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MORE ABELIAN GROUPS WITH FREE DUALS

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Abstract. In answer to a question of A. Blass, J. Irwin and G. Schlitt, a subgroup G of the additive group $\mathbb{Z}^\omega$ is constructed whose dual, $\mathrm{Hom}(G,\mathbb{Z})$, is free abelian of rank $2^{\aleph_0}$. The question of whether $\mathbb{Z}^\omega$ has subgroups whose duals are free of still higher rank is discussed, and some further classes of subgroups of $\mathbb{Z}^\omega$ are noted.

Introduction.

The additive group $\mathbb{Z}^\omega$ (the countable direct power of $\mathbb{Z}$) is a nonfree abelian group G whose rank (maximum number of linearly independent elements) is the cardinality of the continuum, $2^{\aleph_0}$; but its dual, $\mathrm{Hom}(G,\mathbb{Z})$, is known to be free abelian of merely countable rank [13]. Blass, Irwin and Schlitt [3], after examining generally which subgroups of $\mathbb{Z}^\omega$ have duals free of countable rank, ask whether a subgroup of $\mathbb{Z}^\omega$ can have dual free of *uncountable* rank. Such a subgroup is here constructed.

We begin (§1) by sketching briefly an unsuccessful first try, noting why it fails, then sketching how the difficulty can be circumvented. In §§2-4 we develop the resulting successful construction in detail.

In §5 we ask whether there are subgroups of $\mathbb{Z}^\omega$ whose duals are free of still larger ranks, the evident upper bound being $2^{2^{\aleph_0}}$. §6 notes some classes of subgroups of $\mathbb{Z}^\omega$ found while thinking about that question. (For some other unusual subgroups of $\mathbb{Z}^\omega$, see [9, Chapter X], [6], and works cited there.)

A more precise version of the fact that $\mathrm{Hom}(\mathbb{Z}^\omega,\mathbb{Z})$ is free of countable rank, which we shall use below, says that every homomorphism $\varphi:\mathbb{Z}^\omega\to\mathbb{Z}$ factors through the projection to finitely many coordinates. (Regarded as a property of the *codomain* group $\mathbb{Z}$ of φ, this is expressed by saying that $\mathbb{Z}$ is a “slender” abelian group. There is considerable literature on slender groups, modules, and other structures; e.g., [10, §94] [9, Chapter III], [7]. However, since this note focuses on the *domain* group G, we shall not use that language here.)

I am indebted to John Steel for a helpful observation used in §5.

1. A first attempt.

I will sketch here my first try at constructing a group with the desired property. As indicated above, the more complicated construction of §§2-4 will be motivated as a way of patching up the difficulty with this one.

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Clearly, for any countably infinite set Y, the group $\mathbb{Z}^Y$ is isomorphic to $\mathbb{Z}^\omega$; so in place of ω we will use a countable set that is more convenient to our purposes, the set $[0,1]\cap\mathbb{Q}$ of rational numbers between 0 and 1.

Let G be the group of $\mathbb{Z}$-valued functions on $[0,1]\cap\mathbb{Q}$ which are constant in a neighborhood of each *irrational* $x\in[0,1]$. Such functions can have infinitely many jumps. (E.g., consider the function which has the value 0 at 0, while for each natural number n, it has the value n at all rationals in the subinterval $(2^{-n-1},\,2^{-n})$, and $-n$ at 2^{-n} itself.)

For each irrational $x\in[0,1]$, let $h_x\in\mathrm{Hom}(G,\mathbb{Z})$ be the map taking every $g\in G$ to its constant value in the neighborhood of x. If, further, for each rational $x\in[0,1]$ we define h_x to take $g\in G$ to its value at x, it is not hard to see that the set of homomorphisms h_x as x ranges over $[0,1]$ are linearly independent. If they spanned $\mathrm{Hom}(G,\mathbb{Z})$, we would have our desired example.

Why might we hope that the h_x would span $\mathrm{Hom}(G,\mathbb{Z})$?

