Infinite groups arising from partial presentations of finite groups

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Abstract

If G is a finite group, X a generating set for G, and R some set of relations holding among these generators (but not in general a full set of relations presenting G), we establish sufficient conditions for the group $G* = \langle X \mid R \rangle$ to be infinite, and give a lower bound for the rank of the abelianization of the kernel of the natural map $G* \to G$. Related results in the literature and possible directions for generalization are discussed.

1. *Introduction*

Let G be a finite group, let $\alpha: F \to G$ be a homomorphism from the free group on a finite set X onto G, let $N = \ker \alpha$, and let R be a finite subset of N. Denote by $N* \subseteq N$ the normal subgroup of F generated by R, and let $G* = \langle X \mid R \rangle = F/N*$, a group with a canonical surjective homomorphism $\alpha*: G* \to G$.

Clearly, if R is "small enough", $G*$ will be infinite. But how small is "small enough"?

By a standard argument (namely, "abelianize!"), $G*$ is infinite if R has smaller cardinality than X, i.e., if there are fewer relations than generators. R. M. Thomas [**10**] proves the striking fact that this statement remains true if one counts a relator that is a proper power, $r = s^n \in R$, as giving "$1/n$ of a relation", subject to two provisos. First, the hypothesis that the number of generators is greater than the number of relations must be adjusted to say that the former number exceeds the latter (with relations counted as indicated) by at least 1. Second, in every relation $s^n = 1$ that we count in this way, n must actually be the order of the image of s in G, and not a proper multiple thereof.

(The need to have the difference between number of generators and adjusted relation-count at least 1 is illustrated by the case of a finite cyclic group of order n, with one generator and "$1/n$ of a relation", where the difference $1 - 1/n$ is positive, but the group is finite. To see that we also cannot drop the condition that for a relation $s^n = 1$ to be so counted, n must actually be the order of s in G, take any finite presentation of a finite group G with more than one generator, and note that by replacing each relation $s = 1$ by two relations $s^n = 1$, $s^{n+1} = 1$ for sufficiently large n, we can get a presentation of G with a set of relations whose adjusted count, ignoring that restriction, is less than any positive ε, in particular, $\leq$ the number of generators minus 1.)

In this note, we will strengthen the above result in two ways. First, the *cyclic* subgroups of G generated by the images of the elements s will be generalized to *arbitrary* subgroups. Second, our argument will not only prove $G*$ infinite, but will show that the kernel of the natural map $\alpha*: G* \to G$ has infinite abelianization, and will yield a lower bound on the rank of this abelianization in terms of the difference between number of generators and adjusted relation-count.

The first of these improvements is not by itself as strong as it may sound: What we shall show

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(Corollary 7 below) is that for a family $s_1, \ldots, s_m$ of elements of F, if we include in our presentation of $G*$ all relations satisfied by the images of these elements in G, then these can be ''counted'' as $m-1+1/n$ relations, where n is the order of the group generated by the images of $s_1, \ldots, s_m$ in G; in other words, approximately one less than the number of relations we would have imposed if we had set $s_1, \ldots, s_m$ all equal to 1. However, by taking advantage of possible intersection among the families of elements that we treat in this way, we shall get a stronger (if less easily stated) result, Theorem 6. Concerning the second of the above improvements, I am not entirely satisfied with measuring the size of $\ker \alpha*$ via its abelianization; I will suggest a possible stronger result in the last section.

I should say that I have restated Thomas's result slightly: his formulation is that if a group presented using a family of relations of the form $s_i^{n_i} = 1$ is *finite*, and the order of each s_i in that group is in fact n_i, then $1 + \Sigma\, 1/n_i$ is greater than the number of generators. However, he applies this to prove various groups infinite, and in doing so, the canonical way to show that the order of each s_i is actually n_i is, as above, to find a finite homomorphic image G of that group in which this holds.

I know of a few other papers in this area. Allcock [**1**] (an early version of which inspired the present results) considers presentations of the same form as Thomas's; like us, he measures the size of the resulting group by estimating the ranks of the abelianizations of its subgroups.

Baumslag and Shalen [**2**] also prove that certain groups presented using relations of the form $s_i^{n_i} = 1$ are infinite, but instead of assuming each s_i is of order precisely n_i, they require that all the n_i be equal, and be a prime power. (Clearly, this excludes the ''$s^n = 1$, $s^{n+1} = 1$'' counterexamples noted above!) In [**4**] they obtain similar results, without the ''prime power'' half of their hypothesis, but for dimensions of augmented algebras, rather than orders of groups.

A special case of the numerology of sums of reciprocals of exponents that has long been known is the theory of triangle groups, which seems to have been the starting point for most of these investigations. For results on other families of groups, see [**11**].

The diversity of techniques used in proving the results described above is notable. [**10**] and [**1**] use 1- and 2-dimensional topology (graphs and 2-complexes) respectively, [**2**] uses the algebraic geometry of matrix representations, and the present note uses semidirect product groups.

A paper of which I have learned more recently, [**6**], is not very similar to those mentioned above in the form of its results, but is quite close to the present paper in technique. Put in notation parallel to that of this paper, its main result says that if a group $G*$ has a finite homomorphic image G such that some $\mathbf{Q}G$-module has nontrivial first cohomology as a $\mathbf{Q}G*$-module, then $G*$ is infinite, and in fact the abelianization of the kernel of the map $G* \to G$ has rank at least that of an irreducible $\mathbf{Q}G$-module summand of that cohomology group. The results of the present paper could probably be ''piped through'' that of [**6**] by turning them into results about cohomology groups, but I am not the person to do this, and have not made the attempt. [**6**] also recovers Thomas's original result, as Theorem 2.5.

I am indebted to D. Allcock, R. Alperin, M. Conder and W. Dicks for several helpful references to the literature.

