over division rings. For much of what is known about these classes of rings, see [13].

For any positive integer n, a ring R such that every left ideal generated by $\leq n$ elements is free, and free modules of ranks $\leq n$ have unique rank, is called an *n-fir*; this condition turns out to be left-right symmetric. A ring which is an n-fir for all positive integers n is called a *semifir*.

For any n, the class of n-firs is closed under taking direct limits; hence so is the class of semifirs. It follows that if M is a direct limit of monoids each of which is a coproduct of a free group and a free monoid, and D is a division ring, then DM is a semifir. These are the only examples of monoid rings of nontrivial monoids that are known to be semifirs, and Dicks has conjectured that they are the only examples [17, paragraph before Theorem 4.4].

Various conditions on a monoid M have been shown necessary for a monoid ring DM to be a semifir. In §2 we recall these, and note reformulations of some of them. Ferran Cedó made the most recent addition to this list of conditions in [14]. He gives there examples of monoids satisfying the previously known conditions but not his new ones, thus showing that the previous conditions were not sufficient. He then describes a monoid which satisfies all the known necessary conditions, including his, but which is not a direct limit of coproducts of free groups and free monoids. It is not known whether this M has monoid rings DM which are semifirs; if so, it would be a counterexample to Dicks' conjecture. In §5 below, we show that for this M, the rings DM do satisfy a graded version of the semifir condition.

2. Conditions on a monoid M

Necessary and sufficient conditions are known for a monoid ring to be a left or right fir ([26], [20], cf. [15]); so it is only for the semifir condition, and the weaker n-fir conditions, that such questions are open. Moreover, 1-firs are simply the rings without zero divisors, a condition of a different flavor from the n-fir conditions for higher n, so this note will in general only consider conditions for a monoid ring to be an n-fir when $n \geq 2$. Finally, it is known that a monoid ring RM where $M \neq \{1\}$ cannot be a 2-fir unless R is a division ring, so (with a brief exception in §10) we will not discuss monoid rings over non-division-rings. We therefore make

Convention 1. *For the remainder of this note, D will represent an arbitrary division ring. Thus, statements we make which refer to D will be understood to be asserted for all division rings D.*

If M is a monoid, then DM will denote the monoid ring of M with coefficients in D, i.e., the ring of formal finite linear combinations of elements of M with coefficients in D, with the obvious addition, and with multiplication defined so that elements of D commute with elements of M, while the (generally noncommutative) internal multiplicative structures of M and D are retained.

For M a monoid, its *universal group* is the group obtained by universally adjoining inverses to all elements of M. A monoid need not embed in its universal group; easy counterexamples are monoids with instances of non-cancellation, $a\,b = a\,b'$ or $b\,c = b'\,c$ with $b \neq b'$. (Right and left cancellation are, in fact, the simplest of an infinite family of conditions, obtained by A. I. Mal'cev, which together are necessary and sufficient for a monoid to be embeddable in a group. See [22] and [23], or the exposition in [12, §VII.3].)

It is not hard to show that a group G is a direct limit of free groups if and only if every finitely generated subgroup of G is free; such a G is called *locally free*. Dicks and Schofield [17, Theorem 4.4, and the second sentence of the proof of that theorem] obtain the following strong results for any monoid M such that DM is a semifir.

(1) The universal group of M is locally free.

(2) M embeds in its universal group.

The condition that a ring R be an n-fir can be expressed as saying that every n-term relation $\sum_1^n a_i\, b_i = 0$ holding with $a_i,\, b_i \in R$ can be (in a sense that will be recalled in §6) "trivialized". In monoid rings DM, the easy examples of such relations are 2-term relations, arising in one way or another from equalities between two products in M. Hence it is not surprising that most known necessary conditions for DM to be a semifir are consequences of the 2-fir condition. Though (1) and (2) are not so obtained, I don't know any cases where DM is a 2-fir but does not satisfy (1) and (2). Indeed, I do not know the answer to

Question 2. *If M is a monoid such that DM is a 2-fir, must DM be a semifir?*

For commutative rings, the conditions of being a 2-fir and of being semifir are equivalent. Indeed, given an ideal I generated by $n > 2$ elements $a_1, \ldots, a_n$, not all zero, in a commutative 2-fir, we note that the ideal generated by a_1 and a_2 must be free, hence since a commutative ring has no ideals free on more than one generator, must be generated by a single element a, so I is generated by the $n-1$ elements $a, a_3, \ldots, a_n$; and repeating this reduction, we find that it is free on one generator; moreover, over commutative rings, all free modules have unique rank. The same argument works over any right or left Ore ring; so it is only among non-Ore rings that the distinctions among the n-fir conditions for different values of $n > 1$ arise.

It is shown in [16] that if G is a group such that if DG is an n-fir, then G must have the property that every n-generator subgroup is free. If the converse were known to be true, then the fact that there are groups having this property for $n = 2$ but not for larger n [2], [4], [16, Examples on p. 289] would give a negative answer to Question 2. But though many examples are known of rings that are n-firs but not n+1-firs (e.g., [11, Proposition 4.2], [7, Theorems 6.1 and 6.2]), I am aware of none that are group rings or monoid rings.

Nevertheless, let us hedge our bets, and supplement Question 2 with a weaker version, which might have a positive answer if that question does not.

Question 3. *If M is a monoid satisfying* (1) *and* (2)*, such that DM is a* 2*-fir, must DM be a semifir?*

We shall now recall some conditions that have been shown necessary for DM to be a 2-fir. When we give a condition in the form of an implication, e.g., "$(ab = ac) \implies (b = c)$", this will be understood to be quantified universally over the elements of M referred to.

Menal [24] proves that if DM is a 2-fir, then M satisfies the next three conditions, of which the first two are together expressed by saying M is "rigid".

(3) M is cancellative; that is, $(ab = ac) \implies (b = c) \impliedby (bd = cd)$.

(4) $aM \cap bM \neq \emptyset \implies aM \subseteq bM$ or $aM \supseteq bM$.

(5) If $a = cad$, and a is not invertible, then $c = d = 1$.

Note that in the presence of (3), condition (4) is left-right symmetric: it says that any relation $ac = bd$ holding in M can be refined either as $(bf)c = b(fc)$ (i.e., with a having the form bf, so that by cancellation, $d = fc$), or as $a(ed) = (ae)d$, in M.

It is known that (3) and (4) together imply (2). (Indeed, a somewhat stronger statement is proved in [18] – stronger because it assumes a weaker, but more complicated, hypothesis than (4). A proof of the implication as just stated is given in [13, Theorem 0.7.9].) Hence condition (2), stated above as necessary for DM to be a semifir, is in fact necessary for it to be a 2-fir (and hence can be dropped from Question 3).

To the necessary conditions (3)-(5) for DM to be a 2-fir, Cedó [14, Proposition 2] adds the next two.

(6) If $ab = cad$, then either $c = 1$, or there exists b' such that $ab' = c^n$ for some nonnegative integer n.

(6′) If $ba = cad$, then either $d = 1$, or there exists b'' such that $b''a = d^n$ for some nonnegative integer n.

These conditions have some known reformulations, which we will now develop.

Let us begin by looking at the case of (6) where $d = 1$. It is not hard to get from this (we shall do so in the proof of Lemma 4 below) a more detailed statement:

(7) If $ab = ca$, and b, c are not both 1, then there exist $e, f \in M$ and a nonnegative integer n such that $a = (ef)^n e$, $b = fe$, and $c = ef$.

For some insight into this condition, let us see what we can deduce from a relation $ab = ca$ using only (3) and (4). By (4), one of a and c is a left divisor of the other. If a is a left divisor of c, we can write $c = af$, whence $b = fa$, and on setting $e = a$, we find that we have the $n = 0$ case of the conclusion of (7). This leaves the case where c is a left divisor of a; say $a = ca'$. In that case, substituting into the given equation and cancelling c on the left, we get $a'b = ca'$, which, of course, again leads to two possibilities. In one, $c = a'f$, and on taking $e = a'$, we see that $c = ef$, so $a = ca' = efe$, and we have the $n = 1$ case of the conclusion of (7). In the other case, writing $a' = ca''$, we get a new equation $a''b = ca''$. And so on. So what condition (7) says is that this process terminates after finitely many steps; in other words, that a is not left divisible by arbitrarily large powers of c – except (if one examines the argument) when a, b and c are all invertible, in which case one has, at every step, a choice whether to terminate the process or continue it. Let us exclude this case by requiring a to be noninvertible. Thus, the idea of (7) seems to be the condition:

(7′) If $ab = ca$, where $c \neq 1$ and a is not invertible, then a is not left divisible in M by all positive integer powers of c.

This has, of course, a dual:

(7″) If $ab = ca$, where $b \neq 1$ and a is not invertible, then a is not right divisible in M by all positive integer powers of b.

The reader might find it helpful, at this point, to jot down on a piece of paper the statements of the numbered conditions given so far, (1)-(7″), for easy reference as he or she reads further; I will refer to them frequently. (Subsequent numbered displays, in contrast, will mostly be referenced only near where they appear.)

The next result establishes the relations among the last few conditions discussed.

Lemma 4 ([1, Lemma 2])**.** *If M is a monoid satisfying* (3) *and* (4)*, then*

$$(7') \iff (7) \iff (7''),$$

and

$$(6) \iff ((5) \wedge (7)) \iff (6').$$

Proof. To prove that $(7') \Longleftarrow (7)$, suppose (7) holds, and that $ab = ca$, where $c \neq 1$ and a is not invertible. By (7), a will left divide c^{n+1} for some n. Now if a were left divisible by all positive powers of c, it would in particular be left divisible by c^{n+2}, so c^{n+1} would be left divisible by c^{n+2}, so (since M satisfies (3)) c would be invertible; hence so would a, as a left divisor of c^{n+1}, contradicting our assumption.

For the reverse implication, again suppose $ab = ca$, now assuming b and c not both 1. If a is invertible, we can get the conclusion of (7) by taking $n = 0$, $e = a$, and $f = ba^{-1} = a^{-1}c$. If not, then (7′) implies that there is a largest nonnegative n such that c^n left divides a, so we can write $a = c^n e$ where $e \in M$ is *not* left divisible by c. Substituting into our given relation, and cancelling c^n from the left, we get $e\,b = c\,e$, and since e is not left divisible by c, (4) says c is left divisible by e; say $c = ef$. Cancellation now gives $b = fe$, and substituting back, we get $a = c^n e = (ef)^n e$, giving the conclusion of (7).

The implications $(7) \iff (7'')$ follow from the above by left-right symmetry.

To prepare for the proof of the second line of equivalences, let us examine some consequences of (6) for a general relation

$$(8) \qquad a\,b \;=\; c\,a\,d$$

with $c \neq 1$. Condition (6) says that a left divides some nonnegative power of c; let c^n be the least *positive* power of c which a left divides. Thus, c^n will be a common right multiple of a and c^{n-1}, and a will not be a proper left divisor of c^{n-1} (by our minimality assumption on n if $n \geq 2$, or by the fact that $c^0 = 1$ has no proper left divisors if $n = 1$). Hence by (4), a must be a right multiple of c^{n-1}; say $a = c^{n-1}e$. On the other hand, as noted, c^n is a right multiple of a, say $c^n = af$. This gives $c^n = c^{n-1}ef$; so by (3),

$$(9) \qquad c \;=\; e\,f,$$

and substituting into $a = c^{n-1}e$,

$$(10) \qquad a \;=\; (ef)^{n-1}e.$$

Substituting (9) and (10) into (8), and cancelling $(ef)^{n-1}e$ on the left, we get

$$(11) \qquad b \;=\; f\,e\,d.$$

To get $(6) \implies (5)$, we now consider the case of (8) where $b = 1$. If $c = 1$ (the case excluded in the above discussion), then (8) becomes $a = ad$, which by (3) gives $d = 1$, confirming (5). If $c \neq 1$, then we have (11), which for $b = 1$ implies (in view of (3)), that f, e and d are invertible, hence by (10), that a is invertible, the case about which (5) makes no assertion, completing the proof that M satisfies (5).

To see that $(6) \implies (7)$, we note that in the case $d = 1$ of (8), the consequences (9)-(11) are precisely the conclusion of (7).

To show, conversely, that $(5) \wedge (7) \implies (6)$, note that given a relation (8), the elements b and d have a common left multiple, hence by (4) one is a left multiple of the other. If $b = b'd$, then we can cancel d from the two sides of (8), getting a relation to which we can apply (7) to get the desired conclusion. If $d = d'b$, we can similarly cancel a b and apply (5), and conclude either $c = 1$, or a is invertible. The former is one of the alternative conclusions of (6), while the latter is the $n = 0$ case of the other alternative.

Since (5) and (7) are both left-right symmetric, the equivalence of their conjunction with (6) also implies the equivalence of that conjunction with the dual statement, (6′). □

We remark that in monoids M for which DM is a semifir, one *can* have arbitrarily large powers of one noninvertible element left dividing another. For instance, in the direct limit of the maps of free monoids $\langle x, y_0\rangle \to \langle x, y_1\rangle \to \dots \to \langle x, y_n\rangle \to \dots$, where each map sends y_i to $x\, y_{i+1}\, x$, the element y_0 is infinitely divisible by x on both sides. But (7′) and (7″) tell us, somewhat mysteriously, that such infinite divisibility is excluded for elements appearing in certain slots of a relation $a\,b = c\,a$.

Going back to (5), let us record a generalization of that condition, which we will use in §10.

Lemma 5. *Let M be a monoid satisfying* (3)-(5)*. Then for every natural number n, M also satisfies*

(12) *If $a_1 \dots a_n = c_0\, a_1\, c_1\, a_2 \dots c_{n-1}\, a_n\, c_n$, where $a_1, \dots, a_n$ are all noninvertible, then $c_0 = \dots = c_n = 1$.*

Proof. The $n = 0$ case (with the hypothesis understood to be $1 = c_0$) is a tautology, and the $n = 1$ case is (5), so let $n > 1$, assume inductively that the desired result is known for $n - 1$, and suppose we are given $a_1, a_2, \dots, a_n$, and $c_0, c_1, \dots, c_n$, satisfying the conditions stated. We note that if $c_0 = 1$, we can cancel a_1 on the left, and our inductive assumption gives the desired conditions on the remaining c_i; and the analogous argument works if $c_n = 1$.

Now by (4) applied to the relation $a_1 \cdot (a_2 \dots a_n) = (c_0 a_1 c_1) \cdot (a_2 \dots c_{n-1} a_n c_n)$, one of a_1 and $c_0 a_1 c_1$ must left divide the other; so we can either write $a_1 d = c_0 a_1 c_1$ or $a_1 = c_0 a_1 c_1 d$. In the latter case, we can apply (5) to this relation, and, in particular, get $c_0 = 1$, which we have noted gives our desired inductive step. In the former case, we substitute $a_1 d$ for the $c_0 a_1 c_1$ in our given relation and cancel a_1 from the two sides, getting a relation to which our inductive assumption applies. In particular, this gives $c_n = 1$, which we have noted also gives the desired inductive step. □

3. Moving toward Cedó's example

In this section we shall motivate Cedó's example of a monoid M which satisfies the necessary conditions (1)-(7) for DM to be a semifir, but is not a direct limit of coproducts of free monoids and free groups. In §4 we describe it formally, along with a slight variant construction; in §5 we prove that both monoids satisfy (1)-(7), and in §6, obtain some approximations to the statement that the rings DM are semifirs.

