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AN INNER AUTOMORPHISM IS ONLY AN INNER AUTOMORPHISM, BUT AN INNER ENDOMORPHISM CAN BE SOMETHING STRANGE

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Abstract. The inner automorphisms of a group G can be characterized within the category of groups without reference to group elements: they are precisely those automorphisms of G that can be extended, in a functorial manner, to all groups H given with homomorphisms $G \to H$. (Precise statement in §1.) The group of such extended systems of automorphisms, unlike the group of inner automorphisms of G itself, is always isomorphic to G. A similar characterization holds for inner automorphisms of an associative algebra R over a field K; here the group of functorial systems of automorphisms is isomorphic to the group of units of R modulo the units of K.

If one looks at the above functorial extendibility property for endomorphisms, rather than just automorphisms, then in the group case, the only additional example is the trivial endomorphism; but in the K-algebra case, a construction unfamiliar to ring theorists, but known to functional analysts, also arises.

Systems of endomorphisms with the same functoriality property are examined in some other categories; other uses of the phrase "inner endomorphism" in the literature, some overlapping the one introduced here, are noted; the concept of an inner *derivation* of an associative or Lie algebra is looked at from the same point of view, and the dual concept of a "co-inner" endomorphism is briefly examined. Several open questions are noted.

Overview.

You can read this overview if you'd like to know the topics of the various sections to come; but feel free to skip it if you'd prefer to plunge in, and let the story tell itself.

In §1, we motivate the approach of this paper using the case of groups. We obtain the characterization of inner automorphisms of groups that is stated in the abstract, and, modeled on this, we define concepts of inner automorphism and inner endomorphism for an object of a general category.

In §2, these definitions are applied to associative unital algebras over a commutative ring K, and full characterizations of the inner automorphisms and endomorphisms are obtained in the case where K is a field. In §3, counterexamples are given to the obvious generalizations of these results to base rings that are not fields, and the question of what the general inner automorphism and endomorphism might look like in that case is examined.

In §4 we pause to survey concepts that have appeared in the literature under the name "inner endomorphism", with varying degrees of overlap with that of this note.

§5 contains some easy observations on inner automorphisms and endomorphisms (in the sense here defined) on a few other sorts of algebraic objects.

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I discovered the main results of Theorems 1 and 6 several decades ago, at which time I was partly supported by a National Science Foundation grant, but I cannot now reconstruct the date, and hence the grant number. I spoke on those results at the January, 2009 AMS-MAA Joint Meeting in Washington D.C.. After publication of this note, updates, errata, related references etc., if found, will be recorded at http://math.berkeley.edu/~gbergman/papers/ .

In §§6-8 we study inner *derivations* of associative and Lie algebras, and also inner endomorphisms of Lie algebras, pausing in §7 to consider what the general definition of "inner derivation" should be.

It is noted in §9 that our concept of inner endomorphism dualizes to one of *co-inner endomorphism*, and we determine these for the category of G-sets, for G a group.

In §10 we briefly look at the ideas of this paper from the perspective of the theory of representable functors.

Although we were not able to obtain in §3 a full description of the inner endomorphisms of an associative K-algebra when K is not a field, we prove in a final appendix, §11, using the partial results of §3, that such inner endomorphisms are always one-to-one.

Open questions are noted in §§1, 3, 6 and 8.

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1. Inner automorphisms and inner endomorphisms of groups.

Recall that an automorphism α of a group G is called *inner* if there exists an $s \in G$ such that α is given by conjugation by s:

$$\alpha(t) \;=\; s\,t\,s^{-1} \quad (t \in G). \tag{1}$$

Given this definition, it may seem perverse to ask whether the condition that α be inner can be characterized without speaking of group elements. Note, however, that the definition implies the following property, which can indeed be so stated: For every homomorphism f of G into a group H, there exists an automorphism β_f of H making a commuting square with α:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta_f} \\ G & \xrightarrow{f} & H. \end{array} \tag{2}$$

(Namely, we can take β_f to be conjugation by $f(s)$.)

Whether this property alone is equivalent to α being inner, I do not know; but the above conclusion can be strengthened. Let α be as in (1), and for every group H and homomorphism $f : G \to H$ let β_f be, as above, the inner automorphism of H induced by $f(s)$. Then this system of automorphisms is "coherent", in that for every commuting triangle of group homomorphisms

$$\begin{array}{ccc} & \overset{f_1}{\nearrow} & H_1 \\ G & & \downarrow{\scriptstyle h} \\ & \underset{f_2}{\searrow} & H_2 \end{array} \tag{3}$$

one has

$$\beta_{f_2}\, h \;=\; h\, \beta_{f_1}. \tag{4}$$

Let us show that this strengthened statement *is* equivalent to α being inner; and that the family of morphisms β_f does what α alone in general does not: it uniquely determines s.

Theorem 1. *Let G be a group and α an automorphism of G. Suppose we are given, for each group H and homomorphism*

$$f : G \to H, \tag{5}$$

an automorphism β_f of H, with the properties that

(i) $\beta_{\mathrm{id}_G} \;=\; \alpha$, *and*

(ii) *for every commuting triangle* (3) *one has* (4).

Then there is a unique $s \in G$ such that for all H and f as in (5), *one has*

$$\beta_f(t) \;=\; f(s)\,t\,f(s)^{-1} \quad (t \in H). \tag{6}$$

In particular, α is inner. Thus, an automorphism α of a group G is inner if and only if there exists such a system of automorphisms β_f.

Proof. To investigate the system of maps β_f, let us look at their behavior on a "generic" element. Since the domains of these maps are groups with homomorphisms of G into them, a *generic* member of such a group will be the element x of the group $G\langle x\rangle$ obtained by adjoining to G one additional element x and no additional relations. (This group is the coproduct $G \amalg \langle x\rangle$ of G with the free group $\langle x\rangle$ on one generator; in group-theorists' language and notation, the free product $G * \langle x\rangle$.)

So letting η be the inclusion map $G \to G\langle x\rangle$, consider the element $\beta_\eta(x) \in G\langle x\rangle$. By the structure theorem for coproducts of groups, this can be written $w(x)$, where w is a reduced word in x and the elements of G. Note that for any map f of G into a group H, and any element $t \in H$, we can form a triangle (3) with $H_1 = G\langle x\rangle$, $f_1 = \eta$, $H_2 = H$, $f_2 = f$, and $h(x) = t$. (There is a unique such h making (3) commute, by the universal property of $G\langle x\rangle$.) By (4), the element $\beta_f\, h(x) = \beta_f(t)$ is equal to $h\,\beta_\eta(x) = h\,w(x) = w_f(t)$, where w_f denotes the result of substituting for the elements of G in the word w their images under f. Thus, β_f acts by carrying every $t \in H$ to $w_f(t)$.

Conversely, starting with any element $w(x) \in G\langle x\rangle$, the formula $\beta_f(t) = w_f(t)$ clearly gives a *set* map $\beta_f : H \to H$ for each f as in (5), in such a way that (ii) above holds. To determine when these set maps respect the group operation, we should consider the effect of the map induced by $w(x)$ on the *product* of a generic *pair* of elements. So we now take the group $G\langle x_0, x_1\rangle$ gotten by adjoining to G two elements and no relations, let η be the inclusion of G therein, and consider the relation $\beta_\eta(x_0\,x_1) = \beta_\eta(x_0)\,\beta_\eta(x_1)$, i.e.,

$$w(x_0\,x_1) \;=\; w(x_0)\,w(x_1). \tag{7}$$

When we transform the product on the right-hand side of (7) into a reduced word in x_0, x_1 and nonidentity elements of G, the only reduction that can occur is the simplification of the product of the factors from G at the right end of $w(x_0)$ and at the left end of $w(x_1)$; in particular, all occurrences of x_0 continue to occur to the left of all occurrences of x_1. On the other hand, in the left side of (7), each occurrence of x_0 or x_1 is adjacent to an occurrence of the other. It is easy to deduce that $w(x)$ can contain at most one occurrence of x, and that such an x, if it occurs, must have exponent $+1$. Moreover, if there were no occurrences of x, then the functions β_f would be constant, hence could not give automorphisms of nontrivial groups H; so $w(x)$ must have the form $s_0\,x\,s_1$. Substituting into (7), we find that $s_1\,s_0 = 1$; hence letting $s = s_0$, we have $w(x) = s\,x\,s^{-1}$.

Thus, the maps β_f have the form (6). Moreover, distinct elements s give distinct words $w(x)$, hence give distinct systems of automorphisms, since they act differently on $x \in G\langle x\rangle$; so such a system of automorphisms determines s uniquely.

Combining the above description of β_f with condition (i), we see that our original automorphism α is inner. This gives the "if" direction of the final sentence of the theorem; the remarks at the beginning of this section give "only if". □

(In the above theorem, we did not explicitly assume commutativity of the diagrams (2). But in view of condition (i), that commutativity requirement, for given h, is the case of condition (ii) where $H_1 = G$, $f_1 = \mathrm{id}_G$, $H_2 = H$, $h = f_2 = f$.)

For fixed G one can clearly compose two coherent systems of automorphisms of the sort considered in Theorem 1 to get another such system; and we see from the theorem that under composition, these systems form a group isomorphic to G.

There is an elegant formulation of this fact in terms of comma categories. Recall that for any object X of a category $\mathbf{C}$, the category whose objects are objects Y of $\mathbf{C}$ given with morphisms $X \to Y$, and whose morphisms are commuting triangles analogous to (3), is denoted $(X \downarrow \mathbf{C})$ (called a "comma category" because of the older notation $(X, \mathbf{C})$; see [21, §II.6]). A system of maps β_f as in Theorem 1 associates to each object $f : G \to H$ of the comma category $(G \downarrow \mathbf{Group})$ an automorphism, not of that object, but of the group H; i.e., of the value, at that object $f : G \to H$, of the forgetful functor $(G \downarrow \mathbf{Group}) \to \mathbf{Group}$. Our condition (4) says that these automorphisms β_f should together comprise an automorphism of that forgetful functor. In summary,

Theorem 2. *For any group G, the automorphism group of the forgetful functor $U : (G \downarrow \mathbf{Group}) \to \mathbf{Group}$ is isomorphic to G, via the map taking each $s \in G$ to the automorphism of U given by* (6). □

In the proof of Theorem 1 we used the assumption that α and the β_f were automorphisms, rather than simply endomorphisms, only once; to exclude the case where $w(x)$ contained no occurrences of x. In that

case, (7) forces w to be the identity element, whence the β_f are the trivial endomorphisms, $\varepsilon(t) = 1$. So we have

Corollary 3 (to proof of Theorem 1)**.** *If in the hypotheses of Theorem 1 one everywhere substitutes "endomorphism" for "automorphism", then the possibilities for* (β_f) *are as stated there, together with one additional case: where every* β_f *is the trivial endomorphism of* H*. In the language of Theorem 2, the endomorphism monoid of the forgetful functor* $(G \downarrow \mathbf{Group}) \to \mathbf{Group}$ *is isomorphic to* $G \cup \{\varepsilon\}$*, where* ε *is a zero element.* □

(There is nothing exotic about the trivial endomorphism; so the second half of the title of this note does not apply to the category of groups.)