Consider any $a\in\mathrm{Hom}(G,\mathbb{Z})$. If we take any decomposition of $[0,1]$ into countably many (open, closed, or half-open) subintervals with rational endpoints, then we can define elements of G independently on different members of this decomposition. This observation can be used to construct homomorphisms $\mathbb{Z}^\omega\to G$, by starting with an arbitrary $g\in G$, and using the n-th entry of an element of $\mathbb{Z}^\omega$ to "scale" the output of g on the n-th of our subintervals (under some fixed enumeration of those subintervals). Composing such a map $\mathbb{Z}^\omega\to G$ with $a:G\to\mathbb{Z}$, we get a homomorphism $\mathbb{Z}^\omega\to\mathbb{Z}$. But it is known that every homomorphism $\mathbb{Z}^\omega\to\mathbb{Z}$ is a linear combination of the evaluation maps at *finitely many* coordinates [13, Satz III]. I hoped to use this to prove that the action of a would similarly be localized at finitely many points $x_1,\dots,x_n\in[0,1]$, and would in fact be a linear combination of the corresponding homomorphisms h_{x_i}.

Unfortunately, this fails to be true. For example, suppose x is an irrational point in $[0,1]$, and $y_0,\ y_1,\ y_2,\ \dots$ a sequence of distinct points of $[0,1]$, converging to x. I claim that a homomorphism $G\to\mathbb{Z}$ can be defined by taking each $g\in G$ to

$$(1)\qquad (h_{y_0}(g)-h_{y_1}(g))\ +\ (h_{y_2}(g)-h_{y_3}(g))\ +\ \dots\ +\ (h_{y_{2m}}(g)-h_{y_{2m+1}}(g))\ +\ \cdots\ .$$

Indeed, as $n\to\infty$, the y_n approach x, so by definition of G, the sequence of integers $h_{y_n}(g)$ eventually becomes constant. Hence all but finitely many of the parenthesized terms of (1) are zero, so the sum (1) is defined, and clearly gives a homomorphism. But it is not a function of the behavior of g "at" finitely many points of $[0,1]$.

The trouble with the argument that suggested the opposite is that the process of prescribing elements of G independently on countably many intervals cannot be carried out if these intervals converge to an irrational point x : since x itself won't belong to any of the resulting intervals of constancy, functions so constructed will not, in general, belong to G.

The solution we shall take below is to weaken that local constancy requirement, and require instead that each element of the group G we will define be constant on a "perforated" neighborhood of each irrational x, where countably many "perforations" by subintervals with rational endpoints are allowed, as long as the fraction of the space they occupy approaches 0 as we get close enough to x.

This foils the above counterexample, since we can enclose successive y_n in tiny perforations, and on these, let g have values independent of its value at x; thus, the rogue homomorphism (1) will no longer be defined. I expected that this kludge, in solving one problem, would only create more. But it turned out to work, as we shall see in the next three sections.

Between the above sketch and the construction we will present, there are also a few cosmetic improvements. Rather than describing elements of G as functions on $[0,1]\cap\mathbb{Q}$, we will, for most of our development, make them functions on $[0,1]$, i.e., give them genuine values at each x in that interval, and only at the end restrict them to $[0,1]\cap\mathbb{Q}$ to obtain a solution to the original problem. With elements of G expressed as functions on $[0,1]$, we will be able to describe the "perforated constancy" condition as continuity in a certain topology on that interval, finer than the standard topology. Finally, rather than working, throughout, with the particular structures of the set $[0,1]$, its standard topology, and that finer topology, we will posit (in §2) a general situation of a set X with two topologies related in certain ways, describe (in §3) our particular choice of $X=[0,1]$ and its two topologies, showing that these fit this pattern, and, finally (in §4), prove our freeness result in the general context.

2. The general context.

Our construction will assume a set X given with two Hausdorff topologies, which we will call the *coarse* and the *fine topology*, satisfying the following three conditions. (Recall that a topological space is *first-countable* if every point has a countable neighborhood basis.)

(2) Under the coarse topology, X is compact and first-countable.