Part of our development must, of course, be a proof that our monoids M are *not* direct limits of coproducts of free monoids and free groups; so let us begin by obtaining a necessary condition for a monoid to be such a direct limit. The problem of constructing a monoid that fails to satisfy that condition but does satisfy (1)-(7) will then motivate the examples.

We begin with a result about a wider class of coproduct monoids.

Lemma 6. *Let N and N' be cancellative monoids, such that N' has no invertible elements other than* 1, *and suppose that in the coproduct monoid $N \amalg N'$, we have elements satisfying*

(13) $a\,b\,d \;=\; b\,a\,e,$

where d and e lie in N, while a and b do not both lie in N. Then $d = e$.

Proof. Let elements of $N \amalg N'$ be written as alternating strings of nonidentity elements of N and of N'. When we speak of the *initial* (respectively, *final*) N-factor of an element, we will mean the initial (final) term of that string if this is an element of N, or 1 if it is not.

Let us begin by handling the case where one of a, b lies in N. By symmetry, we can assume this is a, so by hypothesis, $b \notin N$. Letting r be the initial N-factor of b, and comparing the initial N-factors of the two sides of (13), we get $ar = r$, so as N is cancellative, $a = 1$, so another application of cancellativity gives $d = e$ as claimed.

Now assume neither a nor b lies in N. Since N' has no invertible elements, multiplication of two elements of $N \amalg N'$ cannot cause N'-factors to cancel, and thus allow N-factors that these had separated to interact; hence from (13) we can conclude that a and b have the same initial N-factor. Call that v, and call their final N-factors w and w' respectively, writing $a = v\,a_0\,w$, $b = v\,b_0\,w'$, where a_0 and b_0 both have initial and final N-factors 1. Note that if $w = w'$, then by considering the final N-factors in (13), and cancelling, we get the desired conclusion; so we will complete the proof by assuming $w \neq w'$, and getting a contradiction.

Substituting our expressions for a and b into (13), and cancelling the common initial factor v, we get

(14) $\quad a_0\, w\, v\, b_0\, w' d \;=\; b_0\, w' v\, a_0\, w\, e.$

By the above equality, the family (set with multiplicity) of internal (i.e., neither initial nor final) N-factors occurring on the two sides of (14) must be the same. Now by assumption, wv and $w'v$ cannot both be 1. If one of them is 1, then we get different numbers of internal N-factors on the two sides of (14), so that case is excluded. If neither is 1, then the family of internal N-factors on the left-hand side is the union (with multiplicity) of the families of internal N-factors of a_0 and of b_0, together with the single additional factor wv, while on the right we have the same with $w'v$ in place of wv. Hence $wv = w'v$, contradicting our assumption that $w \neq w'$, and completing the proof. $\square$

Corollary 7. *Let M be a direct limit of monoids each of which is a coproduct of a free group and a free monoid* (*or more generally, each of which is a coproduct of an arbitrary group, and a cancellative monoid which has no invertible elements other than* 1, *and which admits a homomorphism h into an abelian group, such that h carries no nonidentity element to* 1).

Then in M,

(15) *If $a\,b = b\,a\,g$, where a and b not both invertible, then $g = 1$.*

Proof. To see that the parenthetical generalized hypothesis does indeed cover the original hypothesis, note that on a free monoid, the degree function in the free generators has the properties required of h.

Note further that it suffices to prove (15) when M is itself a free product of a group and a monoid of the indicated sort. For if $a\,b = b\,a\,g$ in a direct limit of such monoids, one can lift a, b and g to one of the monoids whose limit is being taken, find a later step in the limit process where the indicated equality holds, apply (15) in that monoid, and conclude that the given case of (15) holds in M.

So let us assume that M is the coproduct of a group N, and a cancellative monoid N' of the indicated sort. Note that if we apply to both sides of our given relation the homomorphism $f: M \to N'$ that kills N and acts as the identity on N', then apply the homomorphism h of our hypothesis, and then cancel $h(f(a))\,h(f(b)) = h(f(b))\,h(f(a))$, we get $1 = h(f(g))$. Hence $1 = f(g)$, hence no factor from N' in the expression for g can fail to equal 1, hence $g \in N$. Hence we can apply Lemma 6 with $d = 1$, $e = g$ (noting that the statement that a, b are not both invertible means that they don't both belong to N), and the result follows. $\square$

Let us now try to find a monoid M that satisfies (1)-(7), but not (15). A difficulty is that condition (7), which we want to be satisfied, and (15), which we want to fail, are closely related. Given elements of a monoid M satisfying $a\,b = b\,a\,g$ with a and b not both invertible and $g \neq 1$, we can apply (7) with the roles of a, b and c played by b, ag and a, and (naming the e and f of the conclusion of this instance of (7) b' and a'), conclude that there exists a nonnegative integer n and elements $a', b' \in M$ such that

(16) $\quad b = (b'\,a')^n\,b', \qquad a = b'\,a', \qquad a\,g = a'\,b'.$

Substituting the second of these equations into the last one, and interchanging the two sides of the result, we get $a'b' = b'a'g$, an equation of the sort we started with. We can now make a second application of (7), and conclude similarly that for some $n' \geq 0$ and $b'', c'' \in M$, we have $b' = (b''a'')^{n'}b''$, $a' = b''a''$, and $a'g = a''b''$. And so on.

Once we see this pattern, it is not hard to construct examples of this behavior. Note that to get instances of $a\,b = b\,a\,g$ in a *group*, we need merely choose a and b; solving the equation then gives $g = a^{-1}b^{-1}a\,b$. So we can start in the free group on generators x and y, and let M_0 be its submonoid $\langle x,\, y,\, x^{-1}y^{-1}x\,y\rangle_{\mathrm{md}}$ (where $\langle \ldots \rangle_{\mathrm{md}}$ denotes generation as a monoid). Here x, y, and $x^{-1}y^{-1}x\,y$ will play the roles of a, b and g in the preceding discussion. To insure that this relation $a\,b = b\,(a\,g)$ satisfies (7) in the monoid we are aiming for, we should, by the above discussion, map the above monoid into the free group on generators x' and y', by the map which, for some choice of n, sends x and y respectively to $y'\,x'$ and $(y'\,x')^n y'$, and adjoin x' and y' to the image of this monoid M_0 in that group. Not surprisingly (given our preceding calculations where the element g kept its role at successive stages), we find that this homomorphism maps $x^{-1}y^{-1}x\,y$ to $x'^{-1}y'^{-1}x'\,y'$. Moreover, since x and y have been expressed in terms of x' and y', we can now work in the monoid $\langle x',\, y',\, x'^{-1}y'^{-1}x'\,y'\rangle_{\mathrm{md}}$. We can repeat this process indefinitely, using possibly different exponents n, n', etc. at successive stages, and let M be the direct limit of this process. (This is not a direct limit of the form referred to in Dicks' conjecture, since our successive submonoids of free groups are not coproducts of a free group and a free monoid.)

Actually, there is no need to use different copies of the free group on two generators at successive stages; in other words, we can consider the group homomorphisms in the above construction to be a chain of endomorphisms $G \to G \to \dots$, where G is the free group on x and y. We find that these endomorphisms are actually automorphisms of G, since one can express x and y in terms of $y\,x$ and $(y\,x)^n y$. (The generator y can be obtained by left-multiplying $(y\,x)^n y$ by the $(-n)$-th power of $y\,x$, and then x can be recovered from y and yx.) So the direct limit of the groups G under these maps will still be G; and the direct limit of the monoids M_0 will be the union of an increasing chain of isomorphic copies of M_0 within G.

We can get such a direct limit monoid M using any sequence of choices of n, n', n'', etc.. What sequence shall we try?

The simplest choice is $n = n' = \dots = 0$, so that each of our maps carries x to $y\,x$ and y to y, But this is too simple. Although the relation $x \cdot y = y \cdot xz$ behaves well in the limit monoid, if we re-partition it as $x \cdot y = yx \cdot z$, we find that it does not satisfy (4).

But going to the next simplest choice, $n = n' = \dots = 1$, we shall see that the resulting monoid has the properties we want. It is, in fact, the example developed by Cedó in [14, pp. 128-131]. We shall study that example, and a variant, in the next three sections.

4. Cedó's example, and a variant

Below, we shall write $\langle \dots \rangle_{\mathrm{md}}$ for generation or presentation as a monoid, $\langle \dots \rangle_{\mathrm{gp}}$ for generation or presentation as a group.

Let us name the group we will be working in, and the submonoid thereof from which we will build Cedó's example as a direct limit.

(17) G will denote the free group on two generators x and y.
M_0 will denote the submonoid of G generated by x, y, and $z = x^{-1}y^{-1}x\,y$.

We now gather some information about M_0.

Lemma 8. *A presentation for the monoid* M_0 *of* (17) *is* $\langle x,\, y,\, z \mid y\,x\,z = x\,y\rangle_{\mathrm{md}}$.

A normal form for its elements is given by all strings in the symbols x, y *and* z *containing no substring* $y\,x\,z$.

The universal group of M_0 *is* G, *with the inclusion of* M_0 *in* G *as in* (17) *as the universal map.*

Proof. Clearly the relation $y\,x\,z = x\,y$ holds in M_0. Since replacing substrings $y\,x\,z$ by $x\,y$ reduces the length of any string, we can, by successive applications of that reduction, transform any string into one containing no substring $y\,x\,z$. We shall prove next that if two strings containing no such substrings represent the same element of G, then they are equal as strings.

This will imply the presentation and normal-form assertions of the lemma. To get the final assertion, note that since $M_0 = \langle x,\, y,\, z \mid y\,x\,z = x\,y\rangle_{\mathrm{md}}$, its universal group will be $\langle x,\, y,\, z \mid y\,x\,z = x\,y\rangle_{\mathrm{gp}}$. In that presentation, we can solve the relation $y\,x\,z = x\,y$ for z, getting $z = x^{-1}y^{-1}x\,y$, and so eliminate the generator z and that relation from the presentation. Hence the universal group of M_0 is presented by the generators x, y and no relations, and so is, indeed, G.

Turning to what we must prove, suppose $r = s$ were an equality holding between distinct expressions in x, y and z containing no substrings $y\,x\,z$, chosen to minimize, among such examples, the sum of the lengths of r and s. Clearly, the leftmost letters of r and of s cannot be the same. When we map both sides of this relation into G, the two sides must give the same reduced group words in x and y. Since no z in r or s is preceded by the sequence $y\,x$, the only simplifications that can occur, initially, are of the form $x\,z = x \cdot (x^{-1}y^{-1}x\,y) \mapsto y^{-1}x\,y$. We can then have further simplifications, in that the y^{-1} with which one such product begins can cancel the y with which a preceding z or the result of simplifying a preceding $x\,z$ ends. But we can handle this from the start, by applying the simplification $(x\,z)^m \mapsto y^{-1}x^m\,y$ to all maximal strings $(x\,z)^m$, and then, if such a string was preceded by a z, making the additional simplification $z(y^{-1}x^m\,y) \mapsto x^{-1}y^{-1}x^{m+1}y$. We see that when we have done this, the images of r and s will be reduced group words, and hence must coincide.

In this process, we note that the y^{-1} resulting from the leftmost z's in r and in s (if any) will not be affected by any reduction, and so will appear in the final reduced word. It follows that if either of r or s contained no z, the same would have to be true of the other, and they would equal. So they must both contain z. Next, if their common reduced image in G does not begin with x^{-1} or y^{-1}, but with some other letter u (namely, x or y), then the expressions r and s must both begin with that letter u, contradicting our observation that (by minimality) they cannot begin with the same letter. Finally, if their images both begin with x^{-1}, then in M_0, both r and s begin with z, while if both images begin with y^{-1} then they both begin with $x\,z$, contradicting that same observation. This completes the proof of the lemma. $\square$

Let me now introduce the variant construction I have alluded to. The direct limit process (to be described below) by which we get the final monoid will be the same, but instead of starting with $M_0 = \langle x,\, y,\, z\rangle_{\mathrm{md}} \subseteq G$ as in (17), we will use the larger submonoid

(18) $\quad M_1 \ = \ \langle x,\, y,\, z,\, z^{-1}\rangle_{\mathrm{md}} \subseteq G,$ where again, $z = x^{-1}y^{-1}x\,y.$

The idea is that by making z, i.e., the g in our counterexample to (15), invertible, and hence "a little more like 1", we might keep M from violating some as-yet-undiscovered requirement for DM to be a semifir. It is not hard to adapt the proof of Lemma 8, and get the analog of that result for this monoid:

Lemma 9. *A presentation for* M_1 *is* $\langle x,\, y,\, z,\, z^{-1} \mid y\,x\,z = x\,y,\ x\,y\,z^{-1} = y\,x,\ z\,z^{-1} = 1 = z^{-1}z\rangle_{\mathrm{md}}$.

A normal form for its elements is given by all strings in the symbols x, y, z *and* z^{-1} *containing no substrings* $y\,x\,z$, $x\,y\,z^{-1}$, $z\,z^{-1}$ *or* $z^{-1}\,z$.

The universal group of M_1 *is* G, *with the inclusion of* M_1 *in* G *as in* (18) *as the universal map.* $\square$

Essentially all our results about these examples will be proved for both the limit monoid based on M_0 and the limit monoid based on M_1. Here is the description of the map over which we will take our direct limits, as motivated in the last four paragraphs of the preceding section.

(19) $\quad\sigma$ will denote the automorphism of G which acts by $\sigma(x) = y\,x$, $\sigma(y) = y\,x\,y$, and which (it is easy to check) fixes z and hence z^{-1}, and thus carries M_0 and M_1 into themselves.

We now name our direct limit monoids. Because σ is an automorphism of G, we can take the desired direct limits within G. We let

(20)
$M_{(0)}$ = the set of $a \in G$ such that $\sigma^n(a) \in M_0$ for some (hence, for all sufficiently large) n.
$M_{(1)}$ = the set of $a \in G$ such that $\sigma^n(a) \in M_1$ for some (hence, for all sufficiently large) n.

Cedó introduces what we are calling $M_{(0)}$ in the example beginning at [14, p. 128, after proof of Lemma 4.4]. (He writes φ for σ^{-1}, and describes the monoid as the union in G of the chain of submonoids $\varphi^n(M_0)$. His generators r and t are our x and y.)

Note that

(21) $\quad$For $i = 0, 1$, the restriction of σ to the direct limit monoid $M_{(i)}$ is a monoid automorphism.

5. Basic properties of the above monoids

From the final assertions of Lemmas 8 and 9, we have

Lemma 10. $M_{(0)}$ *and* $M_{(1)}$ *satisfy* (1) *and* (2) (*each having* G *as its universal group*), *and hence* (3). $\square$

More work is required to get

Lemma 11. $M_{(0)}$ *and* $M_{(1)}$ *satisfy* (4).