2. *Definitions, and the plan of the proof*

Let G be a finite group, $\alpha: F \to G$ a homomorphism from the free group F on a finite set X onto G, and $N = \ker \alpha$. Let $S_1, \ldots, S_m$ be arbitrary finite sets (not necessarily disjoint), and let us be given a map $\beta: \cup\, S_i \to F$. Using the notation $\langle \ldots \rangle$ for ''subgroup generated by'', let $N*$ denote the normal subgroup of F generated by $\cup_i\, (N \cap \langle \beta(S_i) \rangle)$, and let $G* = F/N*$. Thus, $G*$ is presented by an image of the same generating set X as G, and relations saying that the image in $G*$ of each of the subgroups $\langle \beta(S_i) \rangle$ of F looks like the image of that same subgroup in G. We shall write $\alpha*$ for the canonical map $G* \to G$.

To study the abelianization of $\ker\alpha*$ is equivalent to studying the universal homomorphic image of $G*$ in which that kernel becomes abelian. To do this, we shall start with a fairly ''generic'' example of a group with a homomorphism onto G having abelian kernel, namely the semidirect product $kG \rtimes G$, where kG is the group algebra of G over a field k of characteristic zero, and consider X-tuples of elements of this group which map to the generating X-tuple of elements of G, and which satisfy the relations described by the sets $N \cap \langle\beta(S_i)\rangle$, but not necessarily all the relations described by N. An X-tuple with this property yields a factorization of the map $G* \to G$ through $kG \rtimes G$, and the kernel of the map that projects the resulting image of $G*$ in $kG \rtimes G$ down to G will be a torsion-free abelian homomorphic image of $\ker\alpha*$.

How will we prove that there exist X-tuples for which this kernel is nonzero? Each X-tuple of elements of $kG \rtimes G$ that projects to the canonical X-tuple in G is determined by an X-tuple of elements of kG. We shall estimate the dimension of the k-space of X-tuples inducing families of elements of $kG \rtimes G$ that satisfy the relations determined by the sets $N \cap \langle\beta(S_i)\rangle$, and also that of the space of X-tuples inducing families which satisfy the larger set of relations determined by N, and establish conditions under which the former dimension must exceed the latter. A key tool in these dimension counts will be the fact that subgroups of $kG \rtimes G$ that project isomorphically to a given subgroup $H < G$ correspond to derivations $H \to kG$. By Hochschild's Theorem, these derivations are inner (because G is finite), and inner derivations are easy to ''count''. Our dimension-counting will not only show that the abelianization of $\ker\alpha*$ has a nonzero torsion-free homomorphic image, but will give us a lower bound on its rank as an abelian group.

3. *Some lemmas*

In this section, we suspend our assumptions on G and prove some general results.

Recall that if G is a group, and A a group on which G acts on the left by automorphisms, then a (left) derivation from G to A means a set-map $d\colon G \to A$ such that for all $g, h \in G$,

$$d(gh) \;=\; d(g)\; g(d(h)), \tag{1}$$

from which one easily obtains

$$d(1) \;=\; 1, \qquad d(g^{-1}) \;=\; g^{-1}(d(g)^{-1}). \tag{2}$$

The importance of such maps arises from the fact that if we form the semidirect product $A \rtimes G$, then a map $G \to A \rtimes G$ of the form $g \mapsto (d(g), g)$ is a group homomorphism if and only if $d\colon G \to A$ is a derivation. If a is any element of A, then conjugating the canonical embedding, $g \mapsto (1, g)$, by the element $(a, 1)$, we get the homomorphism $g \mapsto (a(g(a))^{-1}, g)$; hence the map $g \mapsto a(g(a))^{-1}$ is a derivation, called the *inner derivation* $d_a\colon G \to A$ determined by a.

In our context A will be abelian (in fact, a vector space) and we will write its operation additively, so that the definition (1) and consequences (2) become

$$d(gh) \;=\; d(g) + g\,d(h), \tag{3}$$

$$d(1) \;=\; 0, \qquad d(g^{-1}) \;=\; -g^{-1}d(g), \tag{4}$$

and the inner derivation determined by a is given by

$$d_a(g) \;=\; a - ga. \tag{5}$$

For A abelian, the derivations $G \to A$ clearly form a subgroup of the additive group of all set-maps $G \to A$. If A is in fact a vector space on which G acts by vector-space automorphisms, we see that the derivations $G \to A$ form a subspace of the vector space of set-maps $G \to A$. A k-vector space on which

G acts is equivalent to a left kG-module, so in particular, G acts on the k-vector space kG, by left multiplication. We can now state

LEMMA 1. *Let G be a finite group and k a field. Then*

(i) *The space of derivations $G \to kG$ has dimension $[G:1] - 1$.*

(ii) *More generally, if H is a subgroup of G, then the space of derivations $H \to kG$ (functions $H \to kG$ satisfying* (3) *for all $g, h \in H$) has dimension $[G:1] - [G:H]$.*

(iii) *Still more generally, if $J < H$ are subgroups of G, then the space of derivations $H \to kG$ that restrict to the zero derivation on J has dimension $[G:J] - [G:H]$.*

Proof. Since G is finite, Hochschild's Theorem says that every derivation $G \to kG$ is inner, i.e., that the map taking each element $a \in kG$ to the derivation d_a is surjective. Moreover, if H is a subgroup of G, then as a kH-module, kG is a direct sum of copies of kH ($[G:H]$ of them), so derivations $H \to kG$ are also inner, and hence are restrictions to H of inner derivations $G \to kG$. Let us now prove assertion (iii), of which (i) and (ii) are special cases.