(3) The coarse topology on X has a basis of open sets whose members are clopen (closed and open) in the fine topology.

(4) For every infinite subset S of X, there exists a decomposition of X into disjoint subsets clopen in the fine topology, infinitely many of which contain members of S in their interiors with respect to the coarse topology.

Condition (3) implies that the fine topology contains the coarse topology, justifying the names.

Conditions (2)-(4) would be satisfied if we took any compact, Hausdorff, first-countable topology for the coarse topology, and the discrete topology for the fine topology. However, our application of this setup will involve finding a subset $Y \subseteq X$ of comparatively small cardinality that is dense in the fine topology. So we will need a fine topology that is not *too* fine.

If we took for X the real unit interval $[0,1]$, for the coarse topology the standard topology, and (following the original idea of the preceding section) defined the fine topology to have for its open sets those sets $U \subseteq [0,1]$ which contain a neighborhood, with respect to the coarse topology, of every *irrational* point $x \in U$, then conditions (2) and (3) would hold (the latter because intervals with rational endpoints are clopen in the fine topology); and the countable subset $[0,1]\cap\mathbb{Q}$ would be dense in the fine topology. But (4) would fail: if we took for S a sequence of points $y_0,\ y_1,\ y_2,\ \dots$ converging to an irrational point x, then for any covering of X by disjoint sets clopen in the fine topology, the member of that covering containing x would contain all but finitely many of the y_n, so it would be impossible for infinitely many other members of our disjoint covering to meet S. In the next section we shall see that a fine topology based on "perforated neighborhoods" does satisfy (4); and in §4, condition (4) will be the key to constructing enough composite maps $\mathbb{Z}^\omega \to G \to \mathbb{Z}$ to establish (as was not true for the G of the preceding section) that $\mathrm{Hom}(G,\mathbb{Z})$ is free on the desired basis.

3. Our two topologies on $[0,1]$.

As sketched in §2, let us take the real unit interval $[0,1]$ for our set X, and for our coarse topology the usual compact topology. The description, in the next definition, of the fine topology will make precise the "small perforations" idea.

In speaking of subintervals of $[0,1]$, we will use the terms "closed interval", "open interval" and "half-open interval" in their conventional senses for real intervals (of which the first two match the topological properties of these intervals under the coarse topology).

Definition 1. *A subset $U \subseteq [0,1]$ will be* open *in the fine topology if for every irrational $x \in U$, and every $\varepsilon > 0$, there exists $\delta > 0$ such that for every closed interval $[r,s] \subseteq [0,1]$ with rational endpoints, which contains x and has length $< \delta$, there exists a finite family of pairwise disjoint closed subintervals, also with rational endpoints, which lie in $U \cap [r,s]$, and have total length at least $(1-\varepsilon)(s-r)$.*

It is not hard to see that the class of sets so defined does indeed constitute a topology. (In verifying closure under pairwise intersection, note that the $(1-\varepsilon)(s-r)$ condition means that the part of $[r,s]$ missed by U can be enclosed in finitely many intervals of length totaling $\leq \varepsilon$ times the length of $[r,s]$. To establish this property at x for an intersection $U \cap V$ of sets both satisfying it there, take δ small enough to get the same condition in each of U and V, with $\varepsilon/2$ in place of ε.)

(The assumption in Definition 1 that the intervals named all have rational endpoints could be omitted without changing the topology defined, as long as we keep the restriction that x be irrational, and specify that x lie strictly between r and s. But the present formulation in terms of intervals with rational endpoints will be convenient. In statements made below, assumptions of rational endpoints cannot necessarily be dropped.)

Two easily verified observations:

(5) Every open, closed, or half-open subinterval of $[0,1]$ with rational endpoints (including the degenerate closed interval $[r,r]=\{r\}$ for each rational $r \in [0,1]$) is clopen in the fine topology.

(6) Under the fine topology, the rational points of $[0,1]$ form a *dense* set of *isolated* points.