Let us abstract, and name, the concepts we have been using.

Definition 4. *If* X *is an object of a category* $\mathbf{C}$*, then an endomorphism (respectively automorphism) of the forgetful functor* $(X \downarrow \mathbf{C}) \to \mathbf{C}$ *will be called an* extended *(or if there is danger of ambiguity, "*$\mathbf{C}$*-extended") inner endomorphism (resp., inner automorphism) of* X*.*

An extended inner endomorphism or automorphism will at times be denoted (β_f)*, where* f *is understood to run over all* $f : X \to Y$ *in* $(X \downarrow \mathbf{C})$*, and the* β_f *are the corresponding endomorphisms or automorphisms of the objects* Y*.*

An endomorphism or automorphism of X *will be called* inner *if it is the value at* id_X *of an extended inner endomorphism or automorphism of* X*. When there is danger of ambiguity, one may add "in the category-theoretic sense" and/or "with respect to* $\mathbf{C}$*".*

So, like "monomorphism" and "epimorphism", the term "inner" will acquire a certain tension between a pre-existing sense and a category-theoretic sense, which will agree in many, but not necessarily in all cases where the former is defined. In subsequent sections we will study the category-theoretic concept in several other categories.

We remark that if $\mathbf{C}$ is a legitimate category (i.e., if its hom-sets $\mathbf{C}(X, Y)$ are small sets – in classical language, sets rather than proper classes – but if its object-set may be large), then the monoids of endomorphisms, respectively, the groups of automorphisms, of the forgetful functors $(X \downarrow \mathbf{C}) \to \mathbf{C}$ are not, in general, *small* monoids or groups. However, there is a set-theoretic approach that handles such size-problems elegantly; see [5, §6.4] (cf. [21, §§I.6-7]). Aside from this point, these constructions behave very nicely: if $f : X_1 \to X_2$ is a morphism, it is easy to see that an extended inner endomorphism or automorphism of X_1 induces via f an extended inner endomorphism or automorphism of X_2 (in contrast with the behavior of ordinary endomorphisms, automorphisms, and inner automorphisms); thus, these constructions give *functors* from $\mathbf{C}$ to the categories of (possibly large) monoids and groups. (However, we shall not use this observation below.)

We end our consideration of these concepts in $\mathbf{Group}$ by recording a question mentioned above, generalized from automorphisms to endomorphisms.

Question 5. *If* α *is an endomorphism of a group* G*, such that for each object* $f : G \to H$ *of* $(G \downarrow \mathbf{Group})$ *there exists an endomorphism* β_f *of* H *making the diagram* (2) *commute, must* α *then be inner in the sense of Definition 4; i.e., is it then possible to choose such endomorphisms* β_f *so as to satisfy* (4) *for all commuting triangles* (3)*? (By the preceding results, this is equivalent to: Must* α *either be an inner automorphism in the classical sense, or the trivial endomorphism?)*

2. The case of K-algebras.

Let us now consider the same ideas for rings.

Let $\mathbf{Ring}^1$ denote the category of all associative unital rings. A general difficulty in the study of universal constructions in this category is the nontriviality of the multilinear algebra of abelian groups, i.e., $\mathbb{Z}$-modules. Often things are no worse if we generalize our considerations to the category $\mathbf{Ring}^1_K$ of associative unital algebras over a general commutative ring K, and they then become much better if we assume K a field. Below, we shall begin the analysis of inner endomorphisms of K-algebras for K a general commutative ring; then, about half-way through, we will have to restrict ourselves to the case where K is a field. In the next section we will examine what versions of our result might be true for general K.

So let K be any commutative ring (where "associative unital" is understood), and R any nonzero object of $\mathbf{Ring}^1_K$.

We will again use generic elements. The extension of R by a single generic element x in $\mathbf{Ring}^1_K$ has the K-module decomposition

$$R\langle x\rangle \;=\; R \,\oplus\, (R\,x\,R) \,\oplus\, (R\,x\,R\,x\,R) \,\oplus\, \ldots \;\cong\; R \,\oplus\, (R\otimes R) \,\oplus\, (R\otimes R\otimes R) \,\oplus\, \ldots\,. \tag{8}$$

Here the tensor products are as K-modules. Tensor products over K will be almost the only tensor products used in this note, so we make the convention that $\otimes$, without a subscript, denotes $\otimes_K$.

The extension of R by two generic elements similarly has form

$$R\langle x_0,\, x_1\rangle \;=\; \bigoplus_{\substack{n\geq 0\\ i_1,\ldots,i_n\in\{0,1\}}} R\,x_{i_1}R\ldots R\,x_{i_n}R \;\cong\; \bigoplus_{\substack{n\geq 0\\ i_1,\ldots,i_n\in\{0,1\}}} R\otimes R\otimes\ldots\otimes R\otimes R\,. \tag{9}$$

Exactly as in the proof of Theorem 1, every extended inner endomorphism of R will be determined by the image under it of $x\in R\langle x\rangle$, which will be some element $w(x)\in R\langle x\rangle$. And again, every $w(x)\in R\langle x\rangle$ induces, for each object $f: R\to S$ of $(R\downarrow \mathbf{Ring}^1_K)$, a set map of S into itself, sending each $r\in S$ to $w_f(r)\in S$, and these maps respect morphisms among such objects. So again, our task is to determine for which $w(x)\in R\langle x\rangle$ the induced set-maps $S\to S$ are K-algebra homomorphisms.

These maps will respect addition if and only if the required equation holds in the generic case, i.e., if and only if, in $R\,\langle x_0,\, x_1\rangle$,

$$w(x_0+x_1) \;=\; w(x_0) \;+\; w(x_1)\,. \tag{10}$$

I claim that the only elements $w(x)\in R\langle x\rangle$ satisfying (10) are those which are homogeneous of degree 1 in x; i.e., lie in the summand $R\,x\,R$ of (8). Indeed, if $w(x)$ had a nonzero component in one of the higher degree summands in (8), then on substituting x_0+x_1 for x, one of the nonzero components we would get in the left-hand side of (10) would lie in a summand of (9) that involved both x_0 and x_1, while this is not true of the right-hand side of (10). On the other hand, if $w(x)$ had a nonzero component a in degree zero, then the degree-zero component of the left-hand side of (10) would be a, while that of the right-hand side would be $2a$. So $w(x)$ is homogeneous of degree 1; i.e., we may write

$$w(x) \;=\; \sum_1^n a_i\,x\,b_i \tag{11}$$

for some $a_1,\ldots,a_n,\ b_1,\ldots,b_n\in R$. This necessary condition for (10) to hold is sufficient as well; in fact, it clearly implies that the functions induced by $w(x)$ respect the K-module structure.

It remains to bring in the conditions that the operation induced by $w(x)$ respect 1, and respect multiplication. The former condition says that

$$w(1) \;=\; 1, \tag{12}$$

i.e.,

$$\sum_1^n a_i\,b_i \;=\; 1, \tag{13}$$

while the latter condition,

$$w(x_0\,x_1) \;=\; w(x_0)\;w(x_1)\,, \tag{14}$$

translates to

$$\sum_{i=1}^n a_i\,x_0\,x_1\,b_i \;=\; \sum_{j=1}^n\sum_{k=1}^n a_j\,x_0\,b_j\,a_k\,x_1\,b_k\,. \tag{15}$$

To study these conditions, let us now assume that K is a field. In that case, if there is any K-linear dependence relation among the coefficients $a_1,\ldots,a_n$ in (11), then we can rewrite one of these elements as a K-linear combination of the rest, substitute into (11), collect terms with the same left-hand factor, and thus transform (11) into an expression of the same form, but with a smaller number of summands. We can do the same if there is a K-linear relation among $b_1,\ldots,b_n$. Hence, if we choose the expression (11) to minimize n, we get

(16) $a_1,\ldots,a_n$ are K-linearly independent, and $b_1,\ldots,b_n$ are K-linearly independent.

Now let A be any K-vector-space basis of R containing $a_1, \dots, a_n$, and B any basis containing $b_1, \dots, b_n$. Then as a K-vector-space, the summand $R\,x_0\,R\,x_1\,R \cong R \otimes R \otimes R$ of (9), in which the two sides of (15) lie, decomposes as a direct sum $\bigoplus_{a\in A,\, b\in B} a\,x_0\,R\,x_1\,b$. If for each j and k we take the component of (15) in $a_j\,x_0\,R\,x_1\,b_k \cong R$, and drop the outer factors $a_j\,x_0$ and $x_1\,b_k$, we get the equation in R,

$$\delta_{jk} \;=\; b_j\,a_k \quad (j,k = 1,\dots,n). \tag{17}$$

What this says is that if we write a for the row vector over R formed by $a_1, \dots, a_n$, and b for the column vector formed by $b_1, \dots, b_n$, then $b\,a$ is the identity matrix I_n. On the other hand, (13) says that $a\,b$ is the 1×1 identity matrix I_1. Thus, regarding these vectors as describing homomorphisms of right R-modules $a : R^n \to R$ and $b : R \to R^n$, these relations say that a and b constitute an isomorphism

$$R^n \;\cong\; R \quad \text{as right } R\text{-modules}. \tag{18}$$

For many sorts of rings R (e.g., any ring admitting a homomorphism into a field), (18) can only hold for $n = 1$. In such cases, a and b become mutually inverse elements, so (11) takes the form $w(x) = a\,x\,a^{-1}$, and our inner endomorphism is an inner automorphism in the classical sense. The element a such that $w(x) = a\,x\,a^{-1}$ is easily seen to be determined up to a scalar factor in K, so the group of extended inner automorphisms of R is isomorphic to the quotient group of the units of R by the units of K.

