ON k-SUBSPACES OF L-VECTOR-SPACES

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ABSTRACT. Let $k \subseteq L$ be division rings, with $[L:k]_{\mathrm{right}} < \infty$, and let $B_k \subseteq A_L$ be right vector spaces. Conditions are established in terms of the k-codimension of B in A, respectively the k-dimension of B, for there to exist a nonzero element $a \in A$ such that $aL \cap B = \{0\}$, respectively an L-linear functional f on A such that $f(B) = L$. Some consequences and related open questions are discussed.

Suppose $k \subseteq L$ are fields, or more generally, division rings, with $[L:k]_{\mathrm{right}} < \infty$. If A is a right L-vector-space and B a k-subspace of A, we may ask under what conditions there will exist a nonzero L-subspace of A having zero intersection with B.

For the commutative algebraist, or the noncommutative algebraist to whom the only division rings of interest are finite-dimensional algebras over fields, the question is answered completely in §1, where it is shown that the obvious necessary condition for the existence of such a subspace, namely that the k-codimension of B in A be at least $[L:k]_{\mathrm{right}}$, is also sufficient if k and L are finite-dimensional algebras over a field k_0. Curiously, our proof uses a bit of algebraic geometry, though the result is purely one of linear algebra. We also obtain a natural dual statement (cf. abstract above), and a number of corollaries.

In §2 we consider the case of general k and L. There the argument by algebraic geometry is not available, and I do not know whether the precise analogs

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of the results of §1 hold. We will see, however, that we can get the same conclusions if the k-codimension (respectively k-dimension) of B is sufficiently large, namely $\geq d(d-1)/2+1$, where $d=[L:k]_{\mathrm{right}}$.

In §3 we examine the minimal cases in which we do not know whether the estimates of §1 hold unmodified in the context of §2.

1. The commutative and close-to-commutative case.

The results of this section will use

Lemma 1. *Let k be a field, and $\beta: A\times B\to C$ a bilinear map of nonzero finite-dimensional k-vector-spaces which is* surjective, *i.e., such that every element of C is of the form $\beta(a,b)$ (and not merely a sum of such elements). Then*

$$\dim_k(C) < \dim_k(A)+\dim_k(B). \tag{1}$$

Proof. The intuitive idea is that an element of $A\times B$ has ‘‘$\dim_k(A)+\dim_k(B)$ degrees of freedom’’, and that at least one degree of freedom is lost in applying β. Let us show that the above idea can be translated into a counting argument if k is finite, and into an argument by algebraic geometry when k is infinite.

If k has cardinality $q<\infty$, then A, B and C have cardinalities $q^{\dim_k(A)}$, $q^{\dim_k(B)}$ and $q^{\dim_k(C)}$ respectively; moreover, the map β is not one-to-one, since it takes all pairs $(a,0)$ and $(0,b)$ to 0. Hence

$$q^{\dim_k(C)} < q^{\dim_k(A)}\, q^{\dim_k(B)},$$

and this is equivalent to (1).

If k is infinite, we regard the map β as the restriction to k-valued points of a map from $\dim_k(A)$-dimensional affine space cross $\dim_k(B)$-dimensional affine space to $\dim_k(C)$-dimensional affine space. Because k is infinite, the k-valued points are dense in these affine spaces, hence the fact that β is surjective implies that the above map of affine spaces has dense image. It also has fibers of dimension at least one, since for all $a\in A\setminus\{0\}$, $b\in B\setminus\{0\}$, the 1-parameter set $\{(ua,u^{-1}b)\}$ $(u\in k\setminus\{0\})$ is mapped to a point. Thus, comparing dimensions of varieties, we get the desired inequality.

