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EVERY MODULE IS AN INVERSE LIMIT OF INJECTIVES

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Abstract. It is shown that any left module A over a ring R can be written as the intersection of a downward directed system of injective submodules of an injective module; equivalently, as an inverse limit of one-to-one homomorphisms of injectives. If R is left Noetherian, A can also be written as the inverse limit of a system of surjective homomorphisms of injectives.

Some questions are raised.

The flat modules over a ring are precisely the direct limits of projective modules [11] [6] [10, Theorem 2.4.34]. Which modules are, dually, inverse limits of injectives?

I sketched the answer in [1], but in view of the limited distribution of that item, it seems worthwhile to make the result more widely available. The construction from [1] is Theorem 2 below; the connecting maps there are inclusions. In Theorem 4, we shall see that the connecting maps can, alternatively, be taken to be onto if R is Noetherian on the appropriate side.

In §2 we ask some questions, in §3 we take some steps toward answering one of them, and in §4 we note what the proofs of our results tell us when applied to not necessarily injective modules.

Throughout, "ring" means associative ring with unit, and modules are unital.

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1. Main results

We will need the following generalization of the familiar observation ([4, Proposition I.3.1], [9, Proposition IV.3.7]) that a direct product of injective modules is injective. (It is a generalization because on taking $\kappa > \mathrm{card}(I)$, it yields that result.)

Lemma 1. *Let R be a ring, κ an infinite regular cardinal such that every left ideal of R can be generated by $< \kappa$ elements, and $(M_i)_{i \in I}$ a family of left R-modules. Let*

(1) $\quad \prod_I^{\kappa} M_i \ = \ \{x \in \prod_I M_i \mid x \text{ has support of cardinality } < \kappa \text{ in } I\}.$

Then if all M_i are injective, so is $\prod_I^{\kappa} M_i$.

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Proof. To show $\prod_I^{\kappa} M_i$ injective, it suffices by [4, Theorem I.3.2] [9, Lemma IV.3.8] to show that for every left ideal L of R, every module homomorphism $h: L \to \prod_I^{\kappa} M_i$ can be extended to all of R. By choice of κ, L has a generating set X of cardinality $< \kappa$, and by definition of $\prod_I^{\kappa} M_i$, the image under h of each member of X has support of cardinality $< \kappa$ in I. Hence by regularity of κ, the union $I_0 \subseteq I$ of these supports has cardinality $< \kappa$. Clearly $h(L)$ has support in I_0, hence h can be looked at as a homomorphism $L \to \prod_{I_0} M_i$. As each M_i is injective, we can now lift h componentwise to a homomorphism $R \to \prod_{I_0} M_i \subseteq \prod_I^{\kappa} M_i$, as desired. □

Theorem 2. *Let R be a ring. Then every left R-module A can be written as the intersection of a downward directed system of injective submodules of an injective module; in other words, as the inverse limit of a system of injective modules and one-to-one homomorphisms.*

Proof. Given A, choose an exact sequence of modules

(2) $$0 \to A \to M \to N$$

with M and N injective, as we may by [4, Theorem I.3.3], and call the second map $f: M \to N$. Taking a cardinal κ as in the preceding lemma (for example, any infinite regular cardinal $> |R|$), and a set I of cardinality $\geq \kappa$, one element of which we will denote 0, we define R-modules M_i $(i \in I)$ by letting $M_0 = M$, and $M_i = N$ for $i \neq 0$.

Now let $P = \prod_I^{\kappa} M_i$, and for each finite subset $D \subseteq I - \{0\}$, let $P_D \subseteq P$ be the submodule of elements $(x_i)_{i \in I}$ such that for all $i \in D$, $x_i = f(x_0)$. Clearly, each element of P_D is determined by its components at the indices in $I - D$, from which we see that $P_D \cong \prod_{I-D}^{\kappa} M_i$; so by Lemma 1, P_D is injective. The family of submodules P_D is downward directed, since $P_{D_1} \cap P_{D_2} = P_{D_1 \cup D_2}$.