On the other hand, there are rings R admitting isomorphisms (18) for $n > 1$ [20], [11], [12], [4]. If in such an R we take a row vector a and column vector b describing such an isomorphism, then by the above computations, the element $w(x) = \sum a_i\,x\,b_i$ determines an unfamiliar sort of extended inner endomorphism of R. It is not hard to verify that this system of maps can be described as follows.

Since (for any ring R) the ring of endomorphisms of the right R-module R^n is isomorphic to the $n \times n$ matrix ring $M_n(R)$, a module isomorphism (18) yields a K-algebra isomorphism $M_n(R) \cong M_1(R)$. Moreover, for every object $f : R \to S$ of $(R \downarrow \mathbf{Ring}^1_K)$, the vectors a and b over R induce vectors $f(a)$, $f(b)$ over S satisfying the same relations, and hence likewise inducing isomorphisms of matrix rings. The endomorphism of S induced by $w(x)$ can now be described as the composite

$$S \;\xrightarrow{\text{diag.}}\; M_n(S) \;\xrightarrow[\cong]{((r_{ij}))\,\mapsto\,\sum f(a_i)\,r_{ij}\,f(b_j)}\; S\,. \tag{19}$$

Since the right-hand arrow in (19) is bijective, the composite arrow will, like the left-hand arrow, always be one-to-one, but will not be surjective for any nonzero S unless $n = 1$; so the latter is the only case where the above construction gives *automorphisms* of the algebras S.

These observations are summarized below, along with a final assertion which the reader should not find hard to verify, which corresponds to a description of the degree of nonuniqueness of the expression for an element $w = \sum a_i \otimes b_i$ in a tensor product of K-vector-spaces, when written using the smallest number of summands (the rank of the element as a tensor); equivalently, using K-linearly independent a_i and b_j. Note that (17), which we deduced using those conditions of K-linear independence, clearly also implies them.

Theorem 6. *Let K be a field, and R a nonzero K-algebra. Then for every extended inner* automorphism *(β_f) of R, there is an invertible element $a \in R$, unique up to a scalar factor, such that for each $f : R \to S$, the automorphism β_f of S is given by conjugation by $f(a)$.*

More generally, each extended inner endomorphism *of R has the form* (19) *for a pair (a, b), where for some n, $a = (a_i)$ is a length-n row vector over R, and $b = (b_i)$ a height-n column vector, satisfying* (13) *and* (17)*, equivalently, describing an isomorphism* (18)*. Two such pairs of vectors (a, b) and (a', b'), associated with integers n and n' respectively, determine the same extended inner endomorphism if and only if $n = n'$ and there exists some $U \in \mathrm{GL}(n, K)$ such that*

$$a' \;=\; a\,U, \quad b' \;=\; U^{-1}b. \tag{20}$$

□

The conclusion $n = n'$ in the above result follows from the uniqueness of $w(x)$, and hence of its rank as a member of $R \otimes R$; but let us note a way to see it directly, and in fact to see that n is determined by the value of our extended inner endomorphism at any nonzero object $f : R \to S$ of $(R \downarrow \mathbf{Ring}^1_K)$. From (19) we see that the *centralizer* in S of the image of our extended inner endomorphism will be isomorphic to

$M_n(Z(S))$ as a $Z(S)$-algebra, where $Z(S)$ is the center of S. In particular, it will be free of rank n^2 as a module over $Z(S)$; and free modules over commutative rings have unique rank.

We have noted that (19) shows that every extended inner endomorphism (β_f) of R consists of *one-to-one* endomorphisms β_f. This too can be seen from elementary considerations: Any K-algebra S can be embedded in a simple K-algebra T; and any endomorphism of S arising from an extended inner endomorphism of R will then extend to an endomorphism of T, which necessarily has trivial kernel. (The embeddability of any K-algebra in a simple K-algebra was proved in [9, Corollary 1 and Remark 2]. A different method of getting such an embedding, noted for Lie algebras in [24, Theorem B], is also applicable to associative algebras.)

It is not hard to add to Theorem 6 the necessary and sufficient condition for two extended inner endomorphisms of R as in the final statement to agree, not necessarily globally, but at R, i.e., to determine the same inner endomorphism of R. The condition has the same form as (20), but with U now taken in $\mathrm{GL}(n, Z(R))$. For $n = 1$, this is the expected condition that the conjugating elements differ by an invertible central factor in R.

(For the reader familiar with [6, Chapter III] we remark that $(R \downarrow \mathbf{Ring}^1_K)$ is the category there called R-$\mathbf{Ring}^1_K$, and that the $R\langle x\rangle$ occurring in the above arguments is the underlying algebra of the coalgebra object representing the *forgetful* functor R-$\mathbf{Ring}^1_K \to \mathbf{Ring}^1_K$. Since the values of that forgetful functor have, in particular, additive group structures, the functor can be regarded as **Ab**-valued, so by [6, Theorem 13.15 and Corollary 14.8], its representing K-algebra is freely generated over R by an (R, R)-bimodule. This is the $R\,x\,R \cong R \otimes R$ of (8). Our extended inner endomorphisms of R correspond to endomorphisms of $R\langle x\rangle$ as a co-ring. Since these are in particular co-abelian-group endomorphisms, they will be induced by bimodule endomorphisms of $R\,x\,R$; this is the content of (11). Our subsequent arguments determine when such an endomorphism respects the counit and comultiplication of $R\langle x\rangle$.)

3. What if K is not a field?

For a general commutative ring K and an arbitrary object R of $\mathbf{Ring}^1_K$, any vectors a, b over R that satisfy (13) and (17) will still yield an element $w(x) = \sum_1^n a_i\,x\,b_i$ inducing an extended inner endomorphism (19) of R in $\mathbf{Ring}^1_K$; but we can no longer say that every extended inner endomorphism has this form. As an easy counterexample, if K is a direct product $K_1 \times K_2$ of two fields, then $\mathbf{Ring}^1_K \cong \mathbf{Ring}^1_{K_1} \times \mathbf{Ring}^1_{K_2}$, and one can show that any extended inner endomorphism of an object $R_1 \times R_2$ of $\mathbf{Ring}^1_K$ $(R_i \in \mathbf{Ring}^1_{K_i})$ is determined by an extended inner endomorphism of R_1 and an extended inner endomorphism of R_2. Now if R_1 and R_2 are both nonzero, and if they respectively admit extended inner endomorphisms $(\beta_{1,f})$ and $(\beta_{2,f})$, associated with distinct positive integers n_1 and n_2, then these together induce an extended inner endomorphism of R which does not have the form (19) for any n.

For a different sort of example, suppose K is a commutative integral domain having a nonprincipal invertible ideal J, and let F be the field of fractions of K. (Recall that an ideal J of K is called invertible if it has an inverse in the multiplicative monoid of *fractional ideals* of K, that is, nonzero K-submodules of F whose elements admit a common denominator. The integral domains K all of whose nonzero ideals are invertible are the Dedekind domains [2, Theorem 9.8]. Thus, any Dedekind domain that is not a PID has a nonprincipal invertible ideal J.) Suppose we form the Laurent polynomial ring in one indeterminate, $F[t, t^{-1}]$, and within this, let R be the subring $K[\,Jt,\, J^{-1}t^{-1}]$. Then in $R\langle x\rangle$, the K-submodule $J\,t\,x\,J^{-1}t^{-1} \cong J \otimes J^{-1} \cong K$ is free on one generator, which we shall call $w(x)$, and which we might write (in)formally as $t\,x\,t^{-1}$, though t itself is not an element of R. One finds that $w(x)$ satisfies (10), (12) and (14), and so induces an extended inner endomorphism; but "$t\,x\,t^{-1}$" does not have the form $s\,x\,s^{-1}$ for any invertible element $s \in R$, so this extended inner endomorphism is not as described in Theorem 6. Incidentally, this extended inner endomorphism has an inverse, induced by "$t^{-1}x\,t$", so it is even an extended inner automorphism (showing that the first half of our title is not *quite* true).

In taking an example of maximal simplicity, we have ended up with a commutative R, so that the automorphism of R itself induced by the above extended inner automorphism is trivial, and can be described as conjugation by $1 \in R$. To avoid this, let us freely adjoin to the F-algebra $F[\,t, t^{-1}]$ another noncommuting indeterminate, u, getting the algebra $F\langle t,\, t^{-1},\, u\rangle$, and within this take $R = K\langle Jt,\, J^{-1}t^{-1},\, u\rangle$. Then the automorphism of R induced by "$t\,x\,t^{-1}$" is now nontrivial, and is still not inner in the classical sense; in particular, it takes u to $t\,u\,t^{-1}$, though conjugation by no invertible element of R can do this.

Our general result for K a field, and the above examples for other sorts of K, can be subsumed in a common construction: Suppose P is a K-module, and R a K-algebra having an isomorphism

$$a : P \otimes R \xrightarrow{\cong} R \tag{21}$$

as right R-modules. (In the case where K was a field, P was an n-dimensional vector space; in our $K_1 \times K_2$ example, it was the module $K_1^{n_1} \times K_2^{n_2}$; in the $K[Jt,\, J^{-1}t^{-1}]$ and $K\langle Jt,\, J^{-1}t^{-1},\, u\rangle$ examples, it is J. In this last case, one has an isomorphism (21) $J \otimes R \cong R$ because $J \otimes R \cong JR = t^{-1}R \cong R$, the middle equality holding because R is closed in $F[\,t, t^{-1}]$ under multiplication by $J^{-1}t^{-1}$ and $J\,t$.) Such a map (21) yields, for every algebra S with a homomorphism $R \to S$, a K-algebra homomorphism

$$S \;\cong\; \mathrm{End}_S(S_S) \xrightarrow{P\otimes -} \mathrm{End}_S(P \otimes S_S) \xrightarrow[\cong]{a\otimes_R -} \mathrm{End}_S(S_S) \;\cong\; S. \tag{22}$$

The K-module P in this construction need not be unique. For instance, if we take an example based on an isomorphism (18), but where our R is an algebra over some epimorph K' of K (in the category-theoretic sense; e.g., a factor-ring or a localization), then regarding R as a K-algebra, we could choose the K-module P of (21) to be either K^n or K'^n.