This argument can be stripped of its algebraic-geometric trappings by considering the corresponding homomorphism of the polynomial rings of which the above affine spaces are spectra:

$$k[z_1,\ldots,z_{\dim_k(C)}] \;\to\; k[x_1,\ldots,x_{\dim_k(A)}]\otimes_k k[y_1,\ldots,y_{\dim_k(B)}].$$

(Functorially, these polynomial rings are the symmetric algebras on the dual spaces to C, A and B.) From the surjectivity of β and the fact that k is infinite, one deduces that this homomorphism is one-to-one. A comparison of transcendence degrees, without further considerations, would give $\dim_k(C)\leq\dim_k(A)+\dim_k(B)$; but if we regard the algebra on the right as bigraded by ‘‘degree in the x’s’’ and ‘‘degree in the y’s’’, then we see that the above map carries the component of degree i in the graded algebra on the left into the component of bidegree (i, i) on the right. Hence the map lands in the subalgebra determined by the diagonal subgroup of the grading group $\mathbf{Z}\times\mathbf{Z}$; and a bigraded integral domain is clearly transcendental over this subalgebra if the latter is proper. □

We can now prove the main results of this section. These will concern the situation described in

(2) $k \subseteq L$ will be finite-dimensional division algebras over a field k_0, A will be a right L-vector space, and B a k-subspace of A.

Note that since k and L are finite-dimensional over k_0, we have $[L:k]_{\text{right}} = [L:k]_{\text{left}} = [L:k_0]/[k:k_0]$. We shall denote this common value $[L:k]$.

Theorem 2. *In the situation of* (2), *there exists a nonzero element $a\in A$ such that $B\cap aL = \{0\}$ if and only if the k-codimension of B in A is at least $[L:k]$.*

Proof. As we noted earlier, one direction is clear: if $B\cap aL = \{0\}$, then B must have k-codimension at least $\dim_k aL = [L:k]$.

In proving the reverse implication, let us assume first that A and B are finite-dimensional. If B has k-codimension $\geq [L:k]$, it has k_0-codimension $\geq [L:k_0]$, and this together with Lemma 1 shows that the k_0-bilinear multiplication map $B\times L \to A$ cannot be surjective. Let a be an element of A not in the image of this map; thus, a cannot be written $b\alpha$ with $b\in B$, $\alpha\in L$. This means that no $b\in B$ can be written $a\alpha^{-1}$, hence $B\cap aL = \{0\}$, as required.

If A and B are not assumed finite-dimensional, take any $[L:k]$ elements of A whose images in A/B are right k-linearly independent, let A_0 be the finite-dimensional L-subspace of A that they span, and let $B_0 = B\cap A_0$. Then B_0 has k-codimension at least $[L:k]$ in A_0, and applying the result of the preceding paragraph, we get an $a\in A_0$ with the desired properties. □

Corollary 3. *In the situation of* (2), *let n be a nonnegative integer. Then*

(i) *If the k-codimension of B in A is $\geq n[L:k]$, there exists an L-subspace $C\subseteq A$ of L-dimension n which has trivial intersection with B.*

(ii) *If the k-codimension of B in A is exactly $n[L:k]$, there exists an L-subspace $C\subseteq A$ of L-dimension n such that $A = B\oplus C$.*

(iii) *If the k-codimension of B is $\leq n[L:k]$, there exists an L-subspace $C\subseteq A$ of L-dimension n such that $A = B + C$.*

Proof. By enlarging B in case (i) or shrinking it in case (iii), we immediately reduce each to case (ii), which we shall prove by induction on n. (In fact, (i)-(iii) are easily seen to be equivalent.)

For $n = 0$, (ii) is trivial. If $n > 0$, the preceding Theorem gives us a 1-dimensional subspace $aL \subseteq A$ having zero intersection with B. Now $B' = B\oplus aL$ has k-codimension in A less than that of B by exactly $[L:k]$, and applying the $n-1$ case to B', we get the asserted result. □

From this we can deduce a statement of a dual sort:

Corollary 4. *In the situation of* (2), *suppose we are given a finite-dimensional* L*-vector-space* C. *Then*

(i) *If* $\dim_k(B) \leq \dim_k(C)$, *there exists an* L*-linear map* $f: A \to C$ *which is one-to-one on* B.

(ii) *If* $\dim_k(B) = \dim_k(C)$, *there exists an* L*-linear map* $f: A \to C$ *which maps* B *bijectively to* C.