An inner derivation d_a on G $(a \in kG)$ restricts to zero on J if and only the coefficients of a are constant on cosets of J, and we see that the space of elements a with this property has dimension $[G:J]$. If we send each a in this space to the restriction of d_a to H, then the kernel of this restriction map can be described as the space of elements of kG inducing derivations that are zero on all of H, which will similarly have dimension $[G:H]$; so the image of the restriction map has dimension $[G:J] - [G:H]$, as claimed. □

From this we can prove

LEMMA 2. *Let G be a finite group, and γ: $S \to G$ a map from a finite set S into G. Then the k-vector space of functions $S \to kG$ which are pullbacks along γ of derivations $\langle \gamma(S) \rangle \to kG$ (i.e., which arise by composing γ with such derivations) has dimension $[G:1] - [G:\langle \gamma(S) \rangle]$.*

More generally, suppose that S as above is written as a union $S_1 \cup ... \cup S_m$ of (not necessarily disjoint) subsets. Then if we denote by U the space of maps $S \to kG$ which, when restricted to each S_i, are pullbacks along γ of derivations $\langle \gamma(S_i) \rangle \to kG$, we have

$$\begin{aligned} &\Sigma_{i \le m} [G : \langle \gamma(\cup_{j<i} S_j \cap \cup_{j \ge i} S_j) \rangle] - [G : \langle \gamma(\cup_{j \le i} S_j \cap \cup_{j \ge i} S_j) \rangle] \\ \le\ &\dim U \qquad\qquad (6) \\ \le\ &\Sigma_{i \le m} [G : \langle \gamma((\cup_{j<i} S_j) \cap S_i) \rangle] - [G : \langle \gamma(S_i) \rangle]. \end{aligned}$$

Proof. From the interpretation of derivations in terms of homomorphisms to semidirect products, we see that the restriction of a derivation d to a subgroup of G is determined by its values on any set of elements generating that subgroup. Hence in the context of the first paragraph of the Lemma, the k-linear map that pulls derivations $\langle \gamma(S) \rangle \to kG$ back to functions on S by composition with γ is one-to-one, so by point (ii) of the preceding lemma, the dimension of its image is $[G:1] - [G:\langle \gamma(S) \rangle]$.

Now assume $S_1, \ldots, S_m$ and U as in the second paragraph of the Lemma, and let us write

$$U = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_m = \{0\},$$

where U_i is the space of members of U that are zero on $S_1 \cup ... \cup S_i$. To estimate the dimension of U, we shall estimate the dimension of each U_{i-1}/U_i, and sum over i. We note that U_{i-1}/U_i can be identified with the space of restrictions of members of U_{i-1} to $S_1 \cup ... \cup S_i$. We shall describe below a space V_i of functions on $S_1 \cup ... \cup S_i$ which is contained in that space of restrictions, and a space W_i

which contains it, and show that the ith summand in the first term of (6) is the dimension of V_i, while the ith summand in the last term is the dimension of W_i.

To construct V_i, we start with the space of all functions $S \to kG$ which are induced by derivations $d\colon G \to kG$ and are zero on $(\cup_{j<i} S_j) \cap (\cup_{j \geq i} S_j)$, restrict these to S_i, extend the resulting functions to $S_1 \cup \ldots \cup S_i$ by making them zero on $\cup_{j<i} S_j$, and let V_i be the space of functions on $S_1 \cup \ldots \cup S_i$ so obtained. To see that every $v \in V_i$ is the restriction to that set of a member of U_{i-1}, assume v to be constructed, as above, from a derivation d, and extend v to all of S by using on $\cup_{j \geq i} S_j$ the values induced by d. Since we required d to give the zero function on $(\cup_{j<i} S_j) \cap (\cup_{j \geq i} S_j)$, there is no conflict of definitions, and we get a function whose restriction to each S_j is indeed induced by a derivation (0 or d according as $j < i$ or $j \geq i$); i.e., which lies in U, hence in U_{i-1}. To determine $\dim V_i$, note that each member of V_i is uniquely determined by its restriction to the set $(\cup_{j \leq i} S_j) \cap (\cup_{j \geq i} S_j) = S_i \cup ((\cup_{j<i} S_j) \cap (\cup_{j \geq i} S_j))$, and that this restriction is an arbitrary derivation-induced function on that set which is zero on $(\cup_{j<i} S_j) \cap (\cup_{j \geq i} S_j)$. Hence the dimension of V_i may be computed using Lemma 1(iii) (in view of the first sentence of this proof), which indeed yields the ith summand of the initial sum of (6).

We construct W_i in a similar manner, but starting with derivations $d\colon G \to kG$ that are merely assumed to be zero on $(\cup_{j<i} S_j) \cap S_i$. We restrict these to S_i, extend the results to $\cup_{j \leq i} S_j$ so as to be zero on all of $\cup_{j<i} S_j$, and observe that the space W_i so obtained is isomorphic to the space of derivation-induced functions on S_i which are zero on $(\cup_{j<i} S_j) \cap S_i$, which has for dimension the ith summand of the final sum of (6). Moreover, W_i is easily seen to *contain* the restrictions of all members of U_{i-1}. This completes the proof of (6). $\square$

Let us note why, for $1 < i < m$, neither of the spaces V_i, W_i constructed above need coincide precisely with the space of restrictions of members of U_{i-1} to $\cup_{j \leq i} S_j$. A function on $\cup_{j \leq i} S_j$ which is zero on $\cup_{j<i} S_j$ belongs to V_i if and only if it can be extended to S using a *common* derivation d to determine its values on all of $S_i, \ldots, S_m$. On the other hand, the general element $u \in U_{i-1}$ may involve different derivations on the images of different sets S_j $(j \geq i)$; and though we might hope that, given any $u \in U_{i-1}$, if we chose a derivation d whose restriction to S_i agrees with u, then the function which is 0 on $\cup_{j<i} S_j$ and d on $\cup_{j \geq i} S_j$ would be a member of V_i, this could in fact fail to be a well-defined function, since d may not be 0 on the sets $S_j \cap S_{j'}$ with $j < i < j'$. As for W_i, though every member of that space can be extended to the pullback of a derivation on *each* S_j, namely the pullback of the derivation 0 if $j < i$, and of the derivation d in the contrary case, these pullbacks may again fail to agree on the sets $S_j \cap S_{j'}$ with $j < i < j'$; so not every element of W_i need be induced by an element of U.