Proof. In verifying (4) for a given relation

(22) $\quad a\,b \ = \ c\,d$

in $M_{(i)}$, we may clearly first apply an arbitrarily high power of σ to the elements of $M_{(i)}$ in question. By doing so we can, to start with, bring such elements into the submonoid M_i, where we can write them in the normal form of Lemma 8 or 9. Further applications of σ may not preserve that normal form, however: If an expression contains a sequence $x\,z$ or $y\,z^{-1}$, we see that on applying σ, we get an expression which can be reduced, thus removing an occurrence of z or z^{-1}. On the other hand, application of σ never introduces new occurrences of z or z^{-1}, so under successive applications of σ to an element, the number of such factors eventually stabilizes. From this fact, and the explicit formula for σ, one sees that after sufficiently many applications of that map, every element of $M_{(i)}$ becomes and subsequently remains an element of M_i in whose normal-form expression

(23) No z is immediately preceded by an x, no z^{-1} is immediately preceded by a y, every x is immediately preceded by a y, and if y is the last letter in the expression, that occurrence of y is immediately preceded by an x.

So suppose we have a relation (22) in $M_{(i)}$, such that a, b, c, d lie in M_i, and their normal forms satisfy (23).

If these normal-form expressions, when multiplied together as on the two sides of (22), still give normal-form expressions, then the normal-form expression for one of a, c must be an initial substring of the other, and we have the conclusion of (4), as desired. Moreover, in view of (23), the only way these product-expressions can fail to be in normal form is if the expression for b or d begins with a z or z^{-1}, and the expression for a, respectively c, ends with a symbol or pair of symbols that interacts with this letter.

It is now easy to dispose of the case $i = 1$ (i.e., the case where our monoid is $M_{(1)}$). Since z and z^{-1} are invertible, and multiplying on the right by an invertible element does not affect right divisibility relations, we can take whatever power of z occurs at the beginning of b or d, and move it to the end of a, respectively c, and then reduce the resulting expressions, and, if necessary, apply σ until (23) again holds. Then by the preceding paragraph, one of these elements will left divide the other in M_0 and hence in $M_{(0)}$, establishing (4).

So suppose $i = 0$. Since a reduction must occur in at least one of the products $a \cdot b$ and $c \cdot d$ of (22), we can assume without loss of generality that

(24) The product $a \cdot b$ is reducible; i.e., we can write $a = a' y\, x$, $b = z\, b'$, where a', b' are in normal form for M_0. Moreover, if the product $c \cdot d$ is also reducible, and we similarly write $c = c' y\, x$, $d = z\, d'$, then we may assume the total degree of a in x and y is at least that of c.

Note that if we simplify $a \cdot b$ by the reduction $a' y\, x \cdot z\, b' \mapsto a' x\, y\, b'$, then the resulting expression is in normal form. For the only way it might not be is if the y shown were immediately followed by $x\, z$ in b'; but that would contradict the assumption that b satisfies (23), in particular, the condition that every x is immediately preceded by a y. Hence

(25) In the notation of (24), the normal form of $a\, b$ is $a' x\, y\, b'$, and if the expression $c \cdot d$ is reducible, its normal form is $c' x\, y\, d'$, where $\deg(c') \leq \deg(a')$.

We now argue by cases. If $c \cdot d$ is already in normal form, then c must be an initial substring of $a' x\, y\, b'$. It will left divide a if as a substring it is contained in a', while it will be a right multiple of a if it contains $a' x\, y = a' y\, x\, z = a\, z$; this leaves only the case $c = a' x$. In this case we apply the automorphism σ once more, and see that $\sigma(c) = \sigma(a' x) = \sigma(a') y\, x$, while $\sigma(a) = \sigma(a' y\, x) = \sigma(a') y\, x\, y\, x\, y$, so $\sigma(c)$ left divides $\sigma(a)$ in M_0, hence c left divides a in $M_{(0)}$, as required.

On the other hand, if $c \cdot d$ is not initially in normal form, then equating the normal forms of the two sides of (22), we get $a' x\, y\, b' = c' x\, y\, d'$, with c' having at most the degree of a'. If c is not to left-divide a, then c' cannot equal a', and we see that $c' x\, y$ must be an initial substring of a'. Hence a, being a right multiple of a', is a right multiple of $c' x\, y = c' y\, x\, z = c\, z$, hence is a right multiple of c. So again, the conclusion of (4) is satisfied. □

Much easier is

Lemma 12. *$M_{(0)}$ and $M_{(1)}$ satisfy* (5).

Proof. Let $a = c\, a\, d$ in $M_{(i)}$, with a non-invertible. Again, we may assume without loss of generality that $a, c, d \in M_i$.

Now the total degree function in x and y, i.e., the homomorphism from G to the additive group of integers that takes x and y to 1, has positive value on the generators x and y of M_i, but value 0 on z and z^{-1}. Hence all elements of M_i have positive degree, except those in the submonoid (if $i = 0$) or subgroup (if $i = 1$) generated by z. Hence applying the degree function to the given relation, we see that both c and d must be (natural number or integer) powers of z; say

(26) $c = z^p, \; d = z^q.$

Let us begin with the case where a is also a power of z. If $i = 0$, this says that our given equation has the form $z^m = z^{p+m+q}$ with all of m, p, q nonnegative, so $p = q = 0$, so $c = d = 1$, as desired. On the other hand, if $i = 1$, this case cannot occur, since it would make a invertible, contrary to our assumption.

Assuming a is not a power of z, its normal form in M_i will involve at least one occurrence of x or y. Now since the rules for reduction to normal form, other than $z\,z^{-1} = 1 = z^{-1}z$, only affect z and z^{-1} when they occur to the *right* of another letter, when we reduce $c\,a\,d$ to normal form in M_i the power (possibly 0) of z occurring to the left of the leftmost x or y will be exactly p more than the corresponding value in the normal form of a. Since $a = c\,a\,d$, this means $p = 0$. So $c = 1$, so $a = a\,d$, so $d = 1$, as required. □

Finally, we want to prove that $M_{(0)}$ and $M_{(1)}$ satisfy (7), equivalently, (7′) or (7″). As with (4), this will require delicate considerations, because the condition we want to prove is similar to the condition (15) which we have arranged will not hold. We will prove (7′), i.e., that if $ab = ca$, then a is not left divisible in $M_{(i)}$ by all positive integer powers of c. In doing so, it is not sufficient to work with the images of our elements in $M_{(i)}$ under a fixed power of σ, since it could happen that as we apply larger and larger powers of σ, the image of a becomes divisible by larger and larger powers of the image of c. (Or differently stated, that larger and larger submonoids $\sigma^{-n}(M_{(i)})$ contain more and more of the elements $c^{-m}a$.)

The trick that will help us will be suggested by the following easily verified observation, though that observation will not be called on in the proof.

Lemma 13. *The automorphism σ of G is the square of an automorphism $\sigma^{1/2}$, which acts by*

(27) $\qquad \sigma^{1/2}(x) = y, \qquad \sigma^{1/2}(y) = y\,x.$

This automorphism sends z to z^{-1}, and vice versa. □

Now writing down the first few terms of the orbit of x under $\sigma^{1/2}$,

(28) $\qquad x,\quad y,\quad yx,\quad yxy,\quad yxyyx,\quad yxyyxyxy,\quad yxyyxyxyyxyyx,\quad \ldots$

we find that each term is the product of the two that precede. Precisely, we have

(29) $\qquad (\sigma^{1/2})^{n+2}(x) \;=\; (\sigma^{1/2})^{n+1}(x)\;(\sigma^{1/2})^{n}(x).$

This can be verified by noting that it holds for $n = 0$, and applying $(\sigma^{1/2})^n$ to the resulting equation. From this we see that the length of the term $(\sigma^{1/2})^n(x)$ is the Fibonacci number F_{n+1}. Now the Fibonacci sequence is a linear combination of the sequences of powers of $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, and this line of thought leads us to the following observation.

Lemma 14. *Let $\tau = (1+\sqrt{5})/2$, and let δ be the homomorphism from G to the additive group of real numbers defined by*

(30) $\qquad \delta(x) = 1, \qquad \delta(y) = \tau \qquad$ *(and thus $\delta(z) = 0$).*

Then for all $a \in G$,

(31) $\qquad \delta(\sigma^{1/2}(a)) = \tau\,\delta(a), \quad$ *and hence* $\quad \delta(\sigma(a)) = \tau^2\,\delta(a).$

Hence δ assumes nonnegative values on all elements of $M_{(i)}$, and positive values on all of these except powers of z.

Proof. The relation (31) holds because it holds on the generators x and y of G. The final assertion is seen by applying (31) to the generators of M_i, and then to $M_{(i)} = \bigcup \sigma^{-n}(M_i)$. □

We can now deduce

Lemma 15. *The only elements $c \in M_{(i)}$ such that there exist elements left or right divisible by all powers of c (i.e., such that $\bigcap c^n M_{(i)}$ or $\bigcap M_{(i)} c^n$ is nonempty) are the invertible elements.*

Proof. It is immediate from the last assertion of Lemma 14 that the only elements c which can possibly have either of the above properties are the powers of z. In $M_{(1)}$, these are invertible, so if $i = 1$, we are through.

In $M_{(0)}$, it remains to show that no element is left or right divisible by arbitrarily large powers of z. This is equivalent to saying that for fixed $a \in M_0$, the powers of z left or right dividing elements $\sigma^n(a)$ in M_0 are bounded as n grows.

Now as noted in the proof of Lemma 12, the string of z's at the left-hand end of a product of the generators of M_0 is not affected by reduction to normal form; and neither is the length of that string changed by applying σ; so that length gives an upper bound on the power of z left dividing a, as required.

This is all we will need in what follows, but for completeness, we have asserted the corresponding result for right divisibility as well, so here is a sketch of the argument in that case.

The deviation from the case we have discussed arises from the ability of some elements, when multiplied on the right by one or more z's, to "absorb" them, so that the product has a normal form not showing these right factors. An element whose normal form is $a\,y\,x$ can "absorb" one z in this way, since $a\,y\,x\,z = a\,x\,y$, but since the result ends in y, it can absorb no more; and it is easy to see that its images under all powers of σ still end in y, and so can still absorb no z's. It follows that an element of M_0 whose normal form ends with z^m and no higher power of z can be right divisible by no higher power of z than z^{m+1}, not only in M_0 but in $M_{(0)}$, completing the proof. □

Corollary 16. *$M_{(0)}$ and $M_{(1)}$ satisfy* (7), (7′) *and* (7″).

Proof. By Lemma 4 it suffices to prove condition (7′). The left-divisibility statement of the above lemma proves this for relations $a\,b = c\,a$ in which c is not invertible, while if c is invertible but $\neq 1$, such a relation can be written $a = c^{-1}\,a\,b$, and is then excluded by (5), proved in Lemma 12. □

On the other hand, the relation $y\,x\,z = x\,y$ gives us, as planned,

Lemma 17. *$M_{(0)}$ and $M_{(1)}$ do not satisfy* (15). *Hence neither of them is a direct limit of monoids which are coproducts of a free monoid and a free group.* □

6. Semifirs, and graded semifirs

We defined in §1 the class of rings called n-firs. Let us recall here a useful reformulation of the n-fir condition.

In any ring R, by an *m-term linear relation* we will mean a relation $\sum_{i=1}^{m} u_i v_i = 0$ with $u_i,\, v_i \in R$. We may write such a relation as $(u_i)\cdot(v_i) = 0$, where (u_i) and (v_i) are the length-m row vector and height-m column vector formed from the indicated elements. We shall say our relation is "trivial" if for each $i \in \{1,\dots,m\}$, either $u_i = 0$ or $v_i = 0$. Given an m-term linear relation $(u_i)\cdot(v_i) = 0$ and an invertible $m\times m$ matrix T, let $(u'_i) = (u_i)\,T$, $(v'_i) = T^{-1}(v_i)$, so that the given relation is equivalent to $(u'_i)\cdot(v'_i) = 0$. If this new relation is trivial, then we shall say that T *trivializes* the relation $(u_i)\cdot(v_i) = 0$. It is not hard to show that a ring R is an n-fir if and only if every m-term linear relation with $m \leq n$ can be trivialized by some invertible $m \times m$ matrix [13, Theorem 2.3.1].

Let us now generalize these ideas to rings graded by a group A. Because the grading groups I have in mind are $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$, I will write the operation of A as addition, though in this general discussion, we do not need A to be commutative. If R is an A-graded ring, let us understand a *homogeneous* linear relation in R to be a relation $\sum_{i=1}^{m} u_i v_i = 0$ such that for some $\alpha_1,\dots,\alpha_m$ and β in A, we have $u_i \in R_{\alpha_i}$ and $v_i \in R_{-\alpha_i+\beta}$ for each i; and let us say such a relation is *homogeneously trivializable* if it is trivializable by an invertible matrix $T = ((T_{ij}))$ with $T_{ij} \in R_{-\alpha_i+\alpha_j}$ for each i and j. I shall call the graded ring R a *homogeneous n-fir* if every homogeneous linear relation of $\leq n$ terms is homogeneously trivializable. (Though the name might suggest a condition stronger than being an n-fir, it is, roughly speaking, weaker. It would be truly weaker if we merely required that every homogeneous linear relation be trivializable, rather than homogeneously trivializable. I don't know whether homogeneous trivializability is a stronger condition on a homogeneous relation than trivializability.)

If $h : M \to A$ is a homomorphism from a monoid to a group, this induces an A-grading on the monoid ring DM. Namely, for each $\alpha \in A$, we let the homogeneous component of DM of degree α, which we will write DM_α, be the span over D of $\{c \in M \mid h(c) = \alpha\}$.

Now let G be the free group on x and y, and $M_{(0)}$ and $M_{(1)}$ the submonoids of G so named in the preceding three sections. The map $\delta : G \to \mathbb{Z}\times\mathbb{Z}$ of the next proof is, up to isomorphism, the δ of Lemma 14, since the image $\mathbb{Z} + \tau\mathbb{Z}$ of the latter map is isomorphic to $\mathbb{Z}\times\mathbb{Z}$; but since the embedding of that group in the reals, and the resulting ordering, are irrelevant here, we use the more abstract description. Our first step toward our desired result on $\mathbb{Z}$-graded monoid rings will be the following $\mathbb{Z}\times\mathbb{Z}$-graded result.

Lemma 18. *Let $M = M_{(0)}$ or $M_{(1)}$, and let $\delta : G \to \mathbb{Z}\times\mathbb{Z}$ be the homomorphism taking x to $(1,0)$ and y to $(0,1)$. Then for any division ring D, the $\mathbb{Z}\times\mathbb{Z}$-graded ring DM is a δ-homogeneous semifir.*

Proof. Assuming the contrary, let

(32) $\quad \sum_{i=1}^{n} u_i\, v_i \;=\; 0$

be an n-term homogeneous linear relation which cannot be homogeneously trivialized; say with $u_i \in DM_{\alpha_i}$ and $v_i \in DM_{-\alpha_i+\beta}$ $(\alpha_i, \beta \in \mathbb{Z}\otimes\mathbb{Z})$. (From this point on, however, let us, for brevity, abbreviate "trivialize homogeneously" to "trivialize", and understand that any invertible matrices by which we act on our row and column vectors have all entries homogeneous, of the appropriate degrees.)