Now $\bigcap_D P_D \subseteq P$ consists of the elements $x \in P$ such that for all $i \in I - \{0\}$, $x_i = f(x_0)$. Each such x is determined by its coordinate $x_0 \in M$; but to lie in P, such an element must have support of cardinality $< \kappa$, which only happens if $x_0 \in \ker f$. Thus, $\bigcap_D P_D \cong \ker f = A$. □

Note that in the construction of the above proof, if R is left Noetherian then κ can be taken to be $\aleph_0$, and I countable; so the intersection is over the finite subsets of a countable set, giving a countably indexed inverse system. In that situation, $\prod_I^{\kappa} M_i$ is simply $\bigoplus_I M_i$, and Lemma 1 then says that the class of injective R-modules is closed under direct sums (a known result, [12, Proposition 2.1]. In fact, that condition is necessary and sufficient for R to be left Noetherian [13, Theorem 1] [3, Theorem 1.1] [5, Theorem 20.1], a result variously called the Matlis-Papp Theorem, the Cartan-Eilenberg-Bass Theorem, and by other combinations of these names.) We shall use this closure under direct sums in the proof of our next theorem, along with the following fact.

(3) There exists an inverse system, indexed by the first uncountable ordinal ω_1, of nonempty sets S_α and surjective maps $f_{\alpha\beta}: S_\beta \to S_\alpha$ $(\alpha \leq \beta \in \omega_1)$, which has empty inverse limit [7] [8, §2] [2].

Again, we begin with a general lemma.

Lemma 3 (after [8, §3]; cf. [2, Corollary 8])**.** *Suppose $(S_\alpha, f_{\alpha\beta})_{\alpha \leq \beta \in \omega_1}$ is an inverse system of sets with the properties stated in* (3)*, and N a left module over* *a ring* R. *To each* $\alpha \in \omega_1$, *let us associate the direct sum* $\bigoplus_{S_\alpha} N$ *of an* S_α*-tuple of copies of* N; *and for* $\alpha \leq \beta$, *let* $\varphi_{\alpha\beta} : \bigoplus_{S_\beta} N \to \bigoplus_{S_\alpha} N$ *be the map sending* $(x_j)_{j \in S_\beta}$ *to the element* $(y_i)_{i \in S_\alpha}$ *with components* $y_i = \sum_{f_{\alpha\beta}(j)=i} x_j$.

Then each $\varphi_{\alpha\beta}$ *is surjective, but the inverse limit of the above system is the zero module.*

Sketch of proof. We imitate the argument of [8] (where R was a field and N was R). Surjectivity of the $\varphi_{\alpha\beta}$ is clear. Now suppose x belongs to the inverse limit, and let us write its components $x^{(\alpha)} \in \bigoplus_{S_\alpha} N$ $(\alpha \in \omega_1)$. For each $\alpha \in \omega_1$, let $T_\alpha \subseteq S_\alpha$ be the (finite) support of $x^{(\alpha)}$. We see that the cardinalities of the T_α are monotonically nondecreasing in α; hence, since ω_1 has uncountable cofinality, the supremum of those cardinalities must be finite. (Indeed, for each n such that some $|T_\alpha|$ equals n, let us choose an α_n realizing this value. Then the at most countably many indices α_n have a supremum, $\alpha_{\sup} \in \omega_1$, and the finite value $|T_{\alpha_{\sup}}|$ will bound all $|T_\alpha|$.)

Calling this supremum n, we see that the set of $\alpha \in \omega_1$ such that $|T_\alpha| = n$ is an up-set in ω_1, and that whenever $\alpha \leq \beta$ are both in this up-set, the connecting map $f_{\alpha\beta}$ gives a bijection $T_\beta \to T_\alpha$. These n-element sets T_α thus lead to an n-tuple of elements of $\varprojlim S_\alpha$. But by assumption, that limit set is empty. Hence $n = 0$, so all $x^{(\alpha)}$ are 0, so $x = 0$. □

We can now prove

Theorem 4. *Let* R *be a left Noetherian ring. Then every left* R*-module* A *can be written as the inverse limit of a system, indexed by* ω_1, *of surjective homomorphisms of injective modules.*

Proof. Again let $f : M \to N$ be a homomorphism of injective left R-modules with kernel A. Let us take the inverse system of direct sums of copies of N described in Lemma 3, and append to each of these direct sums a copy of M, getting modules

$$M \oplus \bigoplus_{S_\alpha} N \quad (\alpha \in \omega_1), \tag{4}$$

which we connect using maps that act on M as the identity, and act on the direct sums of copies of N by the connecting morphisms of Lemma 3. Assuming for notational convenience that none of the S_α contains an element named 0, let us write the general element of (4) as $(x_i)_{i \in \{0\} \cup S_\alpha}$, where $x_0 \in M$ and the other components are in N.