Note that the bound on $\dim U$ given in (6) depends on the order in which the family of sets S_i is listed. One might hope (I did!) for a more symmetric and exact formula for $\dim U$, like the ''inclusion-exclusion'' formula for the cardinality of a union of sets in terms of the cardinalities of their various intersections. However, I was not able to get such a result. (The ''inclusion-exclusion'' principle seems to work for entities that generate a distributive lattice, which more than two subgroups of a group or subspaces of a vector space in general do not.) The next lemma (which we will not use) records, however, a special class of cases where (6) is sharp.

LEMMA 3. *The three terms of* (6) *are equal if* $m \leq 2$, *or more generally, if for every* $s \in S$ *the set* $\{i \mid s \in S_i\}$ *forms a contiguous block* $\{i \mid j \leq i \leq j'\}$ *for some* $j \leq j'$. *Still more generally, for a given* i, *the* i*th summands on the left- and right-hand sides of* (6) *are equal if every* $s \in S$ *which belongs to some* S_j *with* $j < i$ *and to some* $S_{j'}$ *with* $i < j'$ *also belongs to* S_i.

Proof. If the condition on S_i stated in the final sentence above holds, then we see that the intersections-of-unions occurring in the ith summands of the left- and right-hand sides of (6) are the same subsets of S. This yields the final sentence of the Lemma, from which the first sentence follows. (Note, incidentally, that the conditions of the final sentence hold vacuously for $i=1$ and $i=m$.) $\square$

In the next section, Lemma 2 will be used to ''count'' homomorphisms $G* \to kG \rtimes G$ with second component the canonical map $\alpha*$, and we will then compare the class of such homomorphisms with the subclass of those whose images are naturally isomorphic to G. A homomorphism of the first sort will fail to have the latter property if and only if it is nontrivial on the kernel M of $\alpha*$, in which case it induces a nonzero homomorphism $M \to kG$. The corollary to the next lemma (with $H = G*$) will then be used to show that if there are ''many'' such homomorphisms, then M has ''big'' abelianization.

LEMMA 4. *Let G be a group, k a field, and $\pi: kG \to k$ the k-linear functional taking each $x \in kG$ to the coefficient of 1 in x. Then for any left kG-module A, the vector-space map*

$$\mathrm{Hom}_{kG}(A,\, kG) \ \to\ \mathrm{Hom}_k(A,\, k)$$

given by composition on the left with π is one-to-one, and if G is finite, this map is also onto.

Proof. If $v \in \mathrm{Hom}_k(A,k)$ is obtained from $u \in \mathrm{Hom}_{kG}(A,kG)$ by left composition with π, then u can be reconstructed from v; namely, for any $a \in A$, we see that

$$u(a) \ = \ \Sigma_{g\in G}\ v(g^{-1}a)\,g. \tag{7}$$

This gives our first assertion.

Now for arbitrary $v \in \mathrm{Hom}_k(A,k)$, note that if the right-hand side of (7) is defined for all a, i.e., if for each a it has only finitely many nonzero summands, then (7) defines a kG-module homomorphism $u: A \to kG$ with $\pi \circ u = v$. The ''finitely many nonzero summands'' condition is automatic if G is finite, so we have the second assertion. $\square$

COROLLARY 5. *Let $\eta: H \to G$ be a surjective group homomorphism, M a group on which H acts by automorphisms, and k a field of characteristic* 0. *Then*

(i) *If, for some positive integer n, the dimension of the k-vector space of all group homomorphisms $M \to kG$ that respect the action of H (where H acts on kG via η) is $\geq n$, then the rank of the abelianization of M is also $\geq n$ (i.e., the abelianization of M has at least n **Z**-linearly independent elements*).

(ii) *If G is finite and H finitely generated, and we take $M = \ker\eta$, with H acting thereon by conjugation, then the abelianization of M has finite rank,* equal *to the dimension of the k-vector space of group homomorphisms $M \to kG$ respecting the H-action.*

Proof. Since the additive group of kG is abelian, any group homomorphism $f: M \to kG$ factors through the abelianization M^{ab} of M. Now M^{ab} inherits an action of H, and clearly if f respects actions of H, so will this induced homomorphism $M^{\mathrm{ab}} \to kG$. Hence this homomorphism must annihilate all elements of the form $x - hx$ with $h \in \ker\eta$; so if we write $(M^{\mathrm{ab}})_G$ for the quotient of M^{ab} by the subgroup generated by such elements, the action of H on M^{ab} induces an action of G on $(M^{\mathrm{ab}})_G$, and our f corresponds to a homomorphism $(M^{\mathrm{ab}})_G \to kG$ respecting G-actions. Finally, that homomorphism is equivalent to a k-vector-space map $(M^{\mathrm{ab}})_G \otimes_{\mathbf{Z}} k \to kG$ respecting G-actions, that is, a kG-module homomorphism.

Hence the vector space of all homomorphisms $f: M \to kG$ as groups with H-action is isomorphic to $\mathrm{Hom}_{kG}((M^{\mathrm{ab}})_G \otimes_{\mathbf{Z}} k,\, kG)$, which by the preceding lemma embeds in the k-linear dual of the vector space $(M^{\mathrm{ab}})_G \otimes_{\mathbf{Z}} k$. Thus, if the k-dimension of the space of such homomorphisms is $\geq n$, then so is the k-dimension of $(M^{\mathrm{ab}})_G \otimes_{\mathbf{Z}} k$, which equals the rank of $(M^{\mathrm{ab}})_G$, which is less than or equal to the rank of M^{ab}. This gives (i).