(iii) *If* $\dim_k(B) \geq \dim_k(C)$, *there exists an* L*-linear map* $f: A \to C$ *which maps* B *surjectively to* C.

Proof. Again, (i) and (iii) reduce to (ii). In the situation of (ii), B is finite-dimensional, so replacing A by its L-subspace spanned by B, we may assume that A is also finite-dimensional, and hence that B has finite k-codimension in A. This codimension is

$$\dim_k(A) - \dim_k(B) \ = \ \dim_k(A) - \dim_k(C) \ = \ (\dim_L(A) - \dim_L(C))\,[L:k].$$

Hence by statement (ii) of the preceding Corollary, A has an L-subspace C' of L-dimension $\dim_L(A) - \dim_L(C)$ which is a k-linear complement of B. Dividing out by this subspace, we get an L-linear quotient map f_0 from A to an L-vector-space C'' of L-dimension $\dim_L(C)$, such that f_0 maps B bijectively to C''. Composing f_0 with an L-vector-space isomorphism $C'' \to C$, we get the desired f. $\square$

Taking $C = L$ in (iii) above (and noting the obvious converse implication), we get a result dual to Theorem 2:

Theorem 5. *In the situation of* (2), *there exists an* L*-linear functional* $f: A \to L$ *such that* $f(B) = L$ *if and only if* $\dim_k(B) \geq [L:k]$. $\square$

Another interesting consequence of Corollary 4 is the following. (Statement (i) below is known in the commutative case [**3**, Lemma on p.189], [**4**, Lemma 1.1].)

Corollary 6. *Let* $k \subseteq L$ *be as in* (2), *and let* B *and* B' *be right* k*-subspaces of the division ring* L. *Then*

(i) *If* $\dim_k(B) + \dim_k(B') \leq [L:k]$, *there exists nonzero* $\alpha \in L$ *such that* $B \cap \alpha B' = \{0\}$.

(ii) *If* $\dim_k(B) + \dim_k(B') = [L:k]$, *there exists nonzero* $\alpha \in L$ *such that* $L = B \oplus \alpha B'$.

(iii) *If* $\dim_k(B) + \dim_k(B') \geq [L:k]$, *there exists nonzero* $\alpha \in L$ *such that* $B + \alpha B' = L$.

Proof. As before, the other cases reduce to (ii). To prove (ii), let A be a two-dimensional right L-vector-space, with basis $\{x, y\}$, and B'' the k-subspace $xB \oplus yB'$. By Corollary 4(ii) there exists an L-linear functional f on A mapping B'' isomorphically to L. Left-multiplying by $f(x)^{-1}$, we may assume that $f(x) = 1$. The desired α is now given by $f(y)$. $\square$

Caveat: It is not hard to verify that Theorem 2, Theorem 5, and each of the statements of Corollaries 3 and 4 are ''equivalent'', i.e., can easily be deduced from one another. I more than once convinced myself, erroneously, that Corollary 6 was likewise equivalent to these statements; specifically, that from Corollary 6(i), one could get the $C = L$ case of Corollary 4(i), from which in turn the $C = L$ case of part (iii) of that Corollary, and hence Theorem 5, would easily follow. The idea of the ''proof'' was as follows. Assuming the hypothesis of Corollary 4(i) with $C = L$, let h be an L-linear functional on A which does not annihilate B. Then B is the direct sum of $B_0 = B \cap \ker(h)$ and a k-linear complement B_1 thereto, which is k-isomorphic to $h(B) \subseteq L$. Now $\dim_k(B_0) < \dim_k(B)$, so we may assume by induction that there is an L-linear functional f_0 on A which is one-to-one on B_0. Applying Corollary 6(i), we conclude that some functional of the form $f_0 + \alpha h$ is one-to-one on $B = B_0 \oplus B_1$.

The fallacy is that though B_0 is contained in the kernel of h, B_1 cannot be assumed to be contained in the kernel of f_0 (for though it is a k-vector-space complement of B_0 in B, it need not lie in an L-subspace of A complementing $B_0 L$). Hence $f_0 + \alpha h$ does not in general carry B to $f_0(B_0) + \alpha h(B_1)$, and Corollary 6(i) is not applicable.