Assume (32) chosen to minimize n. Then I claim that

(33) No invertible matrix T can make either any component of $(u_i)\,T$ or any component of $T^{-1}(v_i)$ equal to zero.

For if T did so, then $(u_i)\,T\cdot T^{-1}(v_i)=0$ would become, on deleting the trivial term, a linear relation with $n-1$ terms, which by our minimality assumption could be trivialized, yielding a trivialization of our original relation.

Now let us choose from among the finitely many elements of M comprising the supports of $u_1,\dots,u_n$, one element a which is not a proper right multiple of any of the others, i.e., which maximizes $a\,M$. Say a occurs in the support of u_1. Let us write each u_i as $a\,u'_i + u''_i$, where the terms u''_i have support consisting of elements that are not right multiples of a. By choice of a, none of the elements of the supports of the u''_i are proper left divisors of a either, so as M satisfies (4), no right multiple of any of these support-elements coincides with any right multiple of a; so projecting (32) onto $a(DM)$ we get $\sum a\,u'_i\,v_i = 0$, and, cancelling a, we have $\sum u'_i\,v_i = 0$.

Note that in this last relation, at least the summand u'_1 is nonzero; hence if this relation could be trivialized, the trivializing transformation would leave at least one of the left-hand factors nonzero, and hence make one of the right-hand factors zero. But this would contradict (33); so our relation $\sum u'_i\,v_i = 0$ is still non-trivializable. Renaming the u'_i as u_i, and recalling that the homogeneous degree of u_i is denoted α_i, we have reduced ourselves to the situation where in (32),

(34) $\quad \alpha_1 = (0,0)$, and u_1 has $1 \in M$ in its support.

So far, everything we have said would be valid for any monoid M satisfying (3) and (4) and given with a homomorphism δ into a group A. Let us now note that for the two monoids $M_{(0)}$ and $M_{(1)}$ that we are interested in, the submonoids

(35) $\quad N \;=\; \delta^{-1}(\{(0,0)\})$

are, respectively, the cyclic submonoid generated by z, and the cyclic subgroup generated by z; so that the rings DN are the (not necessarily commutative) right and left principal ideal domains $D[z]$ and $D[z,z^{-1}]$. I claim that using this observation and (33), we can complement (34) with the statement

(36) $\quad \alpha_i \neq (0,0)$ for all $i>1$.

Indeed, if this were not true, say we had $\alpha_2=(0,0)$. In the right principal ideal domain DN, the elements u_1 and u_2 must be right linearly dependent, and such a linear dependence relation in a principal right ideal domain (which is, in particular, a right fir, hence a 2-fir) can be trivialized by an invertible 2×2 matrix, which will turn $(u_1,\,u_2)$ into a row with one term zero. Now extending this to an $n\times n$ matrix by attaching an $n{-}2\times n{-}2$ identity matrix, we would get a contradiction to (33). This proves (36).

Let us now examine u_1. If $M=M_{(0)}$, then $u_1\in D[z]$ is a polynomial $p(z)$ having nonzero constant term by (34). If $M=M_{(1)}$, then $u_1\in D[z,\,z^{-1}]$ is a Laurent polynomial; but by applying a matrix T which agrees with the identity except in its $(1,1)$ entry, for which we use an appropriate power of the invertible element z, we may reduce to the case where u_1 is again an ordinary polynomial $p(z)\in D[z]$ with nonzero constant term. Thus, we may assume that this is the case whether M is $M_{(0)}$ or $M_{(1)}$. In either case, let d be the degree of $p(z)$.

We shall now apply to (32) an invertible matrix T whose effect on (u_i) is to subtract from each u_i with $i>1$ a certain right DM-multiple of $u_1=p(z)$. I claim that we can do this so that the supports of the resulting terms consist of elements $a\in M$ in which the power of z occurring at the far left in a lies between 0 and $d-1$. (To make precise what we mean by the power of z occurring at the far left, recall that for all sufficiently large r, $\sigma^r(a)$ will lie in M_0 or M_1, where we have a normal form for elements, so we can speak of the power of z that occurs at the left end in that normal form; and as noted in the last paragraph of the proof of Lemma 12, the application of σ does not change that power.) Indeed, DM is free as a left

DN-module on the basis consisting of those elements $a \in M$ in which the power of z at the far left is z^0; and by subtracting a right multiple of $p(z)$, we can reduce any element of that free module to one in which the coefficient in DN of each such a lies in a set of representatives of the residue classes modulo $p(z)$ in $D[z]$, respectively, $D[z, z^{-1}]$, namely (in either case), the D-linear combinations of $z^0, \dots, z^{d-1}$.

So we may assume below that the support of each of $u_2, \dots, u_n$, consists of elements of M in whose normal forms the factor z^i at the left end satisfies $0 \le i < d$. Moreover, by (36), none of the monomials in the supports of $u_2, \dots, u_n$ are themselves powers of z; hence the number of z's at their left ends is unaffected by *right* multiplying by arbitrary elements of M. This shows that

(37) In all monomials in the support of $\sum_{i=2}^n u_i\, v_i$, the power of z occurring at the far left lies in the range $0, \dots, d-1$.

We will now get a contradiction, and complete the proof, by showing that the same is not true of the term $u_1\, v_1 = p(z)\, v_1$. To do this, let b be any monomial with no z or z^{-1} at its far left, such that at least one term of the form $z^j\, b$ occurs in the support of v_1. Thus, we can write the part of v_1 with support in Nb as $q(z)\, b$, for some $q(z) \neq 0$ in DN (i.e., in $D[z]$ if $M = M_{(0)}$, or in $D[z, z^{-1}]$ if $M = M_{(1)}$). Hence, the projection of $u_1 v_1$ onto DNb will be $p(z)q(z)\, b$. If $M = M_{(0)}$ we see that the degree of $p(z)q(z)$ will be $\ge d$. If $M = M_{(1)}$, the same will be true if $q(z)$ involves at least one term in which z has nonnegative exponent, while in the contrary case, $p(z)q(z)$ will have a term in which z has negative exponent. In either case, comparing with (37) we see that $u_1 v_1$ cannot have zero sum with $\sum_{i=2}^n u_i\, v_i$. This contradiction completes the proof of the lemma. □

We shall now deduce from the above result the corresponding statement for $\mathbb{Z}$-gradings, as promised in the Abstract.

Proposition 19. *Let $M = M_{(0)}$ or $M_{(1)}$, let $\delta : G \to \mathbb{Z} \times \mathbb{Z}$ be as in the preceding lemma, and let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be any nonzero homomorphism, so that $h\delta : G \to \mathbb{Z}$ induces a $\mathbb{Z}$-grading on the monoid ring DM. Then the $\mathbb{Z}$-graded ring DM is an $h\delta$-homogeneous semifir.*

Proof. Let $\sum_{i=1}^n u_i v_i = 0$ be an n-term $h\delta$-homogeneous linear relation in DM. Each of the $h\delta$-homogeneous elements u_i and v_i can be dissected into δ-homogeneous summands, with degrees in $\mathbb{Z} \times \mathbb{Z}$ which differ by members of the cyclic group $\ker(h)$, so we can picture each of these sets of degrees as "laid out along a line" in $\mathbb{Z} \times \mathbb{Z}$, parallel to the line given by $\ker(h)$. Our idea will be to look at one end of each of these lines, use the preceding lemma to trivialize the induced relation among the terms at those ends, and thus "eat away at" the given relation until it is trivialized.

But how do we know that the matrices T that we apply to eat away at one end of our elements will not cause them to grow at the other end, and hence survive indefinitely? The key will be a fact that we saw in the preceding section, but have not yet used in this one: that if we let $\eta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be defined by

(38) $$\eta(r, s) \;=\; r + \tau\, s, \text{ where } \tau = (1 + \sqrt{5})/2,$$

then for every $a \in M$, we have $\eta\delta(a) \ge 0$ in $\mathbb{R}$. (This is the final statement of Lemma 14, though we are now writing $\eta\delta$ for what we there called δ.) Hence if we eat away at the end of each element corresponding to large values of $\eta\delta$, we will hit a barrier when those values try to become negative, and we will thus successfully trivialize our relation.

Here are the details.

The kernel of $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is a cyclic subgroup of $\mathbb{Z} \times \mathbb{Z}$; of its two generators, let g be the one such that $\eta(g)$ is positive.

Let us choose $\alpha_1, \dots, \alpha_n \in \mathbb{Z} \times \mathbb{Z}$ which are preimages under h of the $h\delta$-degrees of $u_1, \dots, u_n$, respectively, and β which is a preimage of the common $h\delta$-degree of $u_1 v_1, \dots, u_n v_n$. The set of possible choices for each α_i, respectively for β, is a coset of $g\mathbb{Z}$; and by the fact that $\eta(g) \neq 0$, we can choose our representatives of those cosets so that each $\eta(\alpha_i)$ is ≤ 0, and $\eta(\beta)$ is $\le$ all the $\eta(\alpha_i)$. Thus, if for each i, we write $u_i = \sum_j u_{ij}$, where u_{ij} is δ-homogeneous of degree $\alpha_i + jg$ $(i \in \mathbb{Z})$, then since every element of M has nonnegative $\eta\delta$-degree, while the α_i are ≤ 0, the nonzero terms in this summation must all have $j \ge 0$. Similarly, v_i is a sum of terms v_{ik} that are δ-homogeneous of degrees $-\alpha_i + \beta + kg$ with $k \ge 0$.

Now for each i, let j_i be the largest value such that $u_{ij_i} \neq 0$ and k_i the largest such that $v_{ik_i} \neq 0$. Here we take j_i or k_i to be $-\infty$ if u_i, respectively v_i, is zero. Let $\ell = \max_i(j_i + k_i)$. We shall use induction on ℓ, which is a nonnegative integer as long as our relation is nontrivial.

By re-indexing, we may assume that the values of i for which $j_i+k_i=\ell$ are $1,\dots,m,$ where $1\leq m\leq n.$ Then under the δ-grading of $DM,$ the component of $\sum_{i=1}^n u_i v_i$ of $\mathbb{Z}\times\mathbb{Z}$-degree $\beta+\ell g$ is $\sum_{i=1}^m u_{ij_i} v_{ik_i}.$ Hence $\sum_{i=1}^m u_{ij_i} v_{ik_i}=0,$ which is a δ-homogeneous linear relation (of degree $\beta+\ell g$), so by the preceding lemma, it can be trivialized. If we extend the trivializing matrix by attaching an $n{-}m\times n{-}m$ identity matrix, and apply this to the original relation $\sum_{i=1}^n u_i v_i=0,$ the terms u_i and v_i with $i>m$ are not changed, while for each $i\leq m,$ the terms u_i and v_i are modified so as to decrease j_i or $k_i.$ Thus the new relation has lower $\ell.$ In this way, we eventually reach $\ell<0,$ i.e., $\ell=-\infty,$ and have thus trivialized our given relation. □

I had hoped to take this idea through one more iteration: Given a relation $\sum u_i v_i=0$ holding in $DM,$ without any homogeneity assumption, suppose one chooses an arbitrary nonzero homomorphism $h:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z},$ and grades the terms of this relation by $h\delta: M\to\mathbb{Z}.$ One can look at the components of the u_i and v_i on which $h\delta$ assumes its largest value, take those i where these two values have the greatest sum, $\ell,$ note that the components of those terms with $h\delta$-degree ℓ yield an $h\delta$-homogeneous linear relation, trivialize it using the above proposition, apply the trivializing matrix (extended using an identity matrix) to the original relation, and thus transform it into a relation with smaller ℓ; and repeat this process indefinitely.

The trouble is that there is no reason why this process should terminate. The image of M under δ is $\{(r,s)\in\mathbb{Z}\times\mathbb{Z}\mid r+\tau s\geq 0\},$ i.e., the set of lattice points in a half-plane bounded by a line of irrational slope; so no nonzero homomorphism $h:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ assumes only positive values on that image. Hence the ℓ of the above discussion, unlike the ℓ of the proof of Proposition 19, need not be positive, and there is no reason why the process should not decrease it indefinitely.

I then thought, "Aha! Since we can continue this process indefinitely, it shows that our relation is trivializable in the *completion* (in the negative-degree direction) of DM with respect to any such grading!" But alas, when I tried to write out the argument, I found I could not prove even that. Namely, I can find no reason why the products of the matrices used in the successive steps toward trivialization should converge with respect to the grading. But the idea is still tantalizing; perhaps someone can make it work.

Even if it does work, the semifir condition on a completion is not very close to the condition that the non-completed ring be a semifir. For instance (working over a commutative base field k for the sake of familiarity), the group algebra on a free abelian group on two generators, $k[x,\,x^{-1},\,y,\,y^{-1}]$ is far from being a semifir – ideals such as $(x-1,\,y-1)$ are not free – but its completion with respect to degree in $y,$ which we may write $k[x,\,x^{-1}]((y,\,y^{-1})),$ is a fir.

A variant idea might be to try to complete DM with respect to the homomorphism $\eta\delta,$ which takes values in the ordered group $\mathbb{Z}+\tau\mathbb{Z}\subseteq\mathbb{R},$ either allowing infinite sums in the "upward" direction (toward $+\infty$) or the "downward" direction (toward 0). But completions with respect to non-discrete ordered groups are, I suspect, a messy subject.

Finally, I have played with the idea of trying to trivialize linear relations $\sum u_i v_i=0$ in DM by successively reducing, not the degrees of highest-degree terms, but something like the number of lattice points in the convex hull of the union of the images of the supports of the $u_i v_i$ in $\mathbb{Z}\times\mathbb{Z}.$ These numbers are invariant under the action of $\sigma,$ and since they are nonnegative-integer valued, a process that decreased them would terminate in finitely many steps. But I don't see how to develop such a process. It would require some technique that is applicable to the $\mathbb{Z}\times\mathbb{Z}$-grading of DM determined by the homomorphism $M\to\mathbb{Z}\times\mathbb{Z},$ but not to the $\mathbb{Z}\times\mathbb{Z}$-grading of $k[x,\,x^{-1},\,y,\,y^{-1}]$ determined by the *isomorphism* of the free abelian group on x and y with $\mathbb{Z}\times\mathbb{Z}$ (nor to the corresponding grading on the monoid ring of any rank-2 submonoid of that free abelian group), since $k[x,\,x^{-1},\,y,\,y^{-1}]$ is not a 2-fir (and similarly for monoid rings such as $k[x,\,y]$).

7. A further observation on the above examples

In the discussion by which we motivated the definition of the monoid $M_{(0)},$ we started out with a relation $a\,b=b\,a\,g$ holding in $G,$ saw what sort of elements of G we had to throw into a monoid M containing $a,$ b and g so that (7) would be satisfied for that relation, and then proceeded similarly for an infinite sequence of further relations which that one led to.