In the special context of (ii), note that the action of $\ker\eta = M$ on M^{ab} induced by conjugation is necessarily trivial; so in this case $(M^{\mathrm{ab}})_G$ is just M^{ab}. Moreover, because M has finite index in the finitely generated group H, it is finitely generated, so M^{ab} has finite rank. Hence the final assertion of Lemma 4 is applicable, giving the conclusion of (ii). □

4. *The main theorem*

Let us again assume $\alpha\colon F \to G$, $\beta\colon S_1 \cup \ldots \cup S_m = S \to F$, $N* \lhd N \lhd F$, and $\alpha*\colon G* \to G$ as in the first paragraph of §2. Let us introduce two more pieces of notation: γ will denote the composite map $\alpha\circ\beta\colon S \to F \to G$, and M the kernel of the canonical homomorphism $\alpha*\colon G* \to G$ (so $M \cong N/N*$).

As sketched earlier, we shall consider

$$\text{homomorphisms } G* \to kG \rtimes G \text{ whose second component is } \alpha*. \tag{8}$$

These induce group homomorphisms of M into kG which respect $G*$-action (where $G*$ acts on M by conjugation, and on kG via $\alpha*: G* \to G$). So by the preceding corollary (either part), we can get a lower bound on the rank of the abelianization of M if we can get such a bound on the k-dimension of the space of homomorphisms $M \to kG$ induced by homomorphisms (8).

Now homomorphisms (8) correspond to certain

$$\text{homomorphisms } F \to kG \rtimes G \text{ whose second component is } \alpha \tag{9}$$

(namely, those whose kernels contain $N*$, but we will not consider that condition yet). Homomorphisms (9) are determined by maps from the free generating set X to $kG \rtimes G$ which take each $x\in X$ to an element with second component $\alpha(x)$; hence each such homomorphism is determined by the family of first components of those elements, an X-tuple ξ of elements of kG. We easily see using (3) and (4) that the image, under such a homomorphism, of any given element of F has first component expressible as a certain k-linear function of the components of ξ, i.e., that taken together, these first-component functions comprise a one-to-one k-linear map

$$(kG)^X \to (kG)^F. \tag{10}$$

Composing this with the operation of pulling functions on F back along β to S, we get a k-linear map

$$\theta\colon (kG)^X \to (kG)^S$$

determining first components of images of elements of S under homomorphisms (9) as k-linear functions of the first components of the images of the elements of X.

Now as we noted parenthetically above, homomorphisms (8) correspond to those homomorphisms (9) whose kernels contain $N*$, the normal subgroup of F generated by the subgroups $N\cap\langle\beta(S_i)\rangle$ $(i=1,\ldots,m)$. This condition says that the restriction of the homomorphism (9) to *each* of the subgroups $\langle\beta(S_i)\rangle < F$ factors

$$\langle\beta(S_i)\rangle \xrightarrow{\alpha} \langle\gamma(S_i)\rangle \longrightarrow kG \rtimes \langle\gamma(S_i)\rangle,$$

where the second component of the right-hand map is the identity, and the first component is induced by a derivation $\langle\gamma(S_i)\rangle \to kG$. We see that the elements $\xi\in(kG)^X$ inducing such homomorphisms are just those ξ whose images under θ lie in the subspace of $(kG)^S$ that was called U in Lemma 2. Thus, the space of elements $\xi\in(kG)^X$ inducing homomorphisms (8) has dimension

$$\geq \dim (kG)^X - \dim (kG)^S + \dim U$$

$$= \ [G:1]\,(\mathrm{card}\,X - \mathrm{card}\,S) \ + \ \dim U.$$

Now the natural map from the space of such elements ξ to the space of G-action-respecting homomorphisms $M \to kG$ is, like θ, induced by (10), hence is also k-linear. What is its kernel? By the same reasoning as in the preceding paragraph (with N in place of $N*$), this will consist of those elements $\xi \in (kG)^X$ whose images under θ are functions on S induced by derivations $G \to kG$. By Lemma 1, the space of such derivations has k-dimension $[G:1] - 1$. Combining these observations, we see that the space of maps $M \to kG$ induced by homomorphisms (8) has dimension

$$\begin{aligned} &\geq \ [G:1]\,(\mathrm{card}\,X - \mathrm{card}\,S) + \dim U - ([G:1] - 1) \\ &= \ [G:1]\,(\mathrm{card}\,X - \mathrm{card}\,S - 1) + \dim U + 1. \end{aligned} \tag{11}$$

Let us now substitute into this inequality the lower bound for $\dim U$ obtained in Lemma 2. For convenience in formulation, let us note that the final expression of (11) is positive if and only if the result of deleting the final ‘‘+1’’ is nonnegative, and let us divide that expression by the order of G, everywhere using the formula $[G:H]/|G| = |H|^{-1}$. Then we get:

THEOREM 6. *Let G be a finite group and $\alpha\colon F \to G$ a homomorphism from the free group on a finite set X onto G, with kernel N. Let $S_1, \ldots, S_m$ be finite sets, let $\beta\colon \cup_i S_i \to F$ be a set map, and let $\gamma = \alpha\circ\beta$. Let $N*$ denote the normal subgroup of F generated by $\cup_i (N \cap \langle \beta(S_i)\rangle)$, and let $G* = F/N*$.*

Then $G*$ is infinite if the rational number

$$\begin{aligned} \delta \ = \ &\mathrm{card}\,X - \mathrm{card}\,S - 1 \ + \\ &\Sigma_i \,(|\langle \gamma(\cup_{j<i} S_j \cap \cup_{j\geq i} S_j)\rangle|^{-1} - |\langle \gamma(\cup_{j\leq i} S_j \cap \cup_{j\geq i} S_j)\rangle|^{-1}) \end{aligned} \tag{12}$$

is nonnegative. In this situation, the rank of the abelianization of the kernel M of the canonical map $\alpha\colon G* \to G$ will be at least $|G|\,\delta + 1$. $\square$

(In the last sentence we have inverted the process of subtracting 1 and dividing by $|G|$, to recover our explicit estimate of the rank of M^{ab}.)