I claim now that this pair of topologies satisfies conditions (2)-(4) of the preceding section. Property (2) is well-known, and (3) follows from (5) applied to open intervals with rational endpoints. Less trivial is

Lemma 2. *The pair of topologies on* $[0,1]$ *described above satisfies* (4).

Proof. Let S be an infinite subset of $[0,1]$. Then it must contain an infinite increasing or decreasing sequence; assume without loss of generality that it contains a decreasing sequence $y_1 > y_2 > \dots,$ with greatest lower bound x. By dropping enough terms from this sequence, we can assume that $1 > y_1$ and that for every n, we have

(7) $$y_{n+1} - x \ \leq\ (y_n - x)/2.$$

Now let us surround each of our points y_n by an interval $(r_n, s_n) \subseteq [0,1]$ with rational endpoints, in such a way that the lengths of these intervals shrink much faster than the points y_n approach x :

(8) $$s_n - r_n \ <\ (y_n - x)/2^{n+1}.$$

In view of (7) and (8), the intervals (r_n, s_n) are disjoint, and by (5) they are clopen in the fine topology. Now let

(9) $$V \ =\ [0,1] - \bigcup_{n\geq 1} (r_n, s_n).$$

Since the (r_n, s_n) are open in the fine topology, V is closed in that topology. We claim it is also open. That the condition of Definition 1 holds at all irrational points of V other than, perhaps, x, is immediate: V contains a genuine interval about every such point. Let us show that V also satisfies the required condition at x, if x is irrational. (Definition 1 imposes no such requirement if x is rational.)

Given $\varepsilon > 0$, we can use (7) and (8) to find a δ such that in every interval (r,s) of length $s - r < \delta$ having rational endpoints and containing x, the lengths of the (infinitely many) intervals (r_n, s_n) meeting (r,s) sum to less than $\varepsilon/2$ times the length of (r,s). For any such (r,s), let us take an open subinterval $(r', s') \subseteq (r,s)$ about x with rational endpoints, of length less than $\varepsilon/2$ times the length of (r,s). This will contain all but finitely many of the (r_n, s_n), hence the union of (r', s') and those (r_n, s_n) that meet (r,s) will be a *finite* union of open intervals, of total length less than ε times the length of (r,s); hence its complement in (r,s) will be a finite union of closed intervals of total length $\geq (1-\varepsilon)(s-r)$, as required.

The sets (r_n, s_n) $(n \geq 1)$ and V thus satisfy the conclusion of (4): they are clopen, they form a disjoint covering of $[0,1]$, and infinitely many of them, namely the (r_n, s_n), contain members of S. □

4. Describing G, and proving $\mathrm{Hom}(G,\mathbb{Z})$ free.

We now return to the general assumptions of §2, letting X be an arbitrary set given with two topologies satisfying (2)-(4). Let $G \subseteq \mathbb{Z}^X$ be the group of all $\mathbb{Z}$-valued functions on X that are continuous in the fine topology. (Throughout this section we understand $\mathbb{Z}$ to have the discrete topology, so continuity of a function $X \to \mathbb{Z}$ means that the inverse image of each integer is clopen.) For each $x \in X$,

(10) Let $h_x : G \to \mathbb{Z}$ be the homomorphism of evaluation at x.

We shall show below that $\mathrm{Hom}(G,\mathbb{Z})$ is freely generated by the elements h_x. First, the easy part.

Lemma 3. *The homomorphisms* h_x $(x \in X)$ *are linearly independent.*

Proof. It clearly suffices to show that for any finite family of distinct points $x_1, \dots, x_n \in X$, there exists $g \in \mathrm{Hom}(G,\mathbb{Z})$ with $h_{x_1}(g) = 1$, and $h_{x_m}(g) = 0$ for $m = 2, \dots, n$.

Since X is Hausdorff in the coarse topology, we can find a neighborhood of x_1 in that topology containing none of $x_2, \dots, x_n$. By (3), this will contain a subneighborhood U of x_1 that is clopen in the fine topology. The characteristic function of U, which is continuous in the fine topology, gives our desired g. □

We now begin the process that will decompose every element of $\mathrm{Hom}(G,\mathbb{Z})$ as a linear combination of these maps. First, a result whose proof uses nothing specific to $\mathbb{Z}$-valued functions. By the *support* of an element $g \in \mathbb{Z}^\omega$ we will mean $\{x \in X \mid g(x) \neq 0\}$.