We now define, for each $\alpha \in \omega_1$,

$$P_\alpha \;=\; \{x = (x_i)_{i \in \{0\} \cup S_\alpha} \in M \oplus \bigoplus_{S_\alpha} N \;\mid\; \textstyle\sum_{i \in S_\alpha} x_i = f(x_0)\}. \tag{5}$$

Note that for each α, if we choose any $i_0 \in S_\alpha$, then we can specify an element $x \in P_\alpha$ by choosing its components other than x_{i_0} to comprise an arbitrary member of $M \oplus \bigoplus_{S_\alpha - \{i_0\}} N$. The value of x_{i_0} will then be determined by the relation $\sum_{i \in S_\alpha} x_i = f(x_0)$. Thus, $P_\alpha \cong M \oplus \bigoplus_{S_\alpha - \{i_0\}} N$, a direct sum of injectives, so since R is left Noetherian, each P_α is injective. Clearly, the inverse system of surjective maps among the modules (4) induces an inverse system of surjective maps among the submodules (5).

In a member of $\varprojlim_{\omega_1} P_\alpha$, the $\bigoplus_{S_\alpha} N$-components, as α ranges over ω_1, will form a member of the inverse limit of the system of Lemma 3; hence these components must all be zero. Thus, the corresponding M-components must belong to

$\ker f = A$. Since the connecting maps on these components are the identity map of M, the inverse limit is $A \subseteq M$. $\square$

(Incidentally, Theorem 2 or 4 yields a correct proof of [14, Lemma 3], the statement that $\mathbb{Z}$ is an inverse limit of injective abelian groups. The construction of [14] is similar to our proof of Theorem 2, but since the groups H_j used there are uniquely p-divisible for all odd primes p, their intersection is p-divisible, and so is not $\mathbb{Z}$.)

For further examples of unexpectedly small inverse limits, see [2], [7], [8], [15]. Some questions about these are noted in [2, §§4-5].

2. Questions

Theorem 4 leaves open

Question 5. *For non-left-Noetherian R, which left R-modules are inverse limits of systems of surjective maps of injective R-modules?* (*All?*) *Does the answer change if we restrict ourselves to systems indexed, as in Theorem 4, by ω_1 ?*

We noted following Theorem 2 that for R Noetherian, the construction used there involved a countable inverse system. This suggests

Question 6. *For non-left-Noetherian R, which left R-modules are inverse limits of* countable *systems of* one-to-one *maps of injective R-modules?* (*All?*)

On the other hand, the construction of Theorem 4 used uncountable inverse systems in all cases, and so leaves open

Question 7. *For a* (*left Noetherian or general*) *ring R, which left R-modules are inverse limits of* countable *systems of* surjective *maps of injective left R-modules?*

3. Partial results on Question 7

The answer to Question 7 cannot be either "all modules" or "only the injectives", even for $R = \mathbb{Z}$, as will be shown by Corollary 9 and Example 10, respectively.

In describing inverse limits, we have indexed our inverse systems so that the connecting maps go from higher- to lower-indexed objects. In direct limits, which appear beside inverse limits in the following preparatory lemma, we shall take the connecting maps to go from lower- to higher-indexed objects. (Thus, in each kind of limit, our index-sets are *upward* directed.)