In all the cases looked at so far, our K-module P either was, or (in the above paragraph) could be taken to be, projective over K. But there are examples where this is impossible: Consider any integral domain K which has an epimorph of the form $K_1 \times K_2$ for fields K_1 and K_2 (e.g., $\mathbb{Z}$ has such homomorphic images). Then if we construct, as in the first paragraph of this section, an algebra R over $K_1 \times K_2$ and an extended inner endomorphism of R based on a K'-module $P = K_1^{n_1} \times K_2^{n_2}$ with $n_1 \neq n_2$, this cannot arise from an example based on a projective K-module. This follows from the fact that for a finitely generated projective module over an integral domain K, the rank is constant as a function on the prime spectrum of K [10, Ch.2, §5, n°. 2, Théorème 1, (a)⇒(c)], [19, p.53, Exercise 22].

Question 7. *If K is a commutative ring and R a nonzero object of* $\mathbf{Ring}^1_K$, *can every extended inner endomorphism of R be obtained as in* (22) *from a module isomorphism* (21)*?*

The nonuniqueness of the P in the above construction makes me dubious.

We saw in the preceding section that for K a field, all inner endomorphisms of K-algebras were one-to-one. In an appendix, §11, we show that the same is true for any K.

4. Other concepts of "inner endomorphism" in the literature.

A MathSciNet search for "inner endomorphism" leads to a number of concepts, some of which have interesting overlaps with the one we have been studying.

A striking case, to which we alluded in the abstract, comes from the theory of C^*-algebras. If H is a Hilbert space, and $\mathcal{B}(H)$ the C^*-algebra of bounded operators $H \to H$, it is shown in [1, Proposition 2.1] that *every* endomorphism of the C^*-algebra $\mathcal{B}(H)$ has a form analogous to what we found in Theorem 6, namely

$$A \;\mapsto\; \sum V_i\, A\, V_i^*, \tag{23}$$

where the V_i are a (possibly infinite) family of isometric embeddings $H \to H$ having mutually orthogonal ranges which sum to H, and V_i^* is the adjoint of V_i.

Here is a heuristic sketch for the algebraist of why this is plausible. Since complex Hilbert spaces look alike except for their dimension, it is natural to generalize the problem of characterizing endomorphisms of $\mathcal{B}(H)$ to that of characterizing homomorphisms $\mathcal{B}(H_1) \to \mathcal{B}(H_2)$ for two Hilbert spaces H_1 and H_2. If H_1 and H_2 are finite-dimensional, of dimensions d_1 and d_2, then $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$ are matrix algebras $M_{d_1}(\mathbb{C})$ and $M_{d_2}(\mathbb{C})$. Temporarily ignoring the C^* structure, we know that a $\mathbb{C}$-algebra homomorphism $M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ exists if and only if $d_2 = n\,d_1$ for some integer n, and that in this case, it can be gotten by writing $\mathbb{C}^{d_2}$ as a direct sum of n copies of $\mathbb{C}^{d_1}$, and letting $M_{d_1}(\mathbb{C})$ act in the natural way on each of these. If $a_1, \dots, a_n : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ are the chosen embeddings and $b_1, \dots, b_n : \mathbb{C}^{d_2} \to \mathbb{C}^{d_1}$ the corresponding projections, the induced map $M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ is given by

$$r \;\mapsto\; \sum a_i\, r\, b_i. \tag{24}$$

If one wants this to be a homomorphism of C^*-algebras, one has the additional requirement that the a_i each map H_1 into H_2 isometrically, with orthogonal images; the projections b_i will then be the adjoints of the

a_i. Now if instead of finite-dimensional Hilbert spaces we take an infinite-dimensional Hilbert space H, and let $H_1 = H_2 = H$, then for both finite and infinite n, there exist expressions of H as a direct sum (in the infinite case, a completed direct sum) of n copies of itself. The result of [1] says that all endomorphisms of $\mathcal{B}(H)$ are expressible essentially as in the finite dimensional case, in terms of such direct sum decompositions of H.

For R any C^*-algebra, not necessarily of the form $\mathcal{B}(H)$, a family of elements $V_1, \ldots, V_n \in R$ $(n < \infty)$ satisfying the C^*-algebra relations corresponding to the conditions stated following (23) is equivalent to a homomorphism into R of the C^*-algebra presented by those generators and relations; this C^*-algebra is denoted $\mathcal{O}_n$. The objects $\mathcal{O}_n$ are called Cuntz algebras, having been introduced by J. Cuntz [13]. Since the above construction with $n = 1$ gives inner automorphisms of R in the classical sense, endomorphisms of the form (23) in a general C^*-algebra (where they are not in general the only endomorphisms) are called inner *endo*morphisms.

(For $n = \infty$, things are not as neat. Though in $\mathcal{B}(H)$, the infinite sums (23) converge in a topology obtained from the Hilbert space H, this is not the topology arising from the C^*-norm on $\mathcal{B}(H)$. In defining the C^*-algebra $\mathcal{O}_\infty$ one has to omit the relation $\sum V_i V_i^* = 1$, because the infinite sum will not converge; and maps of this object into a C^*-algebra R do not induce endomorphisms of R, though they are still of interest.)

The next concept I will describe is not called an "inner endomorphism" by the author who studies it, though it did turn up in a MathSciNet search for that phrase. (In the paper in question, "inner endomorphism" is used for "endomorphism of a subalgebra".) Namely, in [23], if A is an algebra in the sense of universal algebra, a *termal endomorphism* of A means an endomorphism α which is expressible by a *term* in one variable x, i.e., in the notation of this note, a word $w(x)$ in the operations of A and constants taken from A.

Note that if such a word $w(x)$ defines an endomorphism of A, i.e., if the set map it determines respects all operations of A, then this fact is equivalent to a family of identities in the operations of A and the constants occurring in w. If $\mathbf{V}$ is some variety containing A, those identities need not be satisfied by all members of $(A \downarrow \mathbf{V})$, so w may not define what we are calling a $\mathbf{V}$-extended inner endomorphism of A. However, if we regard $(A \downarrow \mathbf{V})$ as a variety, with the images of the elements of A as new zeroary operations, then the identities named will define a subvariety $\mathbf{V}_0 \subseteq (A \downarrow \mathbf{V})$, on which $w(t)$ does induce an extended inner endomorphism of id_A, and hence an inner endomorphism of A.

The phrase "inner endomorphism" has in fact been used in the theory of semigroups [25], [27] to describe some particular classes of what [23] calls termal endomorphisms.

A different use of the phrase "inner endomorphism" has occasionally been made in group theory. Observe that if G is a group, and $\alpha : G \to G$ is a set map which in one or another sense can be "approximated" arbitrarily closely by endomorphisms, then in general, α will again be an endomorphism; but that if the approximating endomorphisms are bijective, this does not force α to be bijective. In such situations, if the approximating maps are inner automorphisms, α has been called an "inner endomorphism", preceded by some qualifying adverb. Specifically, if one can find inner automorphisms of G that agree with α on a directed family of subgroups having G as union (though the conjugating elements need not belong to the corresponding subgroups, so that α need not carry those subgroups onto themselves), then α is called (in [3], and [14, p. 201, starting in paragraph before Theorem 5.5.9]) a "locally inner endomorphism", while if α induces inner automorphisms on a class of homomorphic images of G that separates points, it is called in [7] a "residually inner endomorphism". In the same spirit, [18] calls a topological limit of inner automorphisms of a C^*-algebra an "asymptotically inner endomorphism" (a usage apparently unrelated to the sense of "inner endomorphism" of a C^*-algebra described above).

On a somewhat related theme, [16] takes a finite-dimensional associative unital algebra A over a field K, with K-vector-space basis $\{u_1, \ldots, u_n\}$, forms an extension field K_0 of K by adjoining n algebraically independent elements, uses these as coefficients in forming a "generic" element of the K_0-algebra $A \otimes K_0$, and notes that this element will necessarily be invertible, so that conjugation by it may be thought of as a "generic" inner automorphism of A. It is then noted that for certain elements $a \in A$, the specialization of our indeterminates to the coefficients of the u_i in a may turn the above conjugation map into a map that is everywhere defined on A, even if a itself was not invertible. (Intuitively, the map obtained by that specialization is approximated by the operations of conjugation by nearby invertible elements.) The resulting

maps are endomorphisms, but examples are given showing that they may not be automorphisms, and they are named "inner endomorphisms" of A.

I don't see a direct relation between the concepts cited in the last two paragraphs and those of this paper. However, pondering the idea of [16], in which one performs a conjugation $r \mapsto a\,r\,a^{-1}$ for which, from the point of view of A, the pair (a, a^{-1}) "doesn't quite exist", helped lead me to the example of the preceding section, in which a conjugating element t was put out of reach by multiplying by a nonprincipal invertible ideal $J \subseteq K$.

I will note another use of "inner" in the literature, not restricted to endomorphisms, at the end of §7.

We now return to inner endomorphisms in the sense of Definition 4.

5. Extended inner endomorphisms in other categories of algebras – some easy observations.

We have examined extended inner endomorphisms in **Group** and $\mathbf{Ring}^1_K$. What about other categories of algebras?

In the category **Ab** of abelian groups (which we will write additively), the result of adjoining a "generic" element x to an object A is $A \oplus \langle x \rangle$, each element $w(x)$ of which has the form $a + nx$ for unique $a \in A$ and $n \in \mathbb{Z}$. Clearly, the system of operations induced by this element will respect the group operations of arbitrary objects of $(A \downarrow \mathbf{Ab})$ if and only if $a = 0$; so here the general extended inner endomorphism is given by multiplication by a fixed integer n; it will be an extended inner automorphism if and only if $n = \pm 1$. These are very different from the extended inner endomorphisms of the same group A in the larger category **Group**.

Note that the above extended inner endomorphisms of A do not really depend on A. Though we are looking at them as endomorphisms of the forgetful functor $(A \downarrow \mathbf{Ab}) \to \mathbf{Ab}$, they are induced by endomorphisms of the identity functor of **Ab**. We might call such operations *absolute* endomorphisms.

We can answer in the negative the analog of Question 5 with **Ab** in place of **Group**. Let p be a prime, and let $A = Z_{p^\infty}$, the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. Recall that this abelian group is injective, that its nonzero homomorphic images are all isomorphic to it, and that its endomorphism ring is canonically isomorphic to the ring of p-adic integers. It is easy to see that the action of each p-adic integer c on A makes a commuting square with the action of c on every homomorphic image $f(A)$. Now if f is a homomorphism of A into any abelian group B, the injectivity of $f(A)$ implies that B can be decomposed as $f(A) \oplus B_0$; hence the action of c on $f(A)$ can be extended to an action on B; e.g., by using the identity on B_0. It follows that all the endomorphisms of A (including its uncountably many automorphisms) have the one-B-at-a-time extendibility property analogous to the hypothesis of Question 5, though we have seen that only those corresponding to multiplication by integers are inner, as defined in Definition 4. Hence in **Ab**, the one-B-at-a-time extendibility property is strictly weaker than the functorial extendibility property by which we have defined inner endomorphisms and automorphisms.