Lemma 4. *Let* a *be a nonzero member of* $\mathrm{Hom}(G,\mathbb{Z})$. *Then there exists a point* $x \in X$ *such that*

(11) *For every neighborhood* U *of* x *in the coarse topology, there exists* $g \in G$ *with support contained in* U *such that* $a(g) \neq 0$.

Proof. Suppose the contrary. Then every $x \in X$ has a neighborhood U_x in the coarse topology such that every member of g with support in U_x is in the kernel of a. By (3) we can assume that the sets U_x are also clopen in the fine topology on X. By compactness of the coarse topology, finitely many of these, say $U_{x_1}, \ldots, U_{x_n}$, cover X. Hence the sets

$$(12) \qquad U_{x_1}, \quad U_{x_2} - U_{x_1}, \quad \ldots, \quad U_{x_n} - (U_{x_1} \cup \cdots \cup U_{x_{n-1}})$$

constitute a covering of X by finitely many pairwise disjoint sets clopen in the fine topology, such that every member of G with support in one of them is in $\ker(a)$.

But because these sets are clopen, every member of G is a sum of members of G with supports in one or another of them; hence every member of G is in $\ker(a)$, so $a = 0$, contradicting our hypothesis. □

We now want to prove that for each $a \in G$, there are only finitely many $x \in X$ such that (11) holds. The key step will be the first assertion of the next lemma. (The second assertion will be used later.) The proof of the lemma will call on the following known result.

(13) [13, Satz III] Every homomorphism $\mathbb{Z}^\omega \to \mathbb{Z}$ depends on only finitely many coordinates of its argument, i.e., can be factored $\mathbb{Z}^\omega \to \mathbb{Z}^n \to \mathbb{Z}$, where the first map is given by the projections to some n coordinates, and the second is an arbitrary homomorphism.

Lemma 5. *Suppose X is written as the union of a disjoint family of sets U_i $(i \in I)$ each clopen in the fine topology, and let $a \in \mathrm{Hom}(G, \mathbb{Z})$. Then only finitely many $i \in I$ have the property*

(14) *There are elements of G with support in U_i on which a has nonzero value.*

Assume, further, that I is countable, and let U denote the union of the finitely many U_i satisfying (14). *Then the value of a at every $g \in G$ is determined by the restriction of g to U. Equivalently, a has in its kernel all elements of G with support in $X - U$.*

Proof. Suppose, in contradiction to the first assertion, that there are infinitely many i satisfying (14). Then we can write I as the union of a countably infinite family of pairwise disjoint subsets I_n $(n \in \omega)$ each containing at least one i that satisfies (14). Letting $V_n = \bigcup_{i \in I_n} U_i$ for each $n \in \omega$, it follows that for each $n \in \omega$ there exists a $g_n \in G$ with support in V_n such that $a(g_n) \neq 0$.

Since the g_n have disjoint supports, we see that for every $f \in \mathbb{Z}^\omega$, the expression $f' = \sum_{n \in \omega} f(n)\, g_n$ makes sense; and as the V_n are clopen in the fine topology, f' is again continuous in that topology, i.e., belongs to G. Clearly the map $f \mapsto f'$ is a homomorphism $\mathbb{Z}^\omega \to G$. Composing it with the given homomorphism $a : G \to \mathbb{Z}$ we get a homomorphism $\mathbb{Z}^\omega \to \mathbb{Z}$ which for each n takes the element $e_n \in \mathbb{Z}^\omega$ having a 1 in the n-th position and 0 everywhere else to $a(g_n) \neq 0$. This contradicts (13), proving the finiteness of the set of $i \in I$ such that (14) holds.