Lemma 8. *Let R be a ring. Let M be the inverse limit of a countable system of injective left R-modules M_α and surjective homomorphisms $\varphi_{\alpha\beta} : M_\beta \to M_\alpha$ $(\alpha \leq \beta,\ \alpha, \beta \in I)$, and let N be the* direct limit *of a countable system of* projective *left R-modules N_γ and* one-to-one *homomorphisms $\psi_{\delta\gamma} : N_\gamma \to N_\delta$ $(\gamma \leq \delta,\ \gamma, \delta \in J)$.*

Then any homomorphism

(6) $\quad f : N_\gamma \ \to \ M_\alpha,$ *where* $\gamma \in J,\ \alpha \in I$

can be factored

(7) $\quad N_\gamma \ \to \ N \to \ M \to \ M_\alpha,$

where the first and last maps are the canonical ones associated with the given direct and inverse limits (*and the indices γ and α*), *while the middle map is an arbitrary module homomorphism.*

Proof. Let us be given a homomorphism (6).

Recall that every countable directed partially ordered set (or more generally, any directed partially ordered set of countable cofinality) has a cofinal chain isomorphic to ω, and that a direct or inverse limit over the original set is isomorphic to the corresponding construction over any such chain. In our present situation, we can clearly take such a chain in I which begins with the index α of (6), and such a chain in J beginning with the index γ. Hence, replacing the two given systems with the systems determined by these chains, we may assume that our direct and inverse system are both indexed by ω, and name the map we wish to extend $f_0 : N_0 \to M_0$ (see (8) below).

Using the projectivity of N_0 and the surjectivity of $\varphi_{01} : M_1 \to M_0$, we can now factor f_0 as $\varphi_{01}\, g_0$ for some homomorphism $g_0 : N_0 \to M_1$, and then, similarly using the injectivity of M_1 and one-one-ness of $\psi_{10} : N_0 \to N_1$, factor g_0 as $f_1\, \psi_{10}$ for some $f_1 : N_1 \to M_1$. Thus, we get $f_0 = \varphi_{01}\, f_1\, \psi_{10}$.

We now iterate this process, getting $f_2 : N_2 \to M_2$, etc., where each composite $N_{i-1} \to N_i \to M_i \to M_{i-1}$ is the preceding map f_{i-1} :

$$\begin{tikzcd}
\cdots & N_i \arrow[l] \arrow[d, dotted, "f_i"] & \cdots \arrow[l] & N_2 \arrow[l] \arrow[d, dotted, "f_2"] & N_1 \arrow[l, "\psi_{21}"'] \arrow[d, dotted, "f_1"] & N_0 \arrow[l, "\psi_{10}"'] \arrow[d, "f_0"] \\
\cdots \arrow[r] & M_i \arrow[r] & \cdots \arrow[r] & M_2 \arrow[r, "\varphi_{12}"'] & M_1 \arrow[r, "\varphi_{01}"'] & M_0 .
\end{tikzcd} \tag{8}$$

In particular, each composite $N_0 \to N_i \to M_i \to M_0$ is our original map f_0. Using the universal properties of direct and inverse limits, we see that these maps induce a map $N \to M$ such that the composite $N_0 \to N \to M \to M_0$ is f_0, as required. $\square$

Now suppose that R is a commutative principal ideal domain. Then it is easy to verify that an R-module M is injective if and only if it is *divisible*, i.e., if and only if it is a homomorphic image, as an R-module, of some K-module, where K is the field of fractions of R. If, further, $R \neq K$, and R has at most countably many primes, say $p_1, p_2, \ldots$ (where we allow repetitions in this list, in case R has only finitely many), then K is, as an R-module, the direct limit of a chain of inclusions of free R-modules of rank 1

$$R \subseteq p_1^{-1}R \subseteq p_1^{-2}\,p_2^{-2}R \subseteq \cdots \subseteq p_1^{-i}\,p_2^{-i}\ldots p_i^{-i}R \subseteq \cdots . \tag{9}$$

Hence we can apply Lemma 8 with K as N, calling the modules of (9) $N_0 \subseteq N_1 \subseteq \cdots$; but still letting $(M_\alpha)_{\alpha\in I}$ be an arbitary countable inverse system of injectives. For any $\alpha \in I$, every $x \in M_\alpha$ is, of course, the image of the generator $1 \in R = N_0$ under some homomorphism $f : N_0 \to M_\alpha$. Hence Lemma 8 tells us that x lies in the image of a homomorphism $K = N \to M \to M_\alpha$; so the span in M of the images of all homomorphisms $K \to M$ maps surjectively to each M_α. For brevity and concreteness, we state this result below for $R = \mathbb{Z}$.