It would be interesting to investigate inner automorphisms and endomorphisms in still other varieties of groups.

In the category of commutative rings, it is not hard to verify that $\mathbb{Z}$ has no nontrivial extended inner endomorphisms. On the other hand, $\mathbb{Z}/p\mathbb{Z}$ has, for every positive integer n, the extended inner endomorphism given by exponentiation by p^n (the n-th power of the Frobenius map). These endomorphisms are trivial on $\mathbb{Z}/p\mathbb{Z}$ itself; but on every other integral domain of characteristic p, the Frobenius map is a nontrivial inner endomorphism.

If A is an object of the variety of abelian semigroups (written multiplicatively), and e an idempotent element of A, then multiplication by e is an inner endomorphism; the same is true in the category of nonunital commutative rings. Similarly, if D is an object of the category of distributive lattices, then for any $a, b \in D$, the operators $a \vee -$, $b \wedge -$, and $a \vee (b \wedge -)$ are inner endomorphisms.

If A is an object of the category of all semigroups (not necessarily abelian), and e is a *central* idempotent of A, then the word $w(x) = ex$ gives a *termal* endomorphism of A in the sense of [23] (see preceding section), but not an inner endomorphism in our sense. However, following the idea noted in that section, if we form the subvariety of $(A \downarrow \mathbf{Semigroup})$ defined by the identity making the image of e central, then $w(x) = ex$

does determine an inner extended endomorphism in that category. The analogous observations hold for nonunital commutative rings, and for not necessarily distributive lattices.

6. Derivations of associative algebras.

Alongside inner automorphisms of groups and rings, there is another pair of cases where the modifier "inner" is classical: inner *derivations* of associative and Lie algebras. We shall examine the case of associative algebras in this section, that of Lie algebras in §8.

If K is a commutative ring and R an object of $\mathbf{Ring}^1_K$, we recall that a *derivation* of R as a K-algebra means a set-map $d : R \to R$ satisfying

$$d(r+s) \;=\; dr \;+\; ds \quad (r, s \in R), \tag{25}$$

$$d(c\,r) \;=\; c\,dr \quad (c \in K,\ r \in R), \tag{26}$$

$$d(r\,s) \;=\; d(r)\,s \;+\; r\,d(s) \quad (r, s \in R). \tag{27}$$

In particular, for every $t \in R$, the map d defined by

$$d\,r \;=\; t\,r - r\,t \tag{28}$$

is a derivation of R, called the *inner derivation* induced by t, and written $tr - rt = [t,\, r]$.

Such an inner derivation d clearly has the analog of the property of inner automorphisms of groups which we abstracted in Definition 4; namely, that to every $f : R \to S$ in $(R \downarrow \mathbf{Ring}^1_K)$ we can associate a derivation d_f of S, in such a way that

$$d_{\mathrm{id}_R} \;=\; d, \tag{29}$$

and that given two objects $f_i : R \to S_i \ \ (i = 1, 2)$ of $(R \downarrow \mathbf{Ring}^1_K)$ and a morphism $h : S_1 \to S_2$ in that category, we have

$$d_{f_2}\,h \;=\; h\,d_{f_1}. \tag{30}$$

What about the converse? Given a system of derivations d_f satisfying (30), let us, as in our investigation of automorphisms and endomorphisms, look at their action on a generic element. Let $\eta : R \to R\langle x\rangle$ be the natural inclusion and write $d_\eta(x) = w(x) \in R\langle x\rangle$. As before, (25) implies that

$$w(x) \;=\; \sum_1^n a_i\,x\,b_i \tag{31}$$

for some $a_1, \ldots, a_n,\ b_1, \ldots, b_n \in R$, and conversely, this condition implies both (25) and (26). To handle (27), we need, as before, an additional assumption; but this time we can get away with much less than K being a field. Let us merely assume that the canonical map $K \to R$ makes K a K-module direct summand in R; i.e., that there exists a K-module-theoretic left inverse $\varphi : R \to K$ to that map. Given such a φ, it is not hard to see that we can obtain from (31) an equation of the same form (possibly with n increased by 1) in which $a_1 = 1$, while $a_2, \ldots, a_n \in \mathrm{Ker}(\varphi)$. So let us assume that (31) has those properties.

Let us now take the generic instance of (27), namely, in $R\langle x_0,\, x_1\rangle$, the equation

$$\sum_1^n a_i\,x_0\,x_1\,b_i \;=\; (\sum_1^n a_i\,x_0\,b_i)\;x_1 \;+\; x_0\;(\sum_1^n a_i\,x_1\,b_i)\,. \tag{32}$$

The two sides of this equation lie in $R\,x_0\,R\,x_1\,R \cong R \otimes R \otimes R$. Let us apply φ to the leftmost of the three tensor factors, getting an equation in $x_0\,R\,x_1\,R$, and take the right coefficient of x_0 therein. This is an equation in $R \otimes R \cong R\,x_1\,R$, namely

$$x_1\,b_1 \;=\; b_1\,x_1 \;+\; \sum_1^n a_i\,x_1\,b_i\,. \tag{33}$$

Solving for the summation, which is $w(x_1)$, and writing x in place of x_1, we get

$$w(x) \;=\; x\,b_1 \;-\; b_1\,x\,. \tag{34}$$

Hence, each map d_f is the inner derivation, in the classical sense, determined by $f(b_1)$. We summarize this result below. The "unique up to ..." assertion in the conclusion is obtained by noting that an element

$b \in R$ satisfies $b \otimes 1 - 1 \otimes b = 0$ in $R \otimes R = (K \oplus \mathrm{Ker}(\varphi)) \otimes (K \oplus \mathrm{Ker}(\varphi))$ if and only if the component of b in $\mathrm{Ker}(\varphi)$ is 0; i.e., if and only if $b \in K$.

Theorem 8. *Let K be a commutative ring and R a K-algebra, and suppose we have a function associating to every $f : R \to S$ in $(R \downarrow \mathbf{Ring}^1_K)$ a derivation d_f of the K-algebra S, such that* (30) *holds for every morphism h of $(R \downarrow \mathbf{Ring}^1_K)$. Suppose also that the canonical map $K \to R$ has a K-module-theoretic left inverse.*

Then there exists $b \in R$, unique up to an additive constant from K, such that for each $f : R \to S$, d_f is the inner derivation of S induced by $f(b)$. □

We can push this a bit further. Instead of assuming that the canonical map $K \to R$ has a K-module left inverse, assume the K-algebra structure on R extends to a K'-algebra structure for some epimorph K' of K in the category of commutative rings, and that the map of K' into R has a K'-module left inverse. (This is the same as a K-module left inverse to the latter map. On the other hand, when the epimorphism $K \to K'$ is not an isomorphism, the map of K itself into R cannot have a K-module left inverse.) Then we can apply the above theorem to R as a K'-algebra; moreover, it is not hard to show that $(R \downarrow \mathbf{Ring}^1_K) \cong (R \downarrow \mathbf{Ring}^1_{K'})$; so the characterization by Theorem 8 of such systems of derivations parametrized by $(R \downarrow \mathbf{Ring}^1_{K'})$ gives the same result for systems of derivations parametrized by $(R \downarrow \mathbf{Ring}^1_K)$. Note, however, that the element inducing the system will be unique up to an additive constant in K', rather than in K.

I know of no example showing the need for any version of the module-theoretic hypothesis of Theorem 8 for the existence half of the conclusion. So we ask

Question 9. *If K is a commutative ring and R an associative unital K-algebra, must every system of derivations (d_f) satisfying* (30) *be induced, as above, by an element $b \in R$?*

7. How should one define "extended inner derivation"?

We would have stated Theorem 8 and Question 9 in terms of "extended inner derivations of R", if it were clear how to define this concept. We could, of course, make an ad hoc definition of the phrase, as a system of derivations d_f satisfying the hypothesis of those statements; but it would be better if we could make it an instance of a general use of "extended inner —". Derivations are not, in an obvious way, morphisms in a category, so we cannot use Definition 4. Below, we will note several ways that derivations can be put in a more general context, then take the one that seems best as the basis of our definition.

Recall first that there is a well-known characterization of derivations in terms of algebra homomorphisms. If R is a K-algebra, let $I(R)$ denote the K-algebra obtained by adjoining to R a central, square-zero element ε (intuitively, an infinitesimal. Formally, I can be described as the functor of tensoring over K with $K[\,\varepsilon \mid \varepsilon^2 = 0\,]$.) Then it is easy to verify that a set-map $d : R \to R$ is a derivation if and only if the map $R \to I(R)$ given by

$$r \;\mapsto\; r + \varepsilon\, d(r) \tag{35}$$

is a K-algebra homomorphism. Under this correspondence, the *inner* derivations, in the classical sense, correspond to the homomorphisms obtained by composing the inclusion $R \to I(R)$ with conjugation by a unit of the form $1 + \varepsilon\, b$ $(b \in R)$.

Using this characterization of derivations, we could put our condition on families of derivations d_f into category-theoretic language. But I don't see the resulting framework as fitting a natural wider class of constructions.

A second approach begins by asking, "Since the common value of the two sides of (30) is neither a derivation of S_1, nor a derivation of S_2, nor a ring homomorphism, what is it?" It is, in fact, what is known as "a derivation from S_1 to S_2 with respect to the homomorphism $h : S_1 \to S_2$"; i.e., a set-map $d : S_1 \to S_2$ which satisfies (25), (26), and the generalization of (27),

$$d(r\,s) \;=\; d(r)\,h(s) \;+\; h(r)\,d(s). \tag{36}$$

If, now, for every pair of K-algebras S_1, S_2, we let $D(S_1, S_2)$ denote the set of all pairs (h, d), where $h : S_1 \to S_2$ is a K-algebra homomorphism and $d : S_1 \to S_2$ a derivation with respect to h in the above sense, then $D(-,-)$ becomes a bifunctor $(\mathbf{Ring}^1_K)^{\mathrm{op}} \times \mathbf{Ring}^1_K \to \mathbf{Set}$, having a forgetful morphism

$(h, d) \mapsto h$ to the bivariant hom functor $\mathrm{Hom} : (\mathbf{Ring}^1_K)^{\mathrm{op}} \times \mathbf{Ring}^1_K \to \mathbf{Set}$. The set of derivations of a single K-algebra S is the inverse image of the identity endomorphism of S under this forgetful map.