Now in Cedó's proof that every monoid M such that DM is a 2-fir must satisfy (6), he considers a relation $a\,b=c\,a\,d$ that holds in $M,$ notes that in DM this leads to the equation $a\,b-a\,d=c\,a\,d-a\,d,$ writes this as the 2-term linear relation $a\,(b-d)-(c-1)(a\,d)\;=\;0,$ and examines what is required to trivialize that linear relation. Let us restrict attention to the case $d=1$, i.e., the case of a relation $a\,b=c\,a$ as in (7), so that Cedó's linear relation takes the form

(39) $\quad a\,(b-1)-(c-1)\,a \;=\; 0.$

In the case of the relation $x\cdot y=y\cdot(xz)$, around which we built the monoid $M_{(0)}$, it is natural to wonder: Can the corresponding relation in DM actually be trivialized, or do the succession of elements we have to adjoin to $M_{(0)}$ endlessly postpone the completion of this trivialization?

It turns out that the relation in question, namely, $y\,(x\,z-1)-(x-1)\,y=0$, is indeed trivializable in DM. To see this, let us first apply σ to that relation, getting $y\,x\,y\,(y\,x\,z-1)-(y\,x-1)\,y\,x\,y=0$, then reduce the term $y\,x\,z$ to normal form, getting $y\,x\,y\,(x\,y-1)-(y\,x-1)\,y\,x\,y=0$. The relation now has all its terms in the free monoid ring $D\langle x,\,y\rangle_{\mathrm{md}}$, which is a fir, so the relation is indeed trivializable. (If one works out the details, the trivialization process, applied to the pair of left-hand factors, $(y\,x\,y,\;-y\,x+1)$, begins by adding to the first term the second term right-multiplied by y, getting $(y,\,-y\,x+1)$. One then adds to the second term the first right-multiplied by x, getting $(y,\,1)$, and finally adds to the first the second right-multiplied by $-y$, giving $(0,1)$. The corresponding operations on the right-hand factors of our relation necessarily yield a vector with second entry zero, so the relation is indeed trivialized.)

So the scenario is not one where the steps of our construction of M repeatedly postpone the trivialization of this relation; rather, the first step renders that relation trivializable, but creates *another* relation which must also be trivialized, and so on.

In fact, it is not hard to see that any relation of the form $a(b-c)-(d-e)f=0$ holding in $DM_{(i)}$ with $a,\dots,f\in M_{(i)}$ will be homogeneous with respect to some $\mathbb{Z}$-grading $h\delta$. For only two distinct elements of $M_{(i)}$ can appear among ab, ac, df, ef, and we can find a nonzero homomorphism $h:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ such that $h\delta$ has the same value on those two elements. Hence the relation is $h\delta$-homogeneous, so by Proposition 19, it is trivializable.

8. Some monoids that misbehave

In this section, we turn back to the conditions on monoids that we discussed in §2, and give examples of monoids that fail to satisfy one or another of them. But I will start with a quick example of a monoid which does satisfy those conditions – but not in the obvious way.

Example 20. *Let $G=\langle x,\,y\rangle_{\mathrm{gp}}$ be the free group on two generators. Then the submonoid $M=\langle x,\,x\,y,\,x\,y^2\rangle_{\mathrm{md}}\subseteq G$ generates G as a group, and DM is a left and right fir, but the universal group of M is not G, but the free group on three generators, namely, on the images of x, $x\,y$ and $x\,y^2$.*

Proof. It is easily seen that M is the free monoid on those three generators. As a free monoid, it is one of the primordial examples of monoids such that DM is a right and left fir for all D. But though it clearly generates the given group G, its *universal* group is, necessarily, the free group on its free generators. □

Now for examples that don't satisfy all the conditions of §2.

Example 21. *Every group G, regarded as a monoid, satisfies* (2)-(7$''$)*; but if G is not locally free, it will not satisfy* (1)*, and so DG will not be a semifir. For $G=\mathbb{Z}\times\mathbb{Z}$, that ring is not even a 2-fir.*

Proof. It is clear that every group satisfies (2)-(5), (7$'$) and (7$''$), and hence by Lemma 4, (6), (6$'$) and (7).

(The last three conditions can also be obtained directly, by noting that from the equation assumed in the condition in question, and any choice of n, one can solve for the elements that make the conclusion hold. For instance, given group elements satisfying $ab=ca$ as assumed in (7), and any natural number n, we can solve $a=c^ne$ for e, then solve $c=ef$ for f, and verify that $b=fe$ also holds.)

As mentioned earlier, [17] shows that the groups for which DG is a semifir are those that are locally free, i.e., satisfy (1). For $G=\mathbb{Z}\times\mathbb{Z}$, the ring DG has the form $D[x,\,x^{-1},\,y,\,y^{-1}]$, which is not a 2-fir. (For details, cf. next-to-last sentence of proof of Corollary 37 below.) □

At the opposite extreme from groups, we can find monoids M with no invertible elements $\neq 1$ which satisfy all our conditions but (1) and for which the rings DM are not 2-firs. Indeed, it is easy to see

Example 22 (cf. Cedó [14, Example 1])**.** *Let G be any subgroup of the additive group of real numbers, and $M=G^{\geq 0}$ the monoid of nonnegative elements of G. Then M satisfies* (2)-(7$''$)*; but if G contains two elements linearly independent over $\mathbb{Q}$, it will not satisfy* (1)*, and in fact DG will not be a 2-fir.* □

The example of Cedó cited above is, up to isomorphism, the case of Example 22 where G is the additive subgroup of $\mathbb{R}$ generated by $\{1,\tau\}$, where again, $\tau=(1+\sqrt{5})/2$.

Let us now give an example where, as in Example 20 above, M is a submonoid of a free group, but as in Examples 21 and 22, its universal group is not free.

Example 23. *In the free group G on x and y, let M be the submonoid generated by*

(40) $\quad y, \quad w=y\,x, \quad u=x^2, \quad v=y\,x^2y^{-1}.$

Then the universal group of M is the free product of an infinite cyclic group and a free abelian group of rank 2.

This monoid M satisfies (2) (*and hence* (3)), (5), (7′)*, and* (7″)*, but not* (1), (4), (6), (6′) *or* (7).

Proof. Let us begin by showing that M has the presentation

(41) $\quad \langle y,w,u,v \mid vy=yu,\ vw=wu\rangle_{\mathrm{md}}.$

Clearly, the two relations of (41) hold in M. With their help we can reduce every element of the monoid with presentation (41) to a product of y,w,u,v in which

(42) no v immediately precedes a y or a w.

(Indeed, the substitutions $vy\mapsto yu$ and $vw\mapsto wu$ both decrease the number of occurrences of v in a string, so starting with any string, the process of repeatedly applying these transformations terminates.) Hence to prove that the natural homomorphism from (41) to the submonoid of G generated by the elements (40) is an isomorphism, it suffices to show that if we take a string s made out of the symbols y, w, u and v satisfying (42), map it into G via the substitutions (40), and write its image as a reduced group word in x and y, then the string s can be recovered from that word.

I claim that for s satisfying (42), every maximal nonempty string of v's in s, say v^n, yields a string $y\,x^{2n}y^{-1}$ in the reduced form of the image element, in which the y on the left and the y^{-1} on the right are not cancelled by any other terms. This is because our v^n is not preceded by a v, or followed by a y, w or v. Since factors y^{-1} in our reduced expression arise only from these substrings v^n, we can locate all such substrings, and thus write the image of s as a product of strings v^n alternating with words that are products of y, $w=y\,x$, and $u=x^2$. But from any product of y, $y\,x$, and x^2 we can easily recover the factors (using the observation that a y in the product will come from a factor y if the number of x's that follows it is even, but from a factor $y\,x$ if that number is odd). Hence we can reconstruct the reduced word s.

It follows that (41) is a presentation of M, and hence that its universal group has the group-presentation $\langle y,w,u,v \mid vy=yu,\ vw=wu\rangle_{\mathrm{gp}}$. Now the first relation in that presentation can be written as $v=yuy^{-1}$, and we may use this to eliminate v from the presentation. This transforms the other relation into $yuy^{-1}w=wu$. If we simplify this relation by making a change of generators of our group, letting $z=y^{-1}w$ and eliminating w in favor of z, our presentation becomes

(43) $\quad \langle y,\ z,\ u \mid uz=zu\rangle_{\mathrm{gp}}.$

Clearly, this group is, as claimed, the coproduct of the infinite cyclic group on y and the free abelian group on u and z. Since in a free group, every abelian subgroup is cyclic, (43) is not free, hence, being finitely generated, it is not locally free, so M does not satisfy (1). (What happens when we apply the natural map of this group into G? We find that u falls together with z^2.)

On the other hand, since M is given as a submonoid of a group, it satisfies (2). The properties (5), (7′) and (7″) can be seen by applying to the equations assumed in those conditions the homomorphism $G\to\mathbb{Z}$ taking x and y to 1, and noting that each of the generators of M has positive image under that map.

Finally, either of the relations in (41) is a counterexample to conditions (4), (6) (with $d=1$), (6′) (with $c=1$), and (7). □

V. Maltcev (personal communication) has pointed out that there exist examples similar to the above in which M is wholly contained in the free monoid on the given generators. Indeed, one can get such an example by applying to Example 23 the automorphism of G that fixes y, and carries x to xy.

The fact that Example 23 satisfies (7′) and (7″) without (7) shows the need for the hypothesis (4) in Lemma 4. I don't know a similar example which satisfies (4). More precisely, I don't know the answer to

Question 24. *If M is a submonoid of a locally free* (*respectively, a free*) *group, and satisfies* (4)*, must the universal group of M be locally free* (*respectively, free*)*?*

If not, does the answer change if we assume G also satisfies (5) *and* (7)*?*

Let us now look at a couple of submonoids M of free groups G which do have G as their universal groups, and so satisfy (1), (2) and (3), but which fail to satisfy some of our other conditions. An easy one is

Example 25. *Let G be the free group on one generator x, and M the submonoid generated by x^2 and x^3. Then M satisfies* (1)*,* (2) (*hence* (3))*,* (5)*,* (7′)*, and* (7″)*, but not* (4) *or* (7).

Proof. Conditions (1) and (2) are easily verified, the universal group being G itself, and the remaining positive results are seen by looking at degree in x. On the other hand, the relation $x^2 \cdot x^3 = x^3 \cdot x^2$, where neither x^2 not x^3 right divides the other in M, gives the two negative assertions. □

To get a monoid not satisfying (5), we can work in the free group G on generators x and y, take $a = x$, $c = y$, and "solve" for the d in the equation $a = cad$. The arguments establishing the properties of this monoid are similar to those used in previous examples, so I will leave many details to the reader.

Example 26. *In the free group G on x and y, let M be the submonoid generated by*

(44) $\quad x, \quad y, \quad w = x^{-1}y^{-1}x.$

Then the universal group of M is G, with the inclusion as the universal map; and M satisfies (1)*,* (2) (*hence* (3))*,* (4)*,* (7)*,* (7′) *and* (7″)*, but not* (5) (*and hence not* (6) *or* (6′)).

Sketch of proof. Noting that $w^n = x^{-1}y^{-n}x$, one verifies that the reduced expressions in G for the elements of M are the strings in which every x^{-1} is followed immediately by a y^{-1}, and every maximal nonempty string of y^{-1}'s is followed immediately by an x. Noting that $yxw = x$, it is not hard to deduce that a normal form for M is given by all words in x, y, w containing no substring $y\,x\,w$, and from this to get the presentation

(45) $\quad M \ = \ \langle x,\, y,\, w \mid y\,x\,w = x\rangle_{\mathrm{md}}.$

Hence the universal group of M is $\langle x,\, y,\, w \mid x = yxw\rangle_{\mathrm{gp}}$. We can use the one relation to eliminate w, showing that this group is G, and establishing (1) and (2).

The proof of (4) is lengthy, but can be made somewhat shorter by first noting that the automorphism θ of G which fixes y and carries x to $y\,x$ also fixes w, and hence takes M into itself. Moreover, θ^{-1} also carries M into itself, taking x to $y^{-1}x = x\,w$; so θ yields an automorphism of M. By applying a sufficiently high power of θ^{-1} to any relation $a\,c = b\,d$ in M, we can assume that none of the normal form expressions for a, c, b and d involve the sequence $y\,x$, and that in all of them, every x is immediately followed by a w. The conclusion of (4) is immediate unless at least one of the pairs of factors in that relation, say $a\,c$, undergoes a nontrivial reduction to normal form when multiplied together, in which case we find that we can write

(46) $\quad a = a'y^n, \quad c = x\,w^n\,c',$

where $n \geq 1$, and where either a' does not end with a y, or c' does not begin with a w. Thus, $a\,c$ has the normal form $a'\,x\,c'$.

We then consider how the normal form of $a\,c$ can equal that of $b\,d$. If $b\,d$ requires no reduction to bring it to normal form, then since b left divides the element whose normal form is $a'x\,c'$, it either left divides a', in which case it left divides a, or it is a right multiple of $a'x$, in which case it is a right multiple of $a = a'y^n$, since the relation $\theta^n(x) = y^n x$ shows that x is a right multiple of y^n. If $b\,d$ does require reduction, we have $b = b'y^m$, $d = x\,w^m\,b'$, with normal form $b'\,x\,d'$. We then find that either $b'\,x$ left-divides a', or $a'\,x$ left-divides b', or $a' = b'$. In the first case, b left divides a, in the next, a left divides b, while in the last, we have one or the other, depending on whether $m \leq n$ or $m \geq n$.

To get (7)-(7″), one first deduces from the arguments showing the equivalence of these three conditions (Lemma 4) that in a relation $a\,b = c\,a$ contradicting (7), the element a would have to be *both* left divisible by arbitrarily large powers of c and right divisible by arbitrarily large powers of b. However, one can verify that the only elements u of M by which an element can be infinitely left divisible are the nonnegative powers of y, and the only ones by which an element can be infinitely right divisible are the nonnegative powers of w. (Key ideas: by symmetry of (45), it suffices to show the former statement. By considering degree in x, we see that the only possibilities for such a u are words in y and w. But one also sees that reduction of a word to its normal form does not alter the substring to the left of the leftmost x other than by changing the number of y's at its right end. This eliminates all possible u other than powers of y.) Hence in our relation $a\,b = c\,a$, c would have to be a nonnegative power of y and b a nonnegative power of $x^{-1}y^{-1}x$. But this, together with the condition in (7) that b and c not both equal 1, is seen to yield a contradiction regarding the total degree in y (i.e., the image under the homomorphism $G \to \mathbb{Z}$ taking y to 1 and x to 0) of the common value of the two sides.

Finally, to show that M does *not* satisfy (5), we of course use the relation $x = y\,x\,w$, which we built this monoid to satisfy. We only need to verify that x is not invertible, and this is clear by looking at degrees in x. (In fact, M has no invertible elements other than 1, since the elements of degree 0 in x must, as noted, lie in the monoid generated by y and w, and that monoid is free on those generators.) $\square$

One would similarly hope to get a monoid M for which (7) fails but (1)-(5) hold by looking inside the free group on x and y, letting $a = x$ and $b = y$, and solving the equation $ab = ca$ for c. Unfortunately, the close relation between (7) and (4) shows itself again: in the resulting monoid, the equation designed to invalidate (7) also invalidates (4). However, here is an example, due to Cedó, of a monoid satisfying (1)-(5) but not (7). It is constructed, like the earlier example $M_{(0)}$, as a direct limit of copies of the monoid we have called M_0, but the limit is now taken with respect to a different automorphism of G.