However, a weakened version of the above argument will be used in proving the first result of the next section.

2. Results for general division rings. In this section, we will drop from (2) the condition that our division rings be finite-dimensional algebras over a common field, and merely assume

(3) $k \subseteq L$ will be division rings with $[L:k]_{\mathrm{right}} < \infty$, A will be a right L-vector space, and B will be a k-subspace of A.

The algebraic-geometric trick of the preceding section is inapplicable to this situation, but we shall show that if the dimension or codimension of B is large enough, then the same conclusions can be obtained. A quick argument of this sort was given in [**1**], where we only needed the conclusions of Theorems 2 and 5 in the case where the dimension or codimension of B was infinite. It was noted there that if $\dim_k(B) \geq ([L:k]_{\mathrm{right}})^2$, then the L-subspace of A spanned by B must have L-dimension at least $[L:k]_{\mathrm{right}}$, so that an L-linear functional on A could be constructed to carry $[L:k]_{\mathrm{right}}$ L-linearly independent elements of B to the elements of a k-basis of L, and that such a functional would carry B onto L. A dual result was likewise proved for B of k-codimension $\geq ([L:k]_{\mathrm{right}})^2$. As the final assertions of the next two results show, we can improve the above estimates slightly.

Proposition 7. *Let n be a nonnegative integer $< [L:k]_{\mathrm{right}}$. Then if $\dim_k(B) > n(n+1)/2$, there exists an L-linear functional $f: A \to L$ such that $\dim_k(f(B)) > n$.*

In particular, writing $d = [L:k]_{\text{right}}$, *if* $\dim_k(B) \geq d(d-1)/2 + 1$, *then there exists an* L*-linear functional* $f: A \to L$ *such that* $f(B) = L$.

Proof. It suffices to prove the first assertion, which we shall do by induction. The case $n = 0$ is clear, so assume that $0 < n < [L:k]_{\text{right}}$, that the result is true for $n-1$, and that $\dim_k(B) > n(n+1)/2$.

Let $h: A \to L$ be any L-linear functional which is nonzero on B. If $\dim_k(h(B)) > n$ then we are done. If not, then $\dim_k(B \cap \ker(h)) > n(n+1)/2 - n = n(n-1)/2$, so letting $B_0 = B \cap \ker(h)$, our inductive assumption says that there is an L-linear functional f_0 on A such that

$$\dim_k(f_0(B_0)) > n-1.$$

If $\dim_k(f_0(B)) > n$, then again we are done. If not, then this dimension is, in particular, $< [L:k]_{\text{right}}$, so $f_0(B)$ is a proper k-subspace of L. Now take any $x \in B \setminus B_0$; thus $h(x) \neq 0$, so by the preceding sentence we can find $\alpha \in L$ such that $\alpha h(x) \notin f_0(B)$. Let $f = f_0 + \alpha h$.

We first note that $f(B) \supseteq f(B_0) = f_0(B_0)$ since h annihilates B_0. On the other hand, for our chosen element x, in the equation $f(x) = f_0(x) + \alpha h(x)$, the first summand lies in $f_0(B)$ but the second does not, so $f(x)$ does not lie in that space, and in particular does not lie in $f_0(B_0)$. Hence $f(B)$ is properly larger than $f_0(B_0)$, hence by the inequality displayed above, $f(B)$ has dimension $> n$. □

We shall next prove the dual result by a dual argument, though the proof will be a bit longer because I felt some of the steps were not as intuitive, and needed more explanation.

Proposition 8. *Let* n *be a nonnegative integer* $< [L:k]_{\text{right}}$. *Then if the* k*-codimension of* B *in* A *is* $> n(n+1)/2$, *there exists a nonzero element* $a \in B$ *such that the* k*-codimension of* $B \cap aL$ *in* aL *is* $> n$.

In particular, writing $d = [L:k]_{\text{right}}$, *if the* k*-codimension of* B *in* A *is* $\geq d(d-1)/2 + 1$, *there exists a nonzero element* $a \in A$ *such that* $B \cap aL = \{0\}$.