Again, however, I don't know of a natural class of constructions wider than the derivations to which these observations generalize. Moreover, the concept of an h-derivation $d : S_1 \to S_2$ for h a ring homomorphism is in turn a special case of that of an (h, h')-derivation, for $h, h' : S_1 \to S_2$ two homomorphisms; such a derivation is a map satisfying (25), (26), and

$$d(r\,s) \;=\; d(r)\,h'(s) \;+\; h(r)d(s). \tag{37}$$

In this context, we again have the concept of the inner derivation induced by an element $b \in S_2$, namely the operation

$$d\,r \;=\; b\,h'(r) \;-\; h(r)\,b. \tag{38}$$

How this generalization might relate to our concept of "extended inner derivation" is not clear.

A third approach is to start with any variety $\mathbf{V}$ of algebras in the sense of universal algebra (i.e., the class of all sets given with a family of operations of specified arities, satisfying a specified set of identities [5, Chapter 8]), and suppose that we are interested in endomaps m of the underlying sets of objects of $\mathbf{V}$ which satisfy a certain set of identities in the operations of $\mathbf{V}$ and the inputs and outputs of m. (E.g., if the variety is $\mathbf{Ring}^1_K$ and the maps are the derivations, the identities are (25), (26) and (27).) For every $A \in \mathbf{V}$, let $M(A)$ denote the set of all such maps, and let us call these the M-maps of A. Then we may define an *extended inner* M-map of A as a way of associating to each $f : A \to B$ in $(A \downarrow \mathbf{V})$ an $m_f \in M(B)$, so as to satisfy the analog of (30). If we look at the action of such an extended inner M-map on a generic element, namely, the element $x \in A\langle x\rangle$, and call its image $w(x) \in A\langle x\rangle$, we again find that $w(x)$ determines the whole extended inner M-map; so we can study such maps by examining such elements. (The same observations apply, with obvious modifications, if one is interested in associating to each $f : A \to B$ an indexed *family* of operations on B, each of a specified *arity*, satisfying a set of identities relating them with each other and the operations of B. Then each operation of arity n in the family would have a generic instance in $A\langle x_1, \ldots, x_n\rangle$. We shall say a little more about this in §10, but will stick to the case of a single unary operation here.) Formalizing, we have

Definition 10. *Suppose we are given a variety $\mathbf{V}$ of algebras in the sense of universal algebra, and a class of set-maps m of members A of $\mathbf{V}$ into themselves, which consists of all set-maps $A \to A$ that satisfy a certain family of identities in the operations of $\mathbf{V}$, and which we call "M-maps". Then an* extended inner *M-map of an object A of $\mathbf{V}$ will mean a function associating to every object $f : A \to B$ of $(A \downarrow \mathbf{V})$ an M-map m_f of B, such that for every morphism $h : B_1 \to B_2$ in $(A \downarrow \mathbf{V})$, one has*

$$m_{f_2}\,h \;=\; h\,m_{f_1}. \tag{39}$$

Clearly the concept of an *inner endomorphism* of an object of a general category $\mathbf{C}$ given by Definition 4, when restricted to the case where $\mathbf{C}$ is a variety $\mathbf{V}$ of algebras, agrees with the above definition. (The inner *auto*morphisms of an object of a variety are then the inner endomorphisms (β_f) such that all β_f are invertible.) On the other hand, systems of maps as in the hypothesis of Theorem 8 can now, as desired, be described as the *extended inner derivations* of our associative algebra R. In the next section we shall similarly consider extended inner derivations of Lie algebras.

Digression: Having talked about several versions of the concept of a derivation, let me for completeness recall two more, though I will not discuss "extended inner" versions of these.

The most general of the versions mentioned above, that of an (h, h')-derivation, is generalized further by the concept of a derivation from a K-algebra S to an (S, S)-bimodule B, formally defined by our original formulas (25), (26) and (27). In the last of these formulas, and in the analog of (28) defining the concept of an *inner* derivation $S \to B$, the "multiplication" on the right-hand sides of these equations is taken to be that of the bimodule structure. (Thus, (37) and (38) are the cases of (27) and (28) where S_2 is made an (S_1, S_1)-bimodule by letting S_1 act on the left via the images of its elements under h, and on the right via their images under h'.) Such a derivation is equivalent to a homomorphism $S \to S \oplus B$, where $S \oplus B$ is made a K-algebra under a multiplication based on the multiplication of S, the bimodule structure of B, and the trivial internal multiplication of B. Each *inner* derivation $S \to B$ corresponds to conjugation by a unit of the form $1 + b$ $(b \in B)$.

Finally, in group theory, one sometimes speaks of a left or right derivation d from a group G to a group N on which G acts by automorphisms. If we write the action of G on N as left superscripts in the case of left derivations, and as right superscripts in the case of right derivations (requiring it to be a left action in the former case and right action in the latter), and denote the group operation of N by "$\cdot$" (to avoid confusion as to which elements such superscripts are attached to), then the identities characterizing these two sorts of maps are

$$d(a\,b) \;=\; d(a)\cdot {}^{a}d(b), \qquad \text{respectively,} \qquad d(a\,b) \;=\; d(a)^{b}\cdot d(b). \tag{40}$$

In the special case where N is abelian, this concept of derivation can be reduced to the preceding one. Indeed, note that a left or right action of G on N is equivalent to a structure on N of left or right module over the group ring $\mathbb{Z}\,G$. To supply an action on the other side, and so make N a $(\mathbb{Z}\,G, \mathbb{Z}\,G)$-bimodule, we map $\mathbb{Z}\,G$ to $\mathbb{Z}$ by the augmentation map, then use the unique action of $\mathbb{Z}$ on any abelian group. A derivation $G \to N$ in the sense of (40) (written now with "$+$" instead of "$\cdot$") is then a derivation from the ring $R = \mathbb{Z}\,G$ to its bimodule N.

Returning to ring theory, let me note yet another way "inner" has been used, possibly related to that of this note. Automorphisms and derivations of a K-algebra R are *actions* on R of certain Hopf algebras, and students of Hopf algebras have defined what it means for an action of an arbitrary Hopf algebra on an algebra to be inner [8] [22] [26]. I am out of my depth in this situation, and do not know how close that concept is to the concepts of inner automorphisms and derivations defined here; but it appears to me that it would be difficult to embrace under the action of a single Hopf algebra (or bialgebra) the class of constructions (19) with n ranging over all positive integers.

As noted in [8], another case of an inner action of a Hopf algebra on an algebra R gives us a concept of an inner *grading* of R by a given group or monoid. It would be interesting to explore the concept of an "extended inner grading".

8. Inner derivations and inner endomorphisms of Lie algebras.

Let K be a commutative ring, and $\mathbf{Lie}_K$ the variety of Lie algebras over K. Derivations $d : L \to L$ are defined for Lie algebras as for associative algebras, with (25) and (26) unchanged, and with Lie brackets replacing multiplication in the analog of (27):

$$d([r,\,s]) \;=\; [d(r),\,s] \;+\; [r,\,d(s)]. \tag{41}$$

The derivations of any Lie algebra L or associative algebra R themselves form a Lie algebra under *commutator brackets*. For L a Lie algebra, there is a natural homomorphism from L to its Lie algebra of derivations, called the *adjoint map*, taking each $s \in L$ to the derivation $\mathrm{ad}_s : L \to L$ given by

$$\mathrm{ad}_s(t) \;=\; [s,\,t] \quad (t \in L). \tag{42}$$

For each s, ad_s is called the *inner derivation* of L determined by s.

Clearly, each $s \in L$ induces in this way an *extended inner derivation* of L in the sense of Definition 10. To investigate whether these are the only extended inner derivations, let $B : \mathbf{Ring}^1_K \to \mathbf{Lie}_K$ be the functor that sends each associative K-algebra R to the Lie algebra having the same underlying K-module as R, and Lie brackets given by commutator brackets,

$$[s,\,t] \;=\; s\,t - t\,s. \tag{43}$$

We recall that the functor B has a left adjoint, the *universal enveloping algebra* functor $E : \mathbf{Lie}_K \to \mathbf{Ring}^1_K$. (The Poincaré-Birkhoff-Witt Theorem tells us, inter alia, that if L is a Lie algebra over a field, then the natural map $L \to B(E(L))$ is an embedding.)

Now suppose (d_f) is an extended inner derivation of L. The adjointness relation between B and E tells us that for S an associative K-algebra, homomorphisms $E(L) \to S$ as associative algebras correspond to Lie homomorphisms $L \to B(S)$; hence if we apply our extended inner derivation to the latter homomorphisms, we get for every object $f : E(L) \to S$ of $(E(L) \downarrow \mathbf{Ring}^1_K)$ a derivation of the Lie algebra $B(S)$, in a functorial manner. The condition of being a derivation of $B(S)$ as a Lie algebra is weaker than that of being a derivation of S as an associative algebra, so we can't apply Theorem 8 directly to this family of derivations. The family will, however, by the same arguments as before, be determined by an element $w(x) \in E(L)\langle x\rangle$; and will satisfy (25) and (26), which we have seen are equivalent to $w(x)$ having the form $\sum a_i\,x\,b_i$, with

a_i, b_i now taken from $E(L)$. The difference between the associative case and the Lie case rears its head in the equation saying that our induced maps satisfy (41). This involves commutator brackets in $E(L)\langle x_0, x_1\rangle$ in place of its associative multiplication; thus, instead of (32) we get

$$\begin{aligned}(44)\qquad &\textstyle\sum_1^n a_i(x_0\,x_1 - x_1\,x_0)b_i \\ &\qquad = ((\textstyle\sum_1^n a_i\,x_0\,b_i)\,x_1 - x_1\,(\sum_1^n a_i\,x_0\,b_i)) + (x_0\,(\sum_1^n a_i\,x_1\,b_i) - (\sum_1^n a_i\,x_1\,b_i)\,x_0).\end{aligned}$$

The added complexity is illusory, however! Writing R for $E(L)$, note that the terms of (44) lie in the direct sum of two components $R\,x_0\,R\,x_1\,R \oplus R\,x_1\,R\,x_0\,R \subseteq R\langle x_0, x_1\rangle$. If we project (44) onto the first of these, we get precisely (32), and we can repeat the computations that led us to Theorem 8. A left inverse to the canonical map $K \to E(L)$, as needed for the proof of that theorem, is supplied by the algebra homomorphism

$$(45)\qquad E(L) \;\to\; K$$

that we get on applying E to the trivial map $L \to \{0\}$.