Example 27 ([14, Example 4])**.** *Let G be the free group on x and y, and as in §§4-7, let M_0 be the submonoid of G generated by x, y, and $z = x^{-1}y^{-1}x\,y$. Let φ be the automorphism of G which fixes x and carries y to $x\,y$, which we see carries M_0 into itself, and let*

(47) $\quad M \;=\;$ *the set of $a \in G$ such that $\varphi^n(a) \in M_0$ for some* (*hence, for all sufficiently large*) *n.*

Then M satisfies (1)-(5), *and its inclusion in G gives its universal group; but M does not satisfy* (7)-(7$''$).

Sketch of proof. As we saw in Lemma 8, M_0 has the presentation $\langle x,\, y,\, z \mid y\,x\,z = x\,y\rangle_{\mathrm{md}}$, has a normal form given by the words having no subwords $y\,x\,z$, and satisfies (1)-(3). It follows that the direct limit M of copies of M_0 likewise satisfies (1)-(3).

As in the proof of Lemma 11, to obtain (4), we will consider a relation $a\,b = c\,d$ holding in M, and by applying φ sufficiently many times, assume without loss of generality that a, b, c and d all lie in M_0, regard them as written in normal form, and consider possible reductions that occur when $a \cdot b$ and $c \cdot d$ are reduced to normal form. But because of the difference between the automorphism σ we used then and the φ in the construction of this monoid, we will not be able to use further applications of that automorphism to put additional useful restrictions on the normal forms of our factors.

We find that if, on multiplying together the normal forms for a and b, a reduction occurs, these elements must have either the forms $a = a'\,y$ and $b = (x\,z)^m\,b'$, or $a = a'\,y\,x$ and $b = z\,(x\,z)^{m-1}\,b'$, where in either case $m > 0$, $a', b' \in M_0$, and b' has normal form not beginning $x\,z$; and that the normal form of the product is in each case $a'\,x^m\,y\,b'$. The analogous observations hold for c and d. As in the proof of Lemma 11, we can assume that at least the calculation of $a \cdot b$ does involve such a reduction, and subdivide into cases according to whether the calculation of $c \cdot d$ does or does not, and how far the left factor c' (if the calculation does involve reduction) or c (if it does not) extends into the product $a'\,x^m\,y\,b'$.

When the multiplication of $c \cdot d$ does not involve reduction, c must be an initial substring of the normal form $a'\,x^m\,y\,b'$ of $a \cdot b$. Here, the case in which it is not obvious that one of a or c left-divides the other is when c has the form $a'\,x^r$ $(r \leq m)$. In that case, we right multiply $c = a'\,x^r$ by $x^{-r}y \in M$, and thus see that cM contains $a'\,y$, and hence $a'\,y\,x$, one of which is a. When the multiplication of $c \cdot d$ does involve reduction, so that the product has the form $c'\,x^n\,y\,d'$, then after possibly interchanging the roles of a and c, the nontrivial case is where the normal forms of the two equal products align in such a way that $a' = c'\,x^r$ with $0 < r \leq n$. In that case, on right-multiplying $c = c'\,y$, respectively $c = c'\,y\,x$, by $(x\,z)^r$, respectively $z\,(x\,z)^{r-1}$, and then if necessary by x, we again get $a \in cM$.

Condition (5) reduces to that condition on the monoid M_0, and this was obtained in the proof of Lemma 12.

Finally, the relation $y \cdot (xz) = x \cdot y$ witnesses the failure of (7$'$), since y is infinitely left divisible by x in M. $\square$

9. A result of Dicks and Menal, and division-closed submonoids

Dicks and Menal prove in [16] that if G is a group such that DG is an n-fir, then every n-generator subgroup of G is free. Their proof generalizes easily to give the following result, which we shall use in the next section. Recall that a ring R is called *left n-hereditary* if every n-generator left ideal is projective as a left R-module.

Theorem 28 (after Dicks and Menal [16])**.** *Let M be a monoid, and n a positive integer such that DM is an n-fir, or more generally, is a left n-hereditary* 1*-fir. Then*

(i) *For every subgroup H of the group of invertible elements of M, DH is again a left n-hereditary* 1*-fir.*

(ii) *Every n-generator subgroup H of the group of invertible elements of M is free.*

Proof. The fact that DM is a 1-fir, i.e., a ring without zero divisors, implies, on the one hand, that M satisfies the cancellation condition (3), and on the other hand, that its group of invertible elements has no elements of finite order. Now given H as in (i), since M is cancellative, the right cosets of H in M, i.e., the orbits in M of the left action of H, are each isomorphic as H-sets to H. Of course, the same is true of the left cosets as right orbits; but this will not be immediately useful to us, except for the consequence that the orbit H itself, and its complement within M, are closed under both actions. Combining the preceding observation with this fact, we get

(48) DM is free as a left DH-module, and DH is a direct summand therein as DH-bimodules.

Now let $A \subseteq DH$ be any n-element subset. Since DM is n-hereditary, $(DM)A$ is projective as a left DM-module, so as DM is left free over DH, $(DM)A$ is also projective as a left DH-module. Since, by (48), the projection of DM onto DH respects right multiplication by the elements of A, it carries $(DM)A$ onto $(DH)A$, so $(DH)A$ is a DH-direct summand in the projective left DH-module $(DM)A$. Hence it is projective as a left DH-module, giving the conclusion of (i).

To get (ii), note that if $h_1, \dots, h_n$ generate H, then $h_1 - 1, \dots, h_n - 1$ generate the augmentation ideal I of DH as a left ideal, so by (i), that ideal is projective as a left DH-module. Thus H is a finitely generated group without elements of finite order, whose augmentation ideal is projective as a left module. The Swan-Stallings Theorem [9, Theorem A] says that any such group H is free. □

Can we get a similar result with H replaced by a more general submonoid N of M? In general, for a submonoid N of a monoid M, there is no nice analog of the coset decomposition used above. But for certain submonoids of a monoid M that satisfies (3) and (4), we can get such a decomposition.

Definition 29. *A submonoid N of a monoid M will be called* left division-closed *if for elements $a, b \in M$ with $a\,b \in N$, we have $a \in N \implies b \in N$. It will be called* right division-closed *if the reverse implication holds.*

If M is a monoid satisfying (3) *and* (4)*, and N a left division-closed submonoid of M, we shall call elements $x, y \in M$* left N-equivalent *if they lie in a common principal right coset Nz $(z \in M)$.* (*This terminology will be justified by point* (ii) *of the next result.*)

Lemma 30. *Let M be a monoid satisfying* (3) *and* (4)*, and N a left division-closed submonoid of M. Then*

(i) *N also satisfies* (3) *and* (4)*.*

(ii) *Left N-equivalence is an equivalence relation on M.*

(iii) *Each left N-equivalence class is a directed union of principal right cosets Nx;*

(iv) *N is itself a left N-equivalence class in M.*

(v) *If two right cosets Nx and Ny have nonempty intersection, then one of them contains the other; so in particular, x and y are left N-equivalent.*

Proof. To see (i), observe that N inherits (3) from M, and also (4), given that it is left division-closed.

Condition (4) and the definition of left division-closed are also easily seen to imply (v).

In (ii), reflexivity and symmetry are clear. To see transitivity, suppose x left N-equivalent to y, and y to z. Thus, we have $u, v \in M$ such that Nu contains the first two of these elements, and Nv the last two. Since the intersection of Nu and Nv contains y, (v) says that one of those two cosets contains the other, and hence contains both x and z, proving these left N-equivalent.

The definition of left N-equivalence, and (v), together show that any left N-equivalence class is a directed union of principal right cosets, i.e., (iii). We shall prove (iv) by showing that any $x \in M$ equivalent to 1 is contained in N. Say 1 and x are both contained in a right coset Ny. Since $1 \in Ny$, we can write $1 = ay$ with $a \in N$. Hence left division-closure of N implies $y \in N$, so $N \supseteq Ny \ni x$, as desired. $\square$

Let us now look at some consequences for monoid rings. These considerations will not require the base ring to be a division ring D, so we give the next result without that assumption.

Lemma 31. *Let M be a monoid satisfying* (3) *and* (4), *N a left division-closed submonoid of M, and R any ring. Then*

(i) *The monoid ring RM is flat as a left RN-module.*

(ii) *The projection map $RM \to RN$ taking $\sum_{a\in M} c_a\, a$ to $\sum_{a\in N} c_a\, a$ is a left RN-module homomorphism.*

If, in addition to being left division-closed, N is right division-closed, and A is a subset of RN, then

(iii) *If $(RM)A$ is flat as a left RM-module, then $(RN)A$ is flat as a left RN-module.*

(iv) *If A is finite and $(RM)A$ is projective as a left RM-module, then $(RN)A$ is projective as a left RN-module.*

Proof. By Lemma 30 (ii)-(iii), RM is the direct sum, over the left N-equivalence classes $E \subseteq M$, of the R-submodules RE, and each of those is a left RN-submodule, isomorphic to a direct limit of (rank-1) free RN-modules, hence flat over RN. This gives (i). By Lemma 30 (iv), RN is one of the direct summands, and the module of elements with support in the complement of N is the sum of the others, so we have (ii).

The additional hypothesis of (iii) and (iv), that N is *right* division-closed, says that the complement of N in M is closed under right multiplication by elements of N. Thus RN (because it is a ring), and its complementary summand (by the above condition) are each closed under right multiplication by elements of RN, hence in particular, by elements of A; so the left RM-module $(RM)A$ is the direct sum of an RN-submodule lying in RN and one lying in that complementary left RN-submodule.

If $(RM)A$ is flat as a left RM-module, then it is a direct limit of free RM-modules, and by (i) each of these is flat as an RN-module, so $(RM)A$ is flat as an RN-module. Hence so is its direct summand $(RN)A$, proving (iii).

To prove (iv) we shall use Jøndrup's Lemma [17, Lemma 1.9], which says that any finitely generated flat module over a ring, which on extension of scalars to some overring gives a projective module over that ring, must already be projective over the given ring. Now if $(RM)A$ is a projective RM-module, it is in particular flat, hence by (iii), $(RN)A$ is flat. To see what happens when we extend scalars to RM, note that since N is assumed right as well as left division-closed, the left-right dual of (i) says that RM is flat as a right RN-module. Hence when we left-tensor the inclusion of RN-modules $(RN)A \hookrightarrow RN$ with RM, we get an embedding $RM \otimes_{RN} (RN)A \hookrightarrow RM$. This says that $RM \otimes_{RN} (RN)A$ is naturally isomorphic to $(RM)A$, so assuming the latter projective as a left RM-module, the hypotheses of Jøndrup's Lemma are satisfied, and we get the desired projectivity of $(RN)A$.

(Warren Dicks has pointed out to me that one can get (iv) directly from (i) and (ii), using the fact that as an RN-module, RM is not merely flat but is a directed union of free modules. One deduces that $(RM)A$, since it is projective, and hence $(RN)A$, as a direct summand therein, are direct summands in such directed unions; hence the latter, being a finitely generated RN-module, must be a direct summand in some free submodule in the limit construction, hence projective. Note that this argument still needs the right division-closed condition to show that $(RN)A$ is the image of $(RM)A$ under the projection $RM \to RN$.) $\square$

We can now partially generalize of the proof of Theorem 28, to get the following result, which will also be used in the next section.

Theorem 32. *Let M be a monoid and n a positive integer such that DM is an n-fir; or if $n > 2$, more generally, such that DM is a 2-fir which is left n-hereditary. Then for every left and right division-closed submonoid $N \subseteq M$, the ring DN is a 1-fir and is left n-hereditary.*

Proof. Whatever the value of n, our hypotheses insure that DM is a 1-fir, i.e., an integral domain; hence so is any subring thereof. In particular, since a 1-fir is 1-hereditary, this gives the required conclusions in the case $n = 1$.

If $n \geq 2$, then since DM is a 2-fir, M satisfies (3) and (4). Hence we can apply Lemma 31(iv) to n-generator left ideals $(DN)A \subseteq DN$, and conclude that they are projective, making DN n-hereditary, as claimed. $\square$

We would, of course, like to know more.

Question 33. *If M is a monoid such that DM is an n-fir, must DN be an n-fir for every right and left division-closed submonoid N of M ? Must this at least be true if N is a subgroup of the group of invertible elements of M ? If it is precisely the group of such invertible elements?*

(The referee has provided an answer to the second sentence of the above question, which will be noted in the published version.)

Another question suggested by the methods we have used is the following.

Question 34. *What can one say about a monoid M such that the augmentation ideal of DM, i.e., the ideal I generated by all elements $a-1$ $(a \in M)$, is flat as a left module, under various hypothesis on M ?*

Warren Dicks (personal communication) notes that results on this question would be of interest even for M a 2-generator group.

One situation where we can get such results is if M is a finitely generated monoid such that DM is embeddable in a division ring D'. For in this situation, $D' \otimes_{DM} I$ is clearly free as a D'-vector-space, hence by Jøndrup's Lemma, if I is flat, it is projective as a left DM-module. In particular, if M is a *group* admitting a two-sided invariant ordering, it is known that DM is embeddable in a division ring, and M has no elements of finite order; so if such an M is finitely generated and its augmentation ideal is flat, we can again apply the Swan-Stallings Theorem and conclude that M is a free group.

(It is an open question whether for every group M admitting a *one-sided* invariant ordering, DM is embeddable in a division ring. A positive answer to that question would clearly allow us to strengthen the above observation.)

Returning to the properties of a general left division-closed submonoid N of a monoid M satisfying (3) and (4), note that in contrast to the case of right cosets of a subgroup, the left N-equivalences classes in M are not in general mutually isomorphic as N-sets. (For example, if G is the free group on x and y, and M the submonoid generated by x and all elements $x^{-n}y$ $(n \geq 0)$, let $N = \langle x \rangle_{\mathrm{md}}$, and compare the N-equivalence classes of 1 and y.)

The next result shows a bit of structure that one can extract from this non-isomorphism. (We will not use it below.)

Lemma 35. *Let M be a monoid satisfying* (3) *and* (4), *and N a left division-closed submonoid of M. Then*

(i) *The set of $a \in M$ such that the left N-equivalence class of a is not embeddable as an N-set in N forms a right ideal of M.*

(ii) *If N has the stronger property that $ab \in N \implies a, b \in N$, then the complementary set, of those $a \in M$ such that the left N-equivalence class of a* is *embeddable as an N-set in N, forms a submonoid of M.*

Proof. To get (i), note that the definition of left N-equivalence shows that if a and b are left N-equivalent, then for any $c \in M$, the elements $a\,c$ and $b\,c$ will also be. Hence the left N-equivalence class of any element a is mapped by right multiplication by c into the left N-equivalence class of $a\,c$. Since M is assumed cancellative, this map is an embedding; so if the equivalence class of a is not embeddable in N, neither is that of $a\,c$.

Before proving (ii), let us note that from (3) and (4), it is not hard to deduce that the left N-set structure of the left N-equivalence class of an element of M is determined by the set of elements of N that left divide that element.