Proof. Again we shall prove the first assertion by induction. The case $n = 0$ says that if B is a proper k-subspace of A, then we can choose $a \in A$ such that $B \cap aL$ is a proper k-subspace of aL; and indeed, we may do this by taking $a \in A \setminus B$. So assume that $0 < n < [L:k]_{\text{right}}$, that the result is true for $n-1$, and that the k-codimension of B in A is $> n(n+1)/2$.

Let a_0 be any element of $A \setminus B$. Then the k-codimension of B in A is the sum of the k-codimensions of $B + a_0 L$ in A and of $B \cap a_0 L$ in $a_0 L$, as may be seen from the exact sequence of k-vector-spaces

$$\{0\} \to a_0 L/(B \cap a_0 L) \to A/B \to A/(B + a_0 L) \to \{0\}.$$

Hence either the k-codimension of $B \cap a_0 L$ in $a_0 L$ is $> n$, or the k-codimension of $B + a_0 L$ in A is $> n(n-1)/2$. In the first case, we may take $a = a_0$ and we are done. In the second, our inductive assumption says that there is an element $a_1 \in A$ such that

$$\text{the } k\text{-codimension of } (B + a_0 L) \cap a_1 L \text{ in } a_1 L \text{ is } > n - 1.$$

Now if $B \cap a_1 L = \{0\}$, then $B \cap a_1 L$ has k-codimension $[L:k]_{\mathrm{right}} > n$ in $a_1 L$, and we are also done; so assume $a_1 \alpha \in B$ $(\alpha \in L \setminus \{0\})$. Let

$$a = a_1 + a_0 \alpha^{-1}.$$

We now note that for all $\beta \in L$ such that $a_1 \beta \notin (B + a_0 L)$, one also has $a\beta \notin (B + a_0 L)$ (since a differs from a_1 by a member of $a_0 L$), so in particular, for such β, $a\beta \notin B$. On the other hand, though $a_1 \alpha \in B$ by choice of α, in the element $a\alpha = a_1 \alpha + a_0$ the second summand does not lie in B, so $a\alpha \notin B$. Thus, $\{\beta \mid a\beta \in B\}$ is strictly smaller than $\{\beta \mid a_1 \beta \in (B + a_0 L)\}$, whence the codimension of $B \cap aL$ in aL is strictly larger than the codimension of $(B + a_0 L) \cap a_1 L$ in $a_1 L$, which we have assumed is $> n - 1$. So the codimension of $B \cap aL$ in aL is $> n$, as required. □

Instead of giving a dual argument as we have done above, could we have proved Proposition 8 *from* Proposition 7, by applying the linear duality between right and left vector spaces (followed by a passage to opposite division rings to turn statements about left vector spaces back to statements about right vector spaces)? Unfortunately, though the functor $\mathrm{Hom}_L(-, L)$ takes right L-vector-spaces to left L-vector-spaces, and the functor $\mathrm{Hom}_k(-, k)$ takes right k-vector-spaces to left k-vector-spaces, I see no way to ''dualize'' a k-subspace of a right L-vector-space to get a k-factor-space of a left L-vector-space. Indeed, it seems unlikely that such a way exists, because in general, $[L:k]_{\mathrm{right}} \neq [L:k]_{\mathrm{left}}$ [**2**, §5.9], so one can't have such a duality that simultaneously preserves k-dimensions and L-dimensions.

Though as far as we know, B needs to have the rather large k-dimension given in Proposition 7 to insure the existence of an L-linear function on A which maps B surjectively to a 1-dimensional L-vector-space, the next result shows that it is not as hard to get from there to maps onto higher-dimensional L-vector-spaces.

Proposition 9. *In the situation of* (3), *let* $d = [L:k]_{\mathrm{right}}$, *and let* C *be a finite-dimensional right L-vector-space. Then*

(i) *If* $\dim_k(B) \geq \dim_k(C) + (d-1)(d-2)/2$, *then there exists an L-linear map* $A \to C$ *mapping* B *surjectively to* C.