What those computations now tell us is that there is a $b \in E(L)$ such that $w(x) = x\,b - b\,x$, so that for an object $f : E(L) \to S$ of $(E(L) \downarrow \mathbf{Ring}^1_k)$, the induced derivation on $B(S)$ is given by the operation of commutator bracket with $f(b)$. (This shows, incidentally, that that derivation of the Lie algebra $B(S)$ is in fact a derivation of the associative algebra S.) If we now assume that K is a field, so that every Lie algebra M can be identified with its image in $E(M)$, we see that given any Lie algebra homomorphism $f : L \to M$, the resulting derivation $d_f : M \to M$ can be described within $E(M)$ as commutator brackets with $f(b)$. (Here we are using the fact that by the functoriality of our extended inner derivation, its behavior on M is the restriction of its behavior on $B(E(M))$.) Also, since elements of K induce the zero derivation, we can assume without loss of generality that the constant term of b (its image under (45)) is zero.

This reduces our problem to the question: what elements $b \in E(L)$ with constant term 0 have the property that for every $f : L \to M$, the operation of commutator brackets with the image of b in $E(M)$ carries $M \subseteq E(M)$ into itself? Equivalently, what elements b with constant term zero have the property that the element $w(x) = x\,b - b\,x \in E(L)\langle x\rangle$ lies in the Lie subalgebra of $E(L)\langle x\rangle$ generated by L and x? Clearly, all $b \in L$ have this property. Are they the only ones?

If the field K has positive characteristic p, the answer is no. It is known that in this case the p-th power of a derivation of a Lie or associative algebra is again a derivation, and in particular, that the p-th power of the inner derivation of an associative algebra determined by an element a is the inner derivation determined by a^p. (This, despite the fact that the p-th power map does not in general respect addition on noncommutative K-algebras.) For nonzero $a \in L \subseteq E(L)$, the element $a^p \in E(L)$ will not lie in L; so commutator brackets with such elements give extended inner derivations of L that do not come from inner derivations in the traditional sense.

The abovementioned fact about p-th powers of derivations in characteristic p leads to the concept of a *p-Lie algebra* (or *restricted* Lie algebra of characteristic p [17, §V.7]): a Lie algebra L over a field of characteristic p with an additional operation of "formal p-th power", $a \mapsto a^{[p]}$, satisfying appropriate identities. For this class of objects, one has a "restricted universal enveloping algebra" construction, $E_p(L)$, where relations are imposed making the formal p-th powers of elements in the p-Lie algebra coincide with their ordinary p-th powers in the enveloping algebra. As we shall note below, this leads to a modified version in characteristic p of the question whose unmodified form we just answered in the negative.

When K has characteristic 0, I suspect that a Lie algebra L has no extended inner derivations other than those induced by elements $b \in L$. (If there existed general constructions in this case, like the p-th power operator in the characteristic-p case, one would expect the phenomenon to be well-known!) In any case, we ask

Question 11. *If L is a Lie algebra over a field of characteristic 0, can commutator brackets with elements $b \in E(L)$ of constant term zero, other than elements of L, induce extended inner derivations of L?*

Same question for L a p-Lie algebra over a field of characteristic $p > 0$, and $b \in E_p(L)$.

These are equivalent to the questions of whether there can exist in $E(L)$ (respectively in $E_p(L)$) elements b of constant term zero not lying in L, with the property that the element $w(x) = x\,b - b\,x$ belongs to the Lie subalgebra (respectively the p-Lie subalgebra) of $E(L)\langle x\rangle$ (respectively $E_p(L)\langle x\rangle$) generated by L and x.

Just as we have used, above, our analysis of extended inner derivations on associative algebras in studying extended inner derivations on Lie algebras, so we can do the same for extended inner endomorphisms of

Lie algebras, again assuming K a field. If we copy the development of Theorem 6, taking $R = E(L)$, and using commutator brackets in place of products, we can again get from an extended inner endomorphism of a Lie algebra L an element $w(x) \in E(L)\langle x\rangle$, which we find will have the form $\sum a_i\, x\, b_i$ for $a_i,\ b_i \in E(L)$; and the map it induces will respect commutator brackets on objects of $(E(L) \downarrow \mathbf{Ring}^1_K)$. That property is equivalent to a formula like (15), but with components in both $R\,x_0\,R\,x_1\,R$ and $R\,x_1\,R\,x_0\,R$. Again, projection onto the $R\,x_0\,R\,x_1\,R$ component gives us precisely our old formula, in this case (15). As in §2, this yields (17).

However, homomorphisms of Lie algebras satisfy no analog of the condition of sending 1 to 1; so we do not have (12), and cannot deduce (13). What does (17) tell us without (13)? It says that the identity endomorphism of the free right $E(L)$-module of dimension n factors through the free right $E(L)$-module of dimension 1.

Now $E(L)$ admits a homomorphism to the field K, namely (45), so such a factorization of maps of free modules can only exist if such a factorization exists for modules over K, i.e., if $n \leq 1$. If $n = 0$ then $w(x) = 0$, and in contrast to the case of unital associative rings, this indeed corresponds to an inner endomorphism of L in $\mathbf{Lie}_K$. If $n = 1$, then (17) becomes $b_1 a_1 = 1$. From the fact that the K-algebra $E(L)$ has a filtration whose associated graded ring is a polynomial ring over K, it follows that, like a polynomial ring, it has no 1-sided invertible elements other than the nonzero elements of K; so $a_1,\ b_1 \in K$, and we conclude that $w(x) = x$. Hence,

Theorem 12. *If L is a Lie algebra over a field K, then its only extended inner endomorphisms are the zero endomorphism and the identity automorphism.* □

The above result, even in the characteristic-p case, concerns ordinary Lie algebras, not p-Lie algebras. If K is a field of characteristic $p > 0$, and L a p-Lie algebra over K, we can begin the analysis of extended inner endomorphisms of L as above, with $E_p(L)$ in place of $E(L)$, and go through much the same argument, using as before the fact that $w(x)$ respects Lie brackets, and conclude that every extended inner endomorphism is either zero, or induced by an element $w(x) = a\,x\,b$ for $a,\ b \in E_p(L)$ satisfying $b\,a = 1$. (Note that this automatically implies that $w(x^p) = w(x)^p$.) But we can no longer say that the relation $ba = 1$ implies that $a,\ b \in K$. For example, if u is an element of L such that $u^{[p]} = 0$, then in $E_p(L)$ we have $u^p = 0$, so $1 - u$ is a nonscalar invertible element. Hence we ask

Question 13. *Can a p-Lie algebra L over a field K have a nonzero non-identity extended inner endomorphism?*

Equivalently, can $E_p(L)$ have elements $a,\ b$, not in K, satisfying $b\,a = 1$, and such that in $E(L)\langle x\rangle$, the element $w(x) = a\,x\,b$ lies in the p-Lie subalgebra generated by L and x ?

The first part of the above question can be divided in two: Can such an L have an extended inner *automorphism* that is not the identity? and can it have a nonzero extended inner endomorphism that is not an extended inner automorphism? The latter possibility can in turn be divided in two: There might be an invertible element, conjugation by which carries the p-Lie subalgebra generated by L and x into, but not onto, itself, or the extended inner endomorphism might arise from elements $a,\ b$ such that $ba = 1$ but $ab \neq 1$; I do not know whether an enveloping algebra $E_p(L)$ can contain one-sided but not two-sided invertible elements.

However, we can again say that a nonzero extended inner endomorphism is everywhere one-to-one. For if our $w(x) = a\,x\,b$, and if on mapping x to some element $u \in L'$ under a map $L\langle x\rangle \to L'$, we get $a\,u\,b = 0$, then by multiplying this equation on the left by b and on the right by a, we find that $u = 0$.

It is natural to ask whether the methods we have used to study inner automorphisms, inner endomorphisms, and inner derivations of associative and Lie algebras are applicable to other classes of not-necessarily-associative algebras. Our results for associative algebras used the descriptions (8) and (9) of the free extensions $R\langle x\rangle$ and $R\langle x_0,\, x_1\rangle$ of an algebra R; and our partial results for Lie algebras were based on reduction to the associative case. For most varieties of K-algebras, the descriptions of the universal one- and two-element extensions of an algebra are not so simple. I have not examined what can be proved in such cases.

9. Co-inner endomorphisms.

If A is an object of a category $\mathbf{C}$, there is a construction dual to that of $(A \downarrow \mathbf{C})$, namely $(\mathbf{C} \downarrow A)$, the category whose objects are objects of $\mathbf{C}$ given with maps to A, and morphisms making commuting triangles with those maps. Thus, we may dualize Definition 4, and define an *extended co-inner* endomorphism of an object A of $\mathbf{C}$ to mean an endomorphism E of the forgetful functor $(\mathbf{C} \downarrow A) \to \mathbf{C}$, and a *co-inner* endomorphism of A itself to mean the value of such a morphism on the identity map of A.

I don't know of important naturally occurring examples, and I suspect that if the concept turns out to be useful, it will be so mainly in areas other than algebra; but let us make a few observations on the algebra case.

Let $\mathbf{V}$ be a variety of algebras in the sense of universal algebra. We begin with the weaker concept of an extended co-inner *set-map* of A; that is, an endomorphism E of the composite of forgetful functors

$$(\mathbf{V} \downarrow A) \longrightarrow \mathbf{V} \longrightarrow \mathbf{Set}. \tag{46}$$

To analyze such a mapping, let us, for each $a \in A$, consider the object of $(\mathbf{V} \downarrow A)$ given by the homomorphism from the free $\mathbf{V}$-object on one generator, $\langle x \rangle_{\mathbf{V}}$, to A, that takes x to a. If we apply our co-inner set-map E to this homomorphism, we get a set-map $\langle x \rangle_{\mathbf{V}} \to \langle x \rangle_{\mathbf{V}}$; this will take x to some element $w_a(x) \in \langle x \rangle_{\mathbf{V}}$; thus we get a family of such elements $w_a(x) \in \langle x \rangle_{\mathbf{V}}$, one for each $a \in A$. We see that this family will determine E; namely, for every object $f : B \to A$ of $(\mathbf{V} \downarrow A)$, and every element $b \in B$, E_f will take b to $w_{f(b)}(b)$. Clearly, any A-tuple $(w_a(x))_{a \in A}$ of elements of $\langle x \rangle_{\mathbf{V}}$ yields such an extended "co-inner set map" E. (Remark: though there is an added complexity relative to the case of an extended inner endomorphism of an algebra, in that we now have a family of elements $w_a(x)$ rather than a single element $w(x)$, there is a corresponding decrease in complexity, in that these lie in $\langle x \rangle$, rather than $A\langle x \rangle$.)