Now assume the hypothesis of (ii), and let N' be the set of a such that the left N-equivalence class of a is embeddable in N. Clearly, $1 \in N'$. Suppose $x, y \in N'$. The set of elements of N that left divide $x\,y$ contains those that left divide x. If it contains nothing more, then by the preceding observation, the equivalence classes of $x\,y$ and x are isomorphic, showing that $x\,y \in N'$. If, on the other hand, $x\,y$, but not x, is left divisible by some $a \in N$, let $x\,y = a\,z$. Then (4) and the fact that x is not left divisible by a show that $a = x\,b$ for some $b \in M$. This relation and our hypothesis on N tell us that $x \in N$, so $x\,y \in Ny$, so the left N-equivalence class of $x\,y$ is that of y, which by assumption is embeddable as a left N-set in N. □

I wonder whether some much weaker hypothesis than $ab \in N \implies a, b \in N$ could be used in (ii) above. This is suggested by the fact that when N is a proper subgroup of the group of invertible elements of M, the conclusion of (ii) holds (since then N' is all of M); but that hypothesis does not.

10. Commuting elements

It is well-known that every subgroup of a free group is free.

Hence if two elements of a free group commute, the subgroup they generate must be cyclic. We can say more: the centralizer of every nonidentity element a of a free group is cyclic. For the center of that centralizer contains a, and the only free group with nontrivial center is the free group on one generator. From these facts, one can deduce similar facts about commuting elements and centralizers in direct limits of free groups, i.e., groups G such that DG is a semifir; and using slightly more elaborate arguments, in groups G such that DG is a 2-fir.

In this section, we shall obtain such results for a general monoid M such that DM is a 2-fir. The arguments are unexpectedly complicated, but many of the intermediate results may be of interest. We will use left division-closed submonoids in place of subgroups, so we begin with some observations on left division closures of commutative submonoids.

Lemma 36. *Let M be a monoid satisfying* (3) *and* (4)*, and having no nonidentity elements of finite order. Suppose $a, b \in M$ commute, and are not both* 1*, and let N be the least left division-closed submonoid of M containing a and b. Then*

(i) $N = \{c \in M \mid a^i\, b^j\, c = a^{i'}\, b^{j'}$ *for some natural numbers* $i,\ j,\ i',\ j'\}$.

(ii) *N is isomorphic either to the additive monoid $\mathbb{N}$ of natural numbers, or to the additive group $\mathbb{Z}$ of integers, or to a submonoid of $\mathbb{Z}\times\mathbb{Z}$ in which the image of a is* $(1,0)$ *and the image of b is* $(0,1)$.

Proof. The set given by the right-hand side of (i) is easily seen to be a submonoid of M, and to contain a and b; and it is certainly contained in N, since every c as in the description of that set can be obtained by left-dividing $a^{i'} b^{j'} \in N$ by $a^i b^j \in N$. Hence to prove (i), it will suffice to show that that set is left division-closed. So suppose $c = d\,e$, where c and d belong to that set, and $e \in M$. Writing

$$a^i\, b^j\, c \;=\; a^{i'}\, b^{j'}, \qquad a^k\, b^l\, d \;=\; a^{k'}\, b^{l'},$$

we verify that $e \in N$ by the computation

$$\begin{aligned} a^{i+k'}\, b^{j+l'} e \;&=\; (a^i b^j)(a^{k'} b^{l'})\, e \;=\; (a^i b^j)(a^k b^l)\, d\, e \;=\; (a^i b^j)(a^k b^l)\, c \\ &=\; (a^k b^l)(a^i b^j)\, c \;=\; (a^k b^l)(a^{i'} b^{j'}) \;=\; a^{i'+k}\, b^{j'+k}. \end{aligned}$$

To get (ii), let us first note that if an element $c \in N$ satisfies $a^i b^j c = a^{i'} b^{j'}$, then it also satisfies $a^{i+m}\, b^{j+n} c = a^{i'+m}\, b^{j'+n}$ for all natural numbers m and n. This suggests that we associate to each $c \in N$ the set S_c of pairs $(i'-i, j'-j) \in \mathbb{Z}\times\mathbb{Z}$ such that $a^i b^j c = a^{i'} b^{j'}$. If two such sets are equal, $S_c = S_d$, it is not hard to see that we can find $i,\ j,\ i',\ j'$ which simultaneously satisfy $a^i\, b^j\, c = a^{i'}\, b^{j'}$ and $a^i\, b^j\, d = a^{i'}\, b^{j'}$; so by (3), $c = d$. Hence the map $c \mapsto S_c$ is one-to-one. It is also not hard to verify that for each $c \in N$, the set of differences between members of S_c will form a subgroup of $\mathbb{Z}\times\mathbb{Z}$, and, using (3), that this subgroup will be the same for all $c \in N$. Calling this common subgroup K, we find that we have an embedding $h : N \to (\mathbb{Z}\times\mathbb{Z})/K$.

I claim that K must be a pure subgroup of $\mathbb{Z}\times\mathbb{Z}$, i.e., that for any $x \in \mathbb{Z}\times\mathbb{Z}$ and any positive integer n, if $n\,x \in K$ then $x \in K$. For given such x and n, we can take elements $c = a^i\, b^j$, $d = a^{i'}\, b^{j'}$ such that $(i-i', j-j') = x$. Now the existence of an embedding in $(\mathbb{Z}\times\mathbb{Z})/K$ shows that N is commutative, so any two elements have a common right multiple; so since, by Lemma 30 (i), N satisfies (4), there exists $e \in N$ such that either $c\,e = d$ or $d\,e = c$. Thus, $\pm x \in S_e$, so the image of e under the embedding of N in $(\mathbb{Z}\times\mathbb{Z})/K$ will have exponent n. Since by hypothesis M has no nonidentity elements of finite order, $e = 1$, so $x \in K$, showing that $K \subseteq \mathbb{Z}\times\mathbb{Z}$ is indeed pure.

We now consider the rank of $K \subseteq \mathbb{Z}\times\mathbb{Z}$ as an abelian group.

If K has rank zero, i.e., if $K = \{0\}$, then we have an embedding of N in $\mathbb{Z}\times\mathbb{Z}$, which by construction takes a and b to $(1,0)$ and $(0,1)$ respectively, giving one of the alternative conclusions of (ii).

If K had rank 2, then being pure, it would be all of $\mathbb{Z}\times\mathbb{Z}$, so N would be trivial, contradicting our assumption that a and b are not both 1.

This leaves the case where K has rank 1. Since it is pure, we have $(\mathbb{Z}\times\mathbb{Z})/K \cong \mathbb{Z}$, so $h: N \to (\mathbb{Z}\times\mathbb{Z})/K$ yields an embedding $f: N \to \mathbb{Z}$.

If $f(N) \subseteq \mathbb{Z}$ contains both positive and negative elements, it is clearly a subgroup, hence cyclic, another of our alternative conclusions. In the contrary case, we may assume without loss of generality that it consists of nonnegative integers. Now condition (4), applied to $f(N) \cong N$, says that if $f(N)$ contains both m and n, then their difference, in one order or the other, also lies in $f(N)$. This allows us to apply the Euclidean algorithm to $f(N)$, and conclude that it is a cyclic monoid, the remaining alternative. □

In the situation where DM is a 2-fir, we can exclude one of the above cases.

Corollary 37. *If M is a monoid such that DM is a 2-fir, then every pair of commuting elements of M lies either in a cyclic submonoid of M, or in a cyclic subgroup of the group of invertible elements of M.*

Proof. As noted in §2, M satisfies (3) and (4). Also, M can have no elements of finite order, since these would lead to zero-divisors in DM, so that it would not even be a 1-fir. Hence by the preceding lemma, given commuting $a, b \in M$, they will both be contained in a left division-closed submonoid N of one of the forms described there. As in the proof of that lemma, we see that given any two elements of N, one will divide the other. This implies that N is not only left, but also right division-closed in M. Since DM is a 2-fir, Lemma 31(iii) allows us to conclude that every 2-generator left ideal of DN is projective.

I now claim that if N had the last of the three forms described in Lemma 36(ii), i.e., if it embedded in $\mathbb{Z}\times\mathbb{Z}$ with a and b mapping to linearly independent elements, then the left ideal of DN generated by $a-1$ and $b-1$ would be non-projective. For if it were projective, then passing to the central localization $D[a,a^{-1},b,b^{-1}]$, the same would be true of the left ideal of that ring generated by $a-1$ and $b-1$; but this ideal is not even flat, as shown by the method of [21, Example 4.19]. (The argument is stated there for commutative base ring, but works equally well in the noncommutative case.) Alternatively, one can regard $D[a,a^{-1},b,b^{-1}]$ as a group ring of a finitely generated group, and conclude from the proof of [16, p. 288] that if its augmentation ideal were projective, that group would be free, which it is not. So we must be in one of the other two cases of Lemma 36(ii). □

The next result examines what one can say about families of more than two commuting elements. Since the argument used is applicable to elements of finite as well as infinite order, we do not exclude these until the final sentence.

Proposition 38. *Let M be a monoid satisfying* (3)*, such that every pair of commuting elements of M lies either in a cyclic subgroup of the group of invertible elements of M, or in a cyclic submonoid of M.*

Then every finite set A of commuting elements of M likewise lies either in a cyclic subgroup or a cyclic submonoid.

Hence, every *set A of commuting elements of M lies in a submonoid N which is a directed union either of cyclic subgroups, or of cyclic submonoids. Such an N is isomorphic either to a subgroup of the additive group $\mathbb{Q}$ of rational numbers, or to a subgroup of the factor-group $\mathbb{Q}/\mathbb{Z}$ thereof, or to the monoid of nonnegative elements in a subgroup of $\mathbb{Q}$.*

In particular, this is true in every monoid M such that DM is a 2-fir. In that case, since elements of finite order do not occur, every such $A \not\subseteq \{1\}$ is contained in a directed union either of infinite cyclic groups, or of infinite cyclic monoids.

Proof. Let A be a set of commuting elements of M.

We first note that if some $a \neq 1$ in A is invertible in M, then every $b \in A$ is invertible. This is immediate if b lies in a cyclic subgroup with a, or if $b = 1$. If $b \neq 1$ and b is instead assumed to lie in a cyclic submonoid containing the invertible element a, then some positive power of b must equal a power of a, hence must be invertible, so b is invertible.

So either all members of A or no nonidentity members of A are invertible. Let us start with the first case. The hypothesis on pairs of commuting elements of M shows that the subgroup of M generated by any two commuting invertible elements is cyclic. Given finitely many commuting invertible elements $a_1, \dots, a_n$ with $n \geq 2$, assume inductively that the subgroup generated by $a_1, \dots, a_{n-1}$ is cyclic, say with generator a. Then a_n will commute with a, hence will generate with it a cyclic subgroup, which will be the subgroup generated by $a_1, \dots, a_n$, as desired. For possibly infinite A, the above result shows that each finite subset of A generates a cyclic subgroup, so the subgroup generated by A is the directed union of those groups. Such a directed union is easily shown to be isomorphic either to a subgroup of $\mathbb{Q}$ or to a subgroup of $\mathbb{Q}/\mathbb{Z}$.

In the noninvertible case, we again start with a finite $A=\{a_1,\dots,a_n\}$, whose members we may assume without loss of generality are all $\neq 1$, and we again assume inductively that $a_1,\dots,a_{n-1}$ are powers (natural-number powers in this case) of a common element c. Replacing c by a power of itself if necessary, we can assume that the exponents to which c is raised to get $a_1,\dots,a_{n-1}$ have no common divisor. It is easily deduced that the monoid generated by $a_1,\dots,a_{n-1}$ contains both c^r and c^{r+1} for some $r\geq 0$. Since a_n commutes with each of $a_1,\dots,a_{n-1}$, it commutes with c^r and with c^{r+1}; so $a_n\,c^{r+1}=c^{r+1}\,a_n=c\,a_n\,c^r$. Cancelling c^r, on the right, we see that a_n commutes with c, hence lies in a cyclic submonoid containing c, and this will thus contain $a_1,\dots,a_n$, as required.

We see in fact that the above construction yields the least left division-closed submonoid of M containing A, which is thus also the least cyclic submonoid of M containing A.

Looking again at possibly infinite A, we see that A will be contained in the directed union of the least cyclic submonoids containing its finite subsets. Since the elements of A are not invertible, and M has cancellation, those cyclic submonoids are infinite; and it is not hard to show that a directed union of infinite cyclic monoids has the form claimed.

Corollary 37 shows that the above results apply to monoids M such that DM is a 2-fir. As we have noted, even for DM a 1-fir, M cannot have elements of finite order, giving the final assertion. ☐

We now wish to prove a corresponding result for the *centralizer* of any nonidentity element $c\in M$. From Corollary 37, we see that every nonidentity element of the centralizer of c has a positive or negative power which is a positive power of c; hence, any two elements a and b of that centralizer have nonzero powers which are equal. For elements of a general group or monoid, this does not guarantee that a and b commute. For instance, in the group

$$(51)\qquad \langle x,\,y\mid x^2=y^2\rangle_{\mathrm{gp}},$$

x and y each lie in the centralizer of $x^2=y^2\neq 1$, and each generates, with that element, a cyclic subgroup, but x and y do not commute with each other. But we shall see that this does not happen in the monoids we are interested in. The proofs of this fact will be quite different depending on whether a and b are invertible. We start with the noninvertible case.

(Ferran Cedó informs me that the next two results are related to the cited results from the unpublished thesis [25].)

Lemma 39 (cf. [25, Corol·lari 3.2.2])**.** *Let M be a monoid satisfying* (3)-(5)*, and suppose a, b are noninvertible elements of M such that $a^m=b^n$ for some positive integers m and n. Then a and b commute.*

Proof. The products ab and ba have a common right multiple: $a\,b^{2n}=a^{2m+1}=b^{2n}a=b\,b^n\,b^{n-1}\,a=b\,a^m\,b^{n-1}\,a$. Hence by (4), one of them is a right multiple of the other. Without loss of generality say

$$(52)\qquad a\,b\;=\;b\,a\,c.$$

It follows that $a^m b=b(ac)^m$. Since $a^m=b^n$ commutes with b, we can cancel the b in that equation on the left, and get $a^m=(ac)^m$. Since a is noninvertible, Lemma 5 tells us that $c=1$, so (52) says that a and b commute. ☐

The analog of the above lemma fails if a and b are invertible, for every group satisfies (3)-(7), but groups such as (51) have noncommuting elements with commuting powers. But if DM is a 2-fir, that behavior can be excluded using Theorem 28(ii). Let us combine that argument with the preceding results of this section, and prove

Theorem 40 (cf. [25, Teoremas 3.2.2, 3.2.9])**.** *If M is a monoid such that DM is a 2-fir, then the binary relation of mutual commutativity is an equivalence relation on the nonidentity elements of M, and each equivalence class either* (i) *consists entirely of invertible elements, or* (ii) *consists entirely of noninvertible elements.*

Every equivalence class falling under (i) *forms, together with the identity element, a locally infinite-cyclic subgroup of the invertible elements of M; equivalently, a group embeddable in the additive group of $\mathbb{Q}$. Every equivalence class falling under* (ii) *forms, together with the identity element, a locally infinite-cyclic submonoid of M; equivalently, a submonoid isomorphic to the monoid of nonnegative elements in a subgroup of $\mathbb{Q}$.*

Proof. We know from the proof of Proposition 38 that of two commuting nonidentity elements of M, either both or neither are invertible. To show that commutativity is an equivalence relation on such elements, suppose a commutes with b, and b with c. By Proposition 38, a and b will lie in a common cyclic subgroup (if b is invertible) or submonoid (otherwise), and likewise for b and c. Hence each of a and c has a nonzero power (positive in the case where b is noninvertible) which equals a power of b. Hence raising these powers to further powers, we have $a^m = c^n$ for some nonzero m and n. If b is noninvertible, Lemma 39 now tells us that a and c commute. In the invertible case, Theorem 28(ii) shows that the subgroup of M generated by a and c is free, so, as it has a nontrivial central element $a^m = c^n$, it must be cyclic, so a and c again commute.