Our considerations have been complicated by our wish to take into account possible intersections among the S_i. If we restrict to the case where the S_i are disjoint, the groups $\langle \gamma(\cup_{j<i} S_j \cap \cup_{j\geq i} S_j)\rangle$ and $\langle \gamma(\cup_{j\leq i} S_j \cap \cup_{j\geq i} S_j)\rangle$ reduce to 1 and $\langle \gamma(S_i)\rangle$ respectively, and we get

$$\begin{aligned} \delta \ &= \ \mathrm{card}\,X - (\Sigma_i\,\mathrm{card}\,S_i) - 1 + \Sigma_i\,(1 - |\langle \gamma(S_i)\rangle|^{-1}) \\ &= \ \mathrm{card}\,X - 1 - \Sigma_i\,(\mathrm{card}\,S_i - 1 + |\langle \gamma(S_i)\rangle|^{-1}). \end{aligned}$$

Moreover, starting with arbitrary S_i, we can always replace these by disjoint sets and map those to the same sets of elements of F; and this will lead to the same group $G*$, by our definition of $N*$. Hence in the context of Theorem 6, the conclusion of that Theorem holds with δ replaced by the value given above. When the S_i are singletons, the above formula yields the result of Thomas [**10**] with which we began this paper. When the S_i are not disjoint, the above formula, because it sums the cardinalities of the S_i's rather than taking the cardinality of their union, generally gives a smaller δ than Theorem 6, and hence a weaker result. On the other hand, the fact that it does not depend on the structures of any subgroups of G other than the $\langle \gamma(S_i)\rangle$ means that the Theorem remains true on replacing G by any finite group generated by an image of X in which the subgroups generated by the images of the S_i have the same form as in G; in particular, groups $G*/L$ for appropriate $L \lhd G*$. The resulting statement (along with a characterization of a class of such subgroups L) is recorded as the second assertion of

COROLLARY 7. *Assume the hypotheses of Theorem 6. Then the conclusion of that Theorem holds with* δ *replaced by*

$$\delta' \;=\; \mathrm{card}\,X - 1 - \Sigma_i\,(\mathrm{card}\,S_i - 1 + |\langle\gamma(S_i)\rangle|^{-1}). \tag{13}$$

Moreover, if we write $\beta*$ *for the composite map* $S \to F \to G*$, *then for every normal subgroup* $L \lhd G*$ *of finite index such that*

$$(\forall i)\;\; L \cap \langle\beta*(S_i)\rangle \;=\; \{1\},$$

(*in particular, for any subgroup* L *of finite index in* M *and normal in* $G*$), *we have*

$$\mathrm{rank}\,L^{\mathrm{ab}} \;\geq\; [G*:L]\,\delta' + 1. \;\;\square \tag{14}$$

The second half of the above corollary was inspired by the corresponding result proved in [**1**] for the case where each S_i is a singleton. Note, incidentally, that as long as δ' is nonnegative, there will be subgroups L as in the above lemma having arbitrarily large index in $G*$; for instance, the inverse images in M of the subgroups $n\,M^{\mathrm{ab}} < M^{\mathrm{ab}}$ $(n \geq 1)$. Thus if δ' is in fact positive, we get subgroups of $G*$ whose abelianizations have arbitrarily large ranks.

5. *Examples*

For our first application, let us consider an example given by Baumslag and Shalen [**2**, p.3], namely the group

$$\langle\, a, b \mid a^4 = b^4 = (ab)^4 = (a^2 b^2)^4 = 1\,\rangle. \tag{15}$$

Their result is applicable because all the exponents are a common prime power. To apply the results of [**10**], [**1**], or this note, we need to find, in some finite group, elements a and b such that a, b, ab and a^2b^2 all have order precisely 4. Such elements a and b can be found in the group of permutations of the set $Z_4 \times Z_4$, namely

$$\begin{gathered} a(x,y) = (x+1, y), \qquad\qquad b(x,y) = (x, y+f(x)), \\ \text{where}\;\; f(0) = 1,\;\; f(1) = -1,\;\; f(2) \;=\; f(3) \;=\; 0. \end{gathered} \tag{16}$$

Thus, by [**10**] we may count each of the four relations in (15) as ‘‘a fourth of a relation’’; so there are effectively two generators and one relation, yielding an infinite group. To translate this construction into an application of our Theorem 6, we take for G the group generated by the permutations (16), and let $X = \{a, b\}$ and $S = S_1 \cup S_2 \cup S_3 \cup S_4 = \{a\} \cup \{b\} \cup \{ab\} \cup \{a^2b^2\}$. Mapping S into F by the inclusion map, we see that $G*$ is (15), and because the S_i are disjoint, (12) simplifies to (13), which gives

$$\delta \;=\; (2-1) - (1/4 + 1/4 + 1/4 + 1/4) \;=\; 0.$$

Hence $G*$ is infinite, and the abelianization of the kernel M of $\alpha*: G* \to G$ has rank at least $|G|\,\delta + 1 = 1$.

To describe an example in which the S_i are not singletons, let us define, in the symmetry group of a regular polyhedron or planar tessellation, the reflection ‘‘along’’ an edge to mean the reflection through the plane perpendicularly bisecting that edge. (Thus, this reflection interchanges the vertices of that edge, but carries each face bounding the edge into itself.)