For most $\mathbf{V}$, few extended co-inner set maps will give endomorphisms of the algebras B. One way to get examples which do so is to take all $w_a(x)$ the same, with value giving what we called in §5 an "absolute endomorphism of $\mathbf{V}$". E.g., for $\mathbf{V} = \mathbf{Ab}$ and A any abelian group, we may take all $w_a(x)$ equal to $n\,x$ for a fixed n. More generally, for $\mathbf{V}$ the variety of modules over a ring R and A any such module, we may take all $w_a(x)$ equal to $c\,x$ for a fixed element c of the center of R.

However, here is a class of cases in which not all co-inner endomorphisms are based on absolute endomorphisms.

Theorem 14. *Let G be a group, let $\mathbf{Set}_G$ be the variety of right G-sets, let A be an object of this variety, and let S be a set of representatives of the orbits of A under G.*

Then every extended co-inner endomorphism E of A in $\mathbf{Set}_G$ is an extended co-inner automorphism*, and may be constructed by choosing, for each $s \in S$, an element g_s of the centralizer in G of the stabilizer G_s of s, and for each $s\,h \in A$ $(s \in S,\ h \in G)$, letting $w_{sh}(x) = x\,h^{-1} g_s\,h$.*

The extended co-inner endomorphisms of A thus form a group, isomorphic to the direct product, over $s \in S$, of the centralizers of the stabilizer subgroups G_s.

Proof. To get an extended co-inner endomorphism of A, we must choose for each $a \in A$ an element $w_a(x)$ of the free G-set on one generator, which we will denote $x\,G$, in a way that makes the resulting extended inner set-map consist of morphisms of G-sets. By the structure of $x\,G$, we see that for each $a \in A$ we have $w_a(x) = x\,g_a$ for a unique $g_a \in G$.

The condition for these maps to induce morphisms of G-sets is that for every $a \in A$ and $h \in G$, $w_a(x)\,h = w_{ah}(x\,h)$, in other words

$$x\,g_a\,h \;=\; (x\,h)\,g_{ah}. \tag{47}$$

The above equality is equivalent to $g_a\,h = h\,g_{ah}$, or solving for g_{ah},

$$g_{ah} \;=\; h^{-1} g_a\,h \quad (a \in A,\ h \in G). \tag{48}$$

If h lies in the stabilizer subgroup G_a, we have $a\,h = a$, so $g_{ah} = g_a$, so in this case (48) says that g_a commutes with h. Hence g_a lies in the centralizer of G_a. For general h, (48) allows us to compute g_{ah} from g_a, hence, the system of elements g_a will be determined by those such that a lies in our set of coset representatives S, and the value at each $s \in S$ will belong to the centralizer of G_s.

For elements g_s so chosen, it is now easy to verify that we indeed get an extended co-inner endomorphism of A. The resulting endomorphisms are clearly invertible, and the description of the group they form is immediate. □

10. Concluding remarks.

The tools used in §§1-8 above are not new from the point of view of category-theoretic universal algebra.

If we consider the general context of "M-maps" as in §7, and then pass to the still more general context, sketched parenthetically there, of a family of additional operations on the underlying set of an object of $\mathbf{V}$, of various arities, subject to some set of identities, we see that this constitutes a structure of algebra in a variety $\mathbf{W}$ whose operations and identities include those of $\mathbf{V}$. An "extended inner system" of such operations on an object A of $\mathbf{V}$ then means a factorization of the forgetful functor $(A \downarrow \mathbf{V}) \to \mathbf{V}$ through the forgetful functor $\mathbf{W} \to \mathbf{V}$. From the point of view of the theory of representable algebra-valued functors ([15], [5, Chapter 9], [6, Chapters I-II]), this corresponds to starting with the representing object for the former forgetful functor, namely, $A\langle x\rangle$ with the canonical system of co-operations that make it a co-$\mathbf{V}$-object of the variety $(A \downarrow \mathbf{V})$, and enhancing that co-$\mathbf{V}$-structure in an arbitrary way to a co-$\mathbf{W}$-structure; i.e., supplying additional co-operations which co-satisfy the identities of $\mathbf{W}$. These co-operations will be determined by their actions on the element x, so by studying the images of x under them, one may attempt to determine the form that the additional co-operations can take.

Thus, what we have been doing falls under the general study of representable functors and the coalgebras that represent them. I consider the contribution of this note not to lie in the maximum generality to which the concepts could be pushed (which comes to that existing general theory), but, inversely, in the focus on a specific class of such problems: those where an added unary operation constitutes a type of additional structure on the objects in question that is already of interest, e.g., an endomorphism, or a derivation. We have gotten exact descriptions of the possibilities for this structure in several such cases, and shown the technique that can be applied to further cases.

Of course, if this note leads some readers to an interest in the general theory of coalgebras and representable functors among varieties of algebras [15], [5, Chapter 9], [6], I will be all the more pleased.

11. Appendix: Inner endomorphisms of associative algebras are one-to-one.

We noted in the second paragraph after Theorem 6 that the one-one-ness of every inner endomorphism of an associative unital algebra R over a field K, which follows from that theorem, also has an elementary proof, using the fact that R can be embedded in a simple K-algebra. We prove below a different embedding result, from which we deduce, more generally, the one-one-ness of all inner endomorphisms of associative unital algebras over arbitrary K.

Below, K is, as usual, a commutative associative unital ring, and $\otimes$ denotes $\otimes_K$. K-algebras are here understood to be associative and unital.

Lemma 15. *Every K-algebra R admits an embedding $f : R \to S$ in a K-algebra S with the property that for every nonzero $r \in R$, the ideal $S\,f(r)\,S$ contains a nonzero element of the center of S.*

Proof. Given R, first form the K-module $R \otimes R$, and note that the two maps $R \to R \otimes R$ given by $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ are one-to-one, since the map $R \otimes R \to R$ induced by the internal multiplication of R gives a left inverse to each of them.

Now regard $R \otimes R$ as a K-algebra in the usual way, i.e., so that $(r_1 \otimes r_2)\cdot(r'_1 \otimes r'_2) = (r_1 r'_1 \otimes r_2 r'_2)$. By the above observation, $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ are embeddings of K-algebras. Note that their images centralize one another, and that the map $\theta : R \otimes R \to R \otimes R$ defined by $\theta(r_1 \otimes r_2) = r_2 \otimes r_1$ is an automorphism of $R \otimes R$. Using this automorphism, let us form the twisted polynomial algebra $(R \otimes R)[t;\theta]$; i.e., adjoin to $R \otimes R$ an indeterminate t satisfying

$$t\,(r_1 \otimes r_2) \;=\; (r_2 \otimes r_1)\,t \quad \text{for } r_1, r_2 \in R. \tag{49}$$

Within $(R \otimes R)[t;\theta]$, we now take the subalgebra of elements whose constant terms lie in $R \otimes 1$, and let S be the quotient of this subalgebra by the ideal of all elements in which t appears with exponent > 2. Thus, as a K-module,

$$S \;=\; (R \otimes 1) \;\oplus\; (R \otimes R)\,t \;\oplus\; (R \otimes R)\,t^2. \tag{50}$$

We now define our algebra embedding $f : R \to S$ by

$$f(r) \;=\; r \otimes 1. \tag{51}$$

For every nonzero $r \in R$, the ideal $S\,f(r)\,S$ contains the element

$$t\,f(r)\,t \;=\; t\,(r \otimes 1)\,t \;=\; (1 \otimes r)\,t^2, \tag{52}$$

which we see from the right-hand side of the above equation is nonzero. Because this element involves t to the second power, it annihilates on both sides the summands of (50) involving t. It also centralizes the summand $R \otimes 1$, since the factors $1 \otimes r$ and t^2 both do so. So (52) gives the desired nonzero central element. □

To make use of this result, recall that in our category of K-algebras, an endomorphism of an object by definition fixes the unit, and that in §2 we translated this to the condition (13) on extended inner endomorphisms. The proof of the next result shows that in this respect, extended inner endomorphisms cannot tell the difference between the unit and other R-centralizing elements.

Lemma 16. *If R is a K-algebra, and (β_f) an extended inner endomorphism of R, then for every homomorphism $f : R \to S$ of K-algebras, the endomorphism β_f of S fixes all elements of S that centralize $f(R)$, hence, in particular, all elements of the center of S.*

Proof. By abuse of notation, let us use the same symbols for elements of R and their images in S. If $c \in S$ centralizes R, then applying (11) to c, and commuting c past the coefficients $b_i \in R$, we get $\beta_f(c) \;=\; \sum_1^n a_i\,b_i\,c$, which by (13) simplifies to c. □

We can now deduce

Proposition 17. *Every inner endomorphism α of an associative unital K-algebra R is one-to-one.*

Proof. Given R, take an embedding $R \to S$ as in Lemma 15. Thus for any nonzero $r \in R$, $S\,f(r)\,S$ contains a nonzero central element c. By Lemma 16, $\beta_f(c) = c$. So

$$0 \;\neq\; c \;=\; \beta_f(c) \;\in\; \beta_f(S\,f(r)\,S) \;\subseteq\; S\,\beta_f(f(r))\,S \;=\; S\,f(\alpha(r))\,S, \tag{53}$$

so $\alpha(r) \neq 0$. □

Incidentally, in Lemma 15, we made our construction satisfy the strong conclusion that $S\,f(r)\,S$ have nonzero intersection with the center of S, since that seemed of independent interest; but for the proof of Proposition 17, it would have sufficed that $S\,f(r)\,S$ have nonzero intersection with the centralizer of $f(R)$. This could have been achieved by the simpler construction

$$S \;=\; (R \otimes R)[t;\theta], \tag{54}$$

with $f : R \to S$ again defined by $f(r) \;=\; r \otimes 1$. Indeed, $t\,f(r)\,t \;=\; (1 \otimes r)\,t^2$ clearly still centralizes $f(R) = R \otimes 1$.

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