Applying Proposition 38 to any equivalence class A of noninvertible elements under the commutativity relation, we see that it will be contained in a commutative subgroup or submonoid N of the asserted form; and since N is itself commutative, we must have $N = A \cup \{1\}$. $\square$

We remark that for M a monoid such that DM is a *semifir*, the fact that commutativity is an equivalence relation on M follows from conditions (1) and (2), while for M a *group* such that DM a 2-fir, the same fact follows from Dicks and Menal's result that every 2-generator subgroup of M is free. But when M is a monoid, the above result gives a new necessary condition for DM to be a 2-fir.

11. Miscellaneous observations

11.1. **Two more questions.** We have mentioned that necessary and sufficient conditions are known for a monoid ring DM to be a right or left fir, and, when M is a group, for DM to be a semifir. In each of these conditions, if M is nontrivial then D must be a division ring, but aside from that, the conditions are restrictions on M alone. It seems likely that for a general monoid M the same will be true of the condition for DM to be a semifir, or more generally, an n-fir. But we don't know this, so we pose it as

Question 41. *Let n be a positive integer. If M is a monoid such that DM is an n-fir for some division ring D, will $D'M$ be an n-fir for every division ring D' ?*

Monoid rings can be generalized to "skew" monoid rings; so we ask

Question 42. *Let M be a monoid, D a division ring, and $\alpha : M \to \mathrm{Aut}(D)$ a homomorphism, with the image under α of $x \in M$ denoted α_x. Let $D[M;\alpha]$ denote the "skew" monoid ring consisting of all left-linear expressions $\sum_{x\in M} c_x x$ $(c_x \in D,$ zero for almost all $x)$, with multiplication and addition defined among elements of D as in D, and multiplication defined among elements of M as in M, while in place of the mutual commutativity of D and M assumed in ordinary monoid rings DM, one uses the relations*

(53) $$x\,c \;=\; \alpha_x(c)\,x \quad (x \in M,\, c \in D).$$

Is it true that $D[M;\alpha]$ will be a left fir, a right fir, a semifir, etc., if and only if the ordinary monoid ring DM has the same property?

There are still other variants of the monoid ring construction. For instance, we can get new multiplications $*$ on DM by defining, for each $x, y \in M$, $x * y = c_{x,y}\,xy$, where $(c_{x,y})_{x,y\in M}$ are nonzero elements of D satisfying the relations required to make this multiplication associative. Rings constructed using such $c_{x,y}$, and possibly also a map $\alpha : M \to \mathrm{Aut}(D)$ as above, are called "crossed products" of D and M; and one can ask when these are n-firs.

11.2. **Some properties of our properties.** If R is a ring and n a positive integer, then the condition that R be an n-fir is what logicians call a *first order* property. This means, in our context, that it is equivalent to the conjunction of a (possibly infinite) set of conditions, each of which is expressible using ring-theoretic equations, logical connectives, and quantifications over elements (but not quantifications over subsets). It is not hard to see that the condition that every m-term linear relation be trivializable may be written in this form. (The formulation of the n-fir condition that we began with, saying that every left ideal of R generated by $\leq n$ elements is free, and free modules of ranks $\leq n$ have unique ranks, takes a little more thought, but is also not hard to express in such a form.) Gathering together, for all n, these first-order statements defining n-firs, we see that the property of being a semifir is also first-order.

Hence it is curious that the condition on a monoid M, that all monoid rings DM be n-firs or semifirs (or, to avoid worrying about whether this condition depends on D, that the monoid ring DM be an n-fir or semifir for a fixed D, e.g., the field of rational numbers), is *not* first-order. For instance, the additive group of $\mathbb{Z}$, and a nonprincipal ultraproduct of countably many copies of that group, are *elementarily equivalent*, that is, they satisfy the same first-order conditions; but such an ultraproduct group will contain two linearly independent elements, and hence fail to satisfy (1).

Here is another observation of the same sort, though not as close to the main topic of this note: Though the class of semifirs is first-order, we shall see that the class of rings in which all finitely generated left ideals are free, but not necessarily of unique rank, is not.

This looks like a justification for considering the class of semifirs "nicer" than the wider class. While I personally prefer the study of semifirs, the above observation is not a justification for that preference; it simply reflects the fact that the property that all finitely generated left ideals be free comprises too complicated a family of cases to be first order. If one specifies for *which* pairs of positive integers m and n one wants to have $R^m \cong R^n$ (which corresponds to specifying a congruence on the additive semigroup of positive integers), and looks at the class of rings R over which those and only those isomorphisms hold, and all finitely generated left ideals are free, the resulting class is first order. The non-first-order class referred to above is the union of this countably infinite family of first-order classes. Each of these classes is, incidentally, nonempty [11], [7, Theorem 6.1].

To see that, as claimed, this union is not first order, suppose we take, for each $d > 1$, a ring R_d whose finitely generated left ideals are free, and such that $R_d^m \cong R_d^n$ for positive integers m and n if and only if $d \,|\, m-n$. Then for each d, the free R_d-module of rank 1 has a decomposition $R_d \cong R_d \oplus R_d^d$. It is easy to deduce that over a nonprincipal ultraproduct R of the R_d, we get an R-module decomposition $R \cong R \oplus P$, where P is a direct summand in R which has free modules of all positive ranks as direct summands. But P cannot itself be free, since if it were, then being finitely generated, it would have to be free of some finite rank n, implying an isomorphism $R \cong R^{n+1}$; but among the R_d, this isomorphism only holds for finitely many of those rings, namely those for which $d \,|\, n$, hence it does not hold in the ultraproduct R.

Let us also note that though the class of semifirs is closed under coproducts with amalgamation of a common division ring [10], [3, p. 201], [6, Corollary 2.13], the class of groups G such that DG is a semifir is not closed under coproducts with amalgamation of a common subgroup. A counterexample is the group (51) above, a coproduct of two infinite cyclic groups with amalgamation of a subgroup of index two in each.

11.3. **On condition** (7)**, and possible variants.** Condition (7) on a monoid M says that relations $a\,b = c\,a$ (with b and c not both 1) in M can only arise in a "generic" way, namely, by taking elements $e,\, f \in M$, and a natural number n, and letting $a = (ef)^n e$, $b = fe$, and $c = ef$.

Yet one can clearly also get such relations, more generally, by taking elements e and f and natural numbers m and n with $m > 0$, and letting

(54) $$a = (ef)^n e, \quad b = (fe)^m, \quad \text{and} \quad c = (ef)^m.$$

The solution to this paradox is that (54) can be rewritten in the form described in (7). Namely, writing $n = qm + r$ with $0 \le r < m$, and setting $\bar{e} = (ef)^r e$ and $\bar{f} = f(ef)^{m-r-1}$, we find that $a = (\bar{e}\bar{f})^q\,\bar{e}$, $b = \bar{f}\bar{e}$, and $c = \bar{e}\bar{f}$, as desired.

Now suppose that for some $a \in M$ we have a *pair* of relations,

(55) $$a\,b \;=\; c\,a, \qquad a\,b' \;=\; c'\,a,$$

with $b,\ c,\ b',\ c' \in M$. I see several "generic" patterns for how families of elements satisfying these equations can arise. In writing the first of these, I will, for visual simplicity, use $[s]^{-1}$ to denote *deletion* of a string s from the beginning or end of an expression. The first pattern is then

(56) $$a = ((ef)^m eg)^n (ef)^m e, \quad b = g(ef)^m e, \quad c = (ef)^m eg, \quad b' = [efe]^{-1}a, \quad c' = a[efe]^{-1},$$
where $m \ge 1$, $n \ge 0$.

This pattern has a dual, in which the formulas given above for b and c are instead used for b' and c', and vice versa. The reader can easily write this down.

The next pattern (which has some intersection with what we get on taking $g = f$ in (56)), is

(57) $$a = (ef)^n\, e, \quad b = (fe)^p, \quad c = (ef)^p, \quad b' = (fe)^q, \quad c' = (ef)^q,$$
where p and q are relatively prime, and $0 \le p \le n$ and $0 \le q \le n$.

(We assume p and q relatively prime because the case where they have a common divisor can be reduced to (57) by a substitution like the one we described following (54) above. We have also excluded the case where $p > n$, because in such cases, a right-divides b, and these fall under (58) below; and the case where $q > n$ for the corresponding reason.)

Finally, we have the following pattern, and the variant we get by interchanging the roles of b, c with those of b', c'.

$$(58)\qquad a = (ef)^n\, e, \quad b = g\,a, \quad c = a\,g, \quad b' = fe, \quad c' = ef.$$

(One can also write down a pattern in which both b and b' are left multiples of a. But this reduces to the $n = 0$ case of (58).)

Question 43. (i) *If M is a monoid such that DM is a 2-fir, is it true that for all a, b, b', c, $c' \in M$ satisfying* (55) *with b, b', c, $c' \neq 1$, either* (56)*, or* (57)*, or* (58)*, or the dual of* (56) *or* (58)*, must hold for some e, f, etc. in M and natural numbers m, n, etc.?*

(ii) *If the answer to* (i) *is no, does there exist a finite list of decompositions which* will *always yield the given relations?*

(iii) *If* (i) *or* (ii) *has a positive answer, i.e., if the existence of decompositions of the relevant sort whenever* (55) *holds is a necessary condition for DM to be a 2-fir, is this condition implied by* (3)-(7) (*as, for instance,* (12) *is*)*, or does adding it to that list give a stronger set of conditions on a monoid M?*

(iv) *Does the answer to any of the above questions change if one strengthens the condition that DM be a 2-fir to say that it is an n-fir, for some $n > 2$?*

To study the consequences of (55) in a 2-fir, one might use the 2-term linear relation analogous to (39),

$$(59)\qquad a\,(b + b' - 1) - (c + c' - 1)\,a \;=\; 0.$$

Instead of fixing an element a, and using more than one pair b, c, as in (55), one might do the opposite, and study pairs of relations,

$$(60)\qquad a\,b = c\,a, \qquad a'\,b = c\,a',$$

in the same spirit.

Another family of "generic" relations, this time ring-theoretic rather than monoid-theoretic, is the list of 2-term linear relations holding in any ring containing ring-elements x, y, z, $\ldots$:

$$(61)\qquad \begin{gathered} 1 \cdot x = x \cdot 1, \\ x \cdot (yx + 1) = (xy + 1) \cdot x, \\ (xy + 1) \cdot (zyx + z + x) = (xyz + x + z) \cdot (yx + 1), \\ (xyz + x + z) \cdot (wzyx + wz + wx + yx + 1) = (xyzw + xy + xw + zw + 1) \cdot (zyx + z + x), \\ \cdots \end{gathered}$$

These are defined and studied in [13, §2.7], and it is shown in [13, Proposition 2.7.3] that in an important class of 2-firs, they give all 2-term linear relations whose pair of left factors has no common left divisor and whose pair of right factors has no common right divisor, equivalently, which can be trivialized to $1 \cdot 0 = 0 \cdot 1$.

By examining the monoid relations holding among the summands in these identities, we can write down families of relations in a monoid, which might (who knows?) be particularly useful in finding conditions for monoid rings DM to be 2-firs. For example, looking at the second of the above identities, $x \cdot (yx + 1) = (xy+1) \cdot x$, we see that if we give x, yx and xy the names a, b and c respectively, then that ring-theoretic equation follows from the monoid relation $a\,b = c\,a$ that these elements satisfy, which is the subject of (7). If we examine the monoid-theoretic relations corresponding in the same way to the next identity of (61), we get, with more work, the same single equation. But moving to the fourth line of (61), we find, among the monomials in the factors comprising that linear relation, eight that are not products of any others, namely x, z, wz, wx, yx, xy, xw and zw; and calling these a, b, c, d, e, f, g and h, we find that the fifteen monoid relations among these elements that underlie that ring identity reduce, after tautologies and repetitions are dropped, to five:

$$(62)\qquad ac = gb, \quad ad = ga, \quad ae = fa, \quad bc = hb, \quad bd = ha.$$

So it might be of interest to search for results to the effect that if this family of relations holds in a monoid M such that DM is a 2-fir, then it can be achieved in some "generic" way; and, if this approach turns out to be productive, the identities further down on the list beginning with (61) might be investigated similarly.

What about conditions for monoid rings to be n-firs for higher values of n? For $n > 2$, the process of trivializing an n-term relation seems to have a much more fluid character than for $n = 2$. E.g., compare the simplicity of the Euclidean algorithm for pairs of real numbers with the complexity, and lack of a single natural choice, among the algorithms for larger families of real numbers discussed in [19]. Nonetheless, one may hope for further results on the conditions for the n-fir property to hold.

11.4. **An observation on $\sigma^{1/2}$.** I claim that the fact that the automorphism $\sigma^{1/2}$ of the free group on x and y introduced in Lemma 13 sends $z = x^{-1}y^{-1}xy$ to z^{-1} is (loosely) related to the fact that $\tau = (1+\sqrt{5})/2$ has algebraic norm -1 over $\mathbb{Q}$. Precisely, let me sketch an argument using the latter fact which shows that in $G/[G,G']$, the image of $\sigma^{1/2}(z)$ must be the inverse of the image of z.

It is not hard to show that the commutator operation on any group G induces an alternating bilinear map $G/G' \times G/G' \to G'/[G,G']$, whose image generates the latter group. Hence for G the free group of rank 2, which has $G/G' \cong \mathbb{Z}\times\mathbb{Z}$ and $G'/[G,G'] \cong \mathbb{Z}$, any endomorphism α of G must induce on $G'/[G,G']$ the operation of exponentiation by the determinant of the map that α induces on G/G'. Now we have seen that the automorphism of G/G' induced by $\sigma^{1/2}$ acts as does multiplication by τ on $\mathbb{Z}+\tau\mathbb{Z} \subseteq \mathbb{R}$; and the determinant of that automorphism of free abelian groups is the norm of τ, namely -1. So since $z \in G'$, the image of z in $G'/[G,G']$ must indeed be sent to its inverse.

For some observations on the limiting behavior of the orbit of x under $\sigma^{1/2}$ under group topologies on G, see [5, §7].

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13. Typographical note

The angle brackets used in this note, $\langle\ \rangle$, are not the standard TEX symbols $\langle\ \rangle$. For the TEX code for the symbols used here, and my thoughts on the subject, see [8]. (However, in this note, since the symbols enclosed in angle brackets are almost all lower-case, I have added `\kern.16em` to the definition of `\lang` shown in [8], so as to get, for example, $\langle x\rangle$ rather than $\langle x\rangle$.)

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