Now if we color the faces of an octahedron alternately black and white, we see that the reflection along any edge of the octahedron will belong to the group G of symmetries preserving this coloring, which may be identified with the symmetry group of the tetrahedron having vertices in the centers of the black (or white) faces, i.e., S_4. It is not hard to see that a generating set S for G will be given by the set of reflections along the four edges of the octahedron coming into the top vertex. Let us name these reflections, in cyclic order around the top vertex, u, v, w and x, and write S as a union of subsets

$$\{u, v\} \cup \{v, w\} \cup \{w, x\} \cup \{x, u\}.$$

Since each doubleton of elements listed above generates a dihedral group of order 6 (the symmetry group of a face of the octahedron), the group $G*$ determined by the above decomposition of S has presentation

$$G* \;=\; \langle\, u,\, v,\, w,\, x \mid u^2 = v^2 = w^2 = x^2 = (uv)^3 = (vw)^3 = (wx)^3 = (xu)^3 = 1\,\rangle. \tag{17}$$

We find that (12) becomes

$$4 - 4 - 1 + (1 - 1/6) + (1/6 - 1/24) + (1/4 - 1/24) + (1/6 - 1/6) \;=\; 1/6.$$

(Here the lone ''1/4'' comes from the subgroup of G generated by $(\{u, v\} \cup \{v, w\}) \cap (\{w, x\} \cup \{x, u\}) = \{u, w\}$, a fours-group.) Hence the abelianization of M has rank at least $|G|\,\delta + 1 = 24 \cdot (1/6) + 1 = 5$.

The next couple of examples will indicate weaknesses in our results. First, suppose we consider the same group, but order the S_i's differently, writing S as

$$\{u, v\} \cup \{w, x\} \cup \{v, w\} \cup \{x, u\}.$$

This time, the value of δ that we get is

$$4 - 4 - 1 + (1 - 1/6) + (1/6 - 1/24) + (1/24 - 1/24) + (1/6 - 1/6) \;=\; -1/24.$$

Since this is negative, this calculation does not prove $G*$ infinite, though we know from the previous calculation that it is.

Next, let us start with a tetrahedron or a dodecahedron instead of our octahedron, and consider the group generated by the set $S = \{u,\, v,\, w\}$ of reflections along the three edges coming into a vertex. (In this case, S generates the full symmetry group of our polyhedron, though we won't need to know this.) Let us decompose S as

$$\{u, v\} \cup \{v, w\} \cup \{w, u\}.$$

Letting p denote the number of sides in a face of our polyhedron (3 or 5 respectively), and N the order of G (24 or 60 respectively, though again we don't need to know this), we get the group

$$G* \;=\; \langle\, u, v, w \mid u^2 = v^2 = w^2 = (uv)^p = (vw)^p = (wu)^p = 1\,\rangle. \tag{18}$$

In this case, (12) becomes

$$3 - 3 - 1 + (1 - 1/(2p)) + (1/(2p) - 1/N) + (1/(2p) - 1/(2p)) \;=\; -1/N \;<\; 0.$$

Nevertheless, the groups (18) are infinite, as we can see for the case $p = 3$ by mapping $G*$ into the symmetry group of the regular tessellation of the plane by hexagons, sending u, v, w respectively to the reflections along the three edges coming into a vertex of that tessellation, and for $p = 5$ by similarly using the regular tessellation of the hyperbolic plane by decagons meeting three at a vertex. (The point here is that for p an odd integer, the group of symmetries of a p-gon and of a $2p$-gon generated by the reflections along a pair of adjacent edges are the same, namely the dihedral group of order $2p$. We can similarly get an alternative geometric proof that the group we constructed from the octahedron is infinite, using the regular tessellation of the hyperbolic plane by hexagons meeting four at a vertex.) In fact, for $p = 3$ the image of $\ker \alpha*$ in this symmetry group is an abelian group of translations, of rank 2; Theorem 6 has simply failed to detect this abelian homomorphic image.

We get even worse estimates if we ignore the fact that our S_i intersect, and use Corollary 7: $\delta' =$

$3 - 1 - 3(2 - 1 + 1/(2p)) = -1 - 3/(2p)$. Similarly, for the group obtained from the octahedron, where Theorem 6 gave a positive result, the δ' of Corollary 7 is $4 - 1 - 4(2 - 1 + 1/6) = -5/3$.

Let me record without proof two easily verified results in the general context of Theorem 6 that I obtained in the course of searching unsuccessfully for variant systems of sets S_i that might lead to nonnegative δ for the groups (18). First, though we have noted that reordering the S_i can change the value of δ, *reversing* this order leaves it unchanged. Second, if S_i is a subset of S_j $(i \neq j)$, then deleting S_i from our family (which clearly does not change $G*$) also does not decrease δ. (One proves this in two steps. First, shifting S_i one step closer to S_j cannot decrease δ; second, if they are adjacent, then deleting S_i does not decrease δ.)

As noted earlier, Baumslag and Shalen's result [**2**, Theorem A] differs from the results of [**10**], [**1**] and this note in that it requires the exponents involved all to be a common prime power $p^e > 1$, but does not require us to know that those orders are ''realizable'' in a finite group. There is also a small but curious numerical difference. Given a group presented by n generators and $m > 0$ relations each of which states that the p^eth power of a word is the identity, the sufficient condition established in [**2**] for the group to be infinite (and in fact to contain an element of infinite order) is