(ii) *If* $\dim_k(B) \leq \dim_k(C) - (d-1)(d-2)/2$, *then there exists an L-linear map* $A \to C$ *mapping* B *injectively to* C.

Proof. Assume the hypothesis of (i). If $C = 0$ the result is trivial, so assume the contrary. Then $\dim_k(C) \geq d$, hence $\dim_k(B) \geq d + (d-1)(d-2)/2 = d(d-1)/2 + 1$. Hence by Proposition 7, there exists an L-linear functional on A carrying B onto L. From this we can get an L-linear map g from A to a 1-dimensional subspace C' of C, which carries B onto C'.

Now let C'' be an L-linear complement of C' in C. We may assume by induction on $\dim_L(C)$ that (i) holds with $B \cap \ker(g)$ in the role of B, and C'' in the role of C. Since these substitutions decrease both sides of the assumed inequality by d, that inequality continues to hold, and we conclude that there exists an L-linear map $h\colon A \to C''$ which maps $B \cap \ker(g)$ onto that space. It is now easy to verify that $(g, h)\colon A \to C' \oplus C'' = C$ maps B surjectively to C.

In the situation of (ii), let us begin by dividing A by a maximal L-subspace having trivial intersection with B. Thus we are reduced to the case where A has no such L-subspace. By Proposition 8, it follows that B has k-codimension $\leq d(d-1)/2$ in A. Combining this with the inequality in our hypothesis, we get $\dim_k(A) \leq \dim_k(C) + d-1$. But this means that $\dim_L(A) - \dim_L(C) < 1$, i.e., $\dim_L(A) \leq \dim_L(C)$, so we can embed A in C as L-vector-spaces. Any such embedding maps B injectively to C. $\square$

Perhaps one can also get results of the form ''If $\dim_k(B)$ is close to $\dim_k(C)$, then there exists an L-linear map $A \to C$ which, when restricted to B, gives a k-linear map whose kernel and cokernel are both fairly small''. We leave these for the interested reader to investigate.

The estimates of this section do not seem strong enough to give an analog of Corollary 6, but there is a weak version thereof that is easy to prove directly.

Lemma 10. *Let* $k \subseteq L$ *be as in* (3), *and let* B *and* B' *be proper nonzero right k-subspaces of* L. *Then there exists nonzero* $\alpha \in L$ *such that neither* B *or* $\alpha B'$ *contains the other; equivalently, such that* $\dim_k(B \cap \alpha B') < \min(\dim_k(B), \dim_k(B'))$, *equivalently, such that* $\dim_k(B + \alpha B') > \max(\dim_k(B), \dim_k(B'))$ (*where* $\dim_k$ *denotes dimension as a right k-subspace of* L).

Proof. Assume without loss of generality that $\dim_k(B) \leq \dim_k(B')$. Take any $\beta \in B$ and any $\gamma \notin B'$, and let $\alpha = \beta\,\gamma^{-1}$. Then $\beta \in B \setminus \alpha B'$, so we do not have $B \subseteq \alpha B'$, while our inequality of right dimensions shows that we cannot have a strict inclusion the other way. $\square$

We remark that for $[L:k]_{\mathrm{right}} \leq 2$, but not for higher values, the final statements of Propositions 7 and 8 are equivalent to the conclusions of Theorems 5 and 2 respectively, and hence yield the conclusions of Corollaries 3 and 4 as well; while for $[L:k]_{\mathrm{right}} \leq 3$, the above Lemma yields the conclusion of Corollary 6(ii), and hence all parts of that Corollary. Thus, if those Theorems could be deduced from that Corollary, Lemma 10 would allow us to get, in the context of (3), the conclusions of those Theorems for $[L:k]_{\mathrm{right}} = 3$, which are stronger than the conclusions of Propositions 7 and 8 in that case.