Corollary 9. *Let M be the inverse limit of a countable system of injective $\mathbb{Z}$-modules M_α and surjective homomorphisms $\varphi_{\alpha\beta} : M_\beta \to M_\alpha$. Let M_{div} be the largest divisible (equivalently, injective) submodule of M, namely, the sum of the images of all $\mathbb{Z}$-module homomorphisms $\mathbb{Q} \to M$. Then M_{div} projects surjectively to each M_α; i.e., the composite maps $M_{\mathrm{div}} \hookrightarrow M \to M_\alpha$ are surjective.* $\square$

This shows that if M is nontrivial, it must have a sizable injective submodule. (In particular, M cannot be a nonzero finitely generated $\mathbb{Z}$-module.) However, the following example shows that that submodule need not be all of M.

Example 10. *A countable inverse system* $\cdots \to M_2 \to M_1 \to M_0$ *of injective* $\mathbb{Z}$*-modules and surjective homomorphisms, whose inverse limit* M *is not injective.*

Construction and proof. For each $n \geq 0$, let

$$M_n = \mathbb{Q} \oplus \ldots \oplus \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Q}/\mathbb{Z}) \oplus \ldots, \tag{10}$$

where the summands $\mathbb{Q}$ are indexed by $i = 0, \ldots, n-1$, and the $\mathbb{Q}/\mathbb{Z}$ by the $i \geq n$. Define connecting maps $\varphi_{mn} : M_n \to M_m$ $(m \leq n)$ to act componentwise; namely, as the identity map of $\mathbb{Q}$, respectively, of $\mathbb{Q}/\mathbb{Z}$, on the components with indices $i < m$ or $i \geq n$, and as the quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ on the $n-m$ intermediate components.

It is not hard to verify that the inverse limit M of these modules can be identified with the submodule of $\mathbb{Q}^\omega$ consisting of those elements all but finitely many of whose components lie in $\mathbb{Z}$. (Given $x \in M$, its image in M_0 will have all but finitely many components $0 \in \mathbb{Q}/\mathbb{Z}$, and these zero components will correspond to the components of x which lie in $\mathbb{Z}$.)

If we take an element $x \in M$ and a positive integer k such that the entries of x in $\mathbb{Z}$ are *not* almost all divisible by k, then x is not divisible by k in M. Hence M is not a divisible group, i.e., is not injective. □

Returning to Corollary 9, we remark that its method of proof, applied to a countable inverse limit M of injective modules and surjective homomorphisms over any integral domain R, shows that M contains many “highly divisible” elements. For most R, this shows that not all R-modules can occur as such inverse limits.

4. Not necessarily injective modules

None of the constructions we have used to get an inverse system of modules from an exact sequence $0 \to A \to M \to N$ are limited to the case where M and N are injective. Let us record what they give us in general.

Corollary 11 (to proofs of Theorems 2 and 4, and Example 10)**.** *Let* R *be a ring,* $\mathbf{M}$ *a class of left* R*-modules,* κ *an infinite regular cardinal such that* $\mathbf{M}$ *is closed under* κ*-restricted direct products* $\prod_I^\kappa M_\alpha$, *and* $0 \to A \to M \to N$ *any exact sequence of left* R*-modules with* $M, N \in \mathbf{M}$. *Then*

(a) A *can be written as the inverse limit of a system of modules in* $\mathbf{M}$ *and one-to-one homomorphisms.*

(b) *If* $\kappa = \aleph_0$ *(so that the hypothesis on* $\mathbf{M}$ *is that it is closed under direct* sums), *then* A *can be written as the inverse limit of an* ω_1*-indexed system of modules in* $\mathbf{M}$ *and* surjective *homomorphisms.*

(c) *If, again,* $\kappa = \aleph_0$, *then the submodule of* M^ω *consisting of those elements with all but finitely many components in* A *can be written as the inverse limit of a* countable *system of modules in* $\mathbf{M}$ *and surjective homomorphisms.* □

So, for instance, by (b), for any ring R, any R-module which can be written as the kernel of a homomorphism of projective modules can also be written as the inverse limit of a system of projective modules and surjective homomorphisms.

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