$$m \ < \ (n-1)(p^e+1), \tag{19}$$

which translates to

$$n - m/(p^e+1) \ > \ 1. \tag{20}$$

Loosely, this says that each of the m relations can be counted as ''$1/(p^e+1)$ of a relation''; though for comparability with the language I used in characterizing the result of [**10**], I should say ''$1/(p^e+1) + \varepsilon$ of a relation'', since the result of [**10**] has a ''$\geq$'' in the hypothesis rather than a ''$>$''. Because of this ''$+\varepsilon$'', this numerical condition is equivalent to that of [**10**] when the number n of generators is 2 (for in that case, (19) says $m < p^e+1$, which can be written $m \leq p^e$). But for higher n, these are weaker hypotheses; e.g., for $n = 3$ the result of [**2**] says that $2p^e+1$ such relations are allowable, while the result of [**10**] only allows $2p^e$. (However, I find it curiously difficult to get a ''natural'' example of, e.g., a 3-generator group presented by 5 relations of the form $s^2 = 1$, or by 7 relations of the form $s^3 = 1$, or by 9 relations of the form $s^4 = 1$, which cannot be proved infinite simply by adding an additional relation making one of the generators equal to a word in the others, and noting that enough relators then vanish or fall together to make the resulting group infinite by the criterion for 2-generator groups.)

6. *Some further observations and questions*

Theorem 6 measures the ''size'' of the group $G*$ using on the one hand the (generally small) rational number δ, and on the other hand the integer $|G|\,\delta + 1$. Which of these numbers – and perhaps we should also include as candidates the intermediate expressions $|G|\,\delta$ and $\delta + |G|^{-1}$ – is it most natural to associate to the group $G*$? One might first think of $|G|\,\delta + 1$, since it bounds the abelianized rank of a subgroup of $G*$. But if we take a hint from the Schreier Index Formula [**5**, Theorem I.8.5], it appears that having a subgroup of index $|G|$ with ''size'' $\geq |G|\,\delta + 1$ should be regarded as saying that $G*$ itself has ''size'' $\geq \delta + 1$. So in fact $\delta + 1$, which in the case originally considered in [**10**] represents the number of generators minus the prorated number of relations, seems the appropriate number to attach to $G*$, while $|G|\,\delta + 1$ is, rather, the number to attach to its subgroup M. Can more be said about M than that its abelianization has at least this rank? One might be daring and ask

QUESTION 8. *In the context of Theorem 6, must* M *have a presentation with at least* $|G|\,\delta+1$ *more generators than relations?*

I also wonder about

QUESTION 9. *In* (14), *can* δ' *be replaced by the* δ *of Theorem 6? Can the condition that* L *is normal in* $G*$ *be dropped? If the answer to either of these questions is negative as stated, can one nevertheless obtain some results in these directions?*

Turning to Baumslag and Shalen's condition that all the relations in a presentation have the form $s_i^{p^e} = 1$ for a fixed prime p and positive integer e, we noted in the Introduction that this neatly excluded the obvious counterexamples to the naive application of the principle ''a relation $s^n = 1$ may be treated as $1/n$ of a relation''. However, those same counterexamples could be excluded by much weaker conditions, so we ask

QUESTION 10. *Can one get a criterion for a group to be infinite based on counting a relation* $s^n = 1$ *as ''*$1/n$ *of a relation'' with the only additional condition being that the exponents* n *that are so treated be powers of a common prime* p*; or alternatively, that they all be equal; or, generalizing both conditions, that the set of exponents so treated be totally ordered under divisibility?*

A finitely presented group is said to have *deficiency* $\geq d$ if it has a presentation with at least d more generators than relations. In what ways, other than properties of its abelianization, is a group with deficiency $\geq d$ like a free group on $\geq d$ generators? Analogy with the theory of abelian groups might suggest that such a group should admit a homomorphism onto a free group on d generators; but this is not so. The question of how large a free group F a group G with a given presentation can be mapped onto is treated in [**9**, p.52 through end of §I.7]. I will sketch one self-contained example. I claim that the 4-generator 1-relator group

$$G = \langle w, x, y, z \mid [w, x] = [y, z] \rangle$$

cannot be mapped onto the free group F of rank 3. To prove this, one identifies $[F, F]/[F, [F, F]]$ with the exterior square of the abelianization of F, and notes that if elements w, x, y, z of a free abelian group (or a vector space) A have the property that in the exterior square of A, $w \wedge x = y \wedge z$, then w, x, y, z span a subgroup (respectively, a subspace) of A of rank ≤ 2.

A finite group must have deficiency ≤ 0; for results on finite groups with deficiency exactly zero, see [**7**], [**8**].

In [**3**], Baumslag and Shalen show that any group G of deficiency ≥ 2 admits a decomposition $G = A *_C B$ as a coproduct of finitely presented groups A and B, with amalgamation of a common finitely presented subgroup C. It is not clear to me how the deficiency of G will relate to those of A, B and C.

Let us note some questions of a more open-ended sort. Suppose we have a rule assigning to presentations of groups certain fractional ''effective relation-counts'', which can be used in determining when the group so presented must be infinite – for example, that of [**2**], that of [**10**] and [**1**], or the one implicit in this note. Then let us define the ''effective deficiency'' of a finitely presented group G as the infimum, over all systems C of additional generators and relations which, when adjoined to G, give a *finite* group, of the numbers

$$(\text{effective number of relations in } C) - (\text{number of generators in } C) + 1.$$

(Such a system C might be called a ''co-presentation'' for G.) What can we say about this invariant on groups (for a given way of counting relations)? Will it be additive on coproduct groups? Will every group for which it is >1 contain a free subgroup on two generators? Will it always be nonnegative? Will it satisfy a Schreier formula (or at least an inequality)?

Or, starting with the idea of the Schreier formula, and the version of the Euler characteristic of a group to which it leads [**5**, p.105], suppose we define the ''Schreierized deficiency'' of a finitely presented group G as the supremum, over all subgroups M of finite index in G, of $(\mathrm{def}(M)-1)/[G:M]$, where $\mathrm{def}(M)$ is the deficiency of M. What good properties, if any, will this function have? (Cf. Question 8 above.)

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