3. Some questions. The central unanswered question of this paper is obviously

Question 11. *Do Theorems 2 and 5* (*and hence the results of §1 deduced from them*) *hold with the hypothesis* (2) *weakened to* (3)*?*

As just noted, $[L:k]_{\mathrm{right}} = 3$ is the smallest value for which the analog of Theorem 2 may fail in the context (3). Suppose we had a counterexample in that case; or more generally, suppose that for some k and L with $[L:k]_{\mathrm{right}} \geq 3$ (preferably finite), we had, in contradiction to the analog of Corollary 4(i), a

3-dimensional k-subspace B of an L-vector-space A, such that no L-linear functional f on A acted on B in a one-to-one fashion.

Note that in this situation, $\dim_L(BL)$ could not be 1 or 3; for in the former case, we could take for f any functional which was one-to-one on that one-dimensional L-subspace of A, while in the latter, we could choose f to map three L-linearly independent elements of B to three k-linearly independent elements of L. So $\dim_L(BL) = 2$. Furthermore, no *two* k-linearly independent elements of B can be L-linearly dependent; for if x and $x\alpha$ $(\alpha \in L)$ were two such elements, and we let y be any element of $B \setminus xL$, then an L-linear functional taking x to 1 and y to a member of $L \setminus (k + \alpha k)$ would satisfy the conclusion of Corollary 4(i).

Taking any two k-linearly independent elements x and y of B, a third member of a k-basis of that space must have the form $x\alpha + y\beta$ for some $\alpha, \beta \in L$. Here if either of the coefficients α or β lay in k, we could modify our choice of this third basis element to make that coefficient 0, which would make that basis element L-linearly dependent on x or on y, which we have just shown impossible. So $\alpha, \beta \in L \setminus k$.

In our assumption that there exists no L-linear functional f on A that is one-to-one on B, we can clearly restrict attention to functionals f taking x to 1. Given a functional f with the latter property, let $\theta = f(y)$. Then f will be one-to-one on the two-dimensional subspace $xk + yk$ if and only if $\theta \notin k$. With a little thought, one concludes from the above observations that the question of whether an example of the sort we have been discussing exists is equivalent to

Question 12. *Do there exist division rings* $k \subseteq L$ *with* $[L:k]_{\mathrm{right}} \geq 3$ (*if possible, with this number finite*), *and elements* $\alpha, \beta \in L \setminus k$, *such that for every* $\theta \in L \setminus k$, *one has* $\alpha + \theta\beta \in k + \theta k$*?*

Finally, turning to Lemma 10, we note that if either B or B' has k-dimension 1, then that Lemma gives the same conclusion as Corollary 6(i), so the first case where the latter conclusion can fail is when both k-subspaces are 2-dimensional. Left-multiplying by appropriate elements of L, we can assume without loss of generality that they have the forms $k + \alpha k$ and $k + \beta k$. So we ask

Question 13. *Do there exist division rings* $k \subseteq L$ *with* $[L:k]_{\mathrm{right}} \geq 4$ (*if possible, with this number finite*), *and elements* $\alpha, \beta \in L \setminus k$, *such that for every* $\theta \in L \setminus \{0\}$, *one has* $(k + \alpha k) \cap \theta(k + \beta k) \neq \{0\}$*?*

An affirmative answer to Question 13 would imply an affirmative answer to Question 12. Indeed, Question 12 with $[L:k]_{\mathrm{right}} \geq 4$ is equivalent to Question 13 with the added restriction that for each θ, the right coefficients of α and β in some nonzero element of $(k + \alpha k) \cap \theta(k + \beta k)$ be 1 and -1 respectively.

REFERENCES

1. George M. Bergman and Ehud Hrushovski, *Linear ultrafilters,* to appear, Communications in Algebra.

2. P. M. Cohn, *Skew fields. Theory of general division rings,* Encyclopedia of Mathematics and its Applications, vol. 57, Cambridge University Press, 1995. MR **97d** : 12003.

3. Michel Matignon, *Genre et genre résiduel des corps de fonctions valués,* manuscripta mathematica **58** (1987) 179-214. MR **89a** : 11063.

4. Jack Ohm, *On the vector space defect for valued field extensions,* to appear in *Real Algebraic Geometry and Ordered Structures: Proceedings of the Special Semester, Spring 1996, Baton Rouge,* Contemp. Math., Amer. Math. Soc..