Now assume, as in the last paragraph of the lemma, that I is countable; without loss of generality we shall take $I = \omega$. Let U be the union of the finitely many U_i for which (14) holds. Because U and $X - U$ are clopen in the fine topology, every element of g is the sum of an element with support in U and an element with support in $X - U$. From this, the equivalence of the last two sentences of the lemma is clear; we shall prove the last of those sentences.

Let $g \in G$ be an element with support in $X - U$, and now for each $n \in \omega$ let $g_n \in G$ be the function which agrees with g on U_n and is zero elsewhere. Again, we map $\mathbb{Z}^\omega$ to $\mathbb{Z}$ by $f \mapsto a(\sum_{n \in \omega} f(n)\, g_n)$. This function is zero on every e_n $(n \in \omega)$: on those with $U_n \subseteq U$ because g has support in $X - U$, and on the others because the corresponding sets U_n do not satisfy (14). But from (13) we can see that a homomorphism $\mathbb{Z}^\omega \to \mathbb{Z}$ which is zero on all the e_n is zero; hence the above map is zero, hence $a(g)$, which is the value of that map on the constant function 1 in $\mathbb{Z}^\omega$, is 0, as claimed. □

(We shall see in Corollary 12 that the countability condition in the second paragraph of the above lemma can be dropped; but the above version suffices for the purposes of this section.)

We deduce

Lemma 6. *For any $a \in \mathrm{Hom}(G, \mathbb{Z})$ there are only finitely many $x \in X$ such that* (11) *holds.*

Proof. If the set S of such points were infinite, then by (4) we could find a covering of X by disjoint subsets U_i clopen in the fine topology, infinitely many of which contained a point of S in their interiors with respect to the coarse topology. It follows from our choice of S that for each of the latter sets, we could find an element $g_i \in G$ with support in U_i such that $a(g_i) \neq 0$, contradicting the first assertion of the preceding lemma. □

From Lemmas 4 and 6, we can now get

Corollary 7. *Every* $a \in \mathrm{Hom}(G,\mathbb{Z})$ *can be written*

(15) $\quad a \;=\; a_1 + \dots + a_n \quad (n \geq 0;\ a_1, \dots, a_n \in \mathrm{Hom}(G,\mathbb{Z})),$

where for each a_m *there is an* $x_m \in X$ *which is the unique point such that* (11) *holds with* a_m *and* x_m *in the roles of* a *and* x; *equivalently, such that*

(16) $\quad a_m$ *is nonzero, but annihilates all elements of* G *whose supports do not have* x_m *in their closure in the coarse topology.*

Proof. Given a, let $x_1, \dots, x_n$ $(n \geq 0)$ be the points described by Lemma 6. Using (3), we can get a covering of X by disjoint sets $U_1, \dots, U_n$ which are clopen in the fine topology, and such that each U_m is a neighborhood of x_m in the coarse topology (cf. the method used to construct (12)). If we define $a_m : G \to \mathbb{Z}$ for $1 \leq m \leq n$ to be the operation that first multiplies $g \in G$ by the characteristic function of U_m, then applies a to the result, we immediately have (15), and it is not hard to verify that x_m is the unique point for which a_m satisfies (11).

It remains to show that this is equivalent to (16). One direction, that (16) implies that x_m is the unique point which, together with the homomorphism a_m, satisfies (11), is easily checked.

Conversely, let us assume that condition and deduce (16). The assumption that there exists x_m which, with a_m, satisfies (11) clearly implies that $a_m \neq 0$. Now let $g \in G$ be any element whose support does not have x_m in its closure under the coarse topology. Thus, x_m has a neighborhood V in that topology disjoint from the support of g. By (3), V has a subset W which is again a neighborhood of x_m in the course topology, and which is clopen in the fine topology. The operation of multiplying by the characteristic function of $X - W$ and then applying a_m will be a member of $\mathrm{Hom}(G,\mathbb{Z})$ having no point x satisfying the condition analogous to (11), since the application of the latter condition with $U = W$ shows that x_m is not such a point, and the fact that a_m has no such point other than x_m shows that the function constructed from it can't either. Hence by Lemma 4, this new function is the zero map. But because g has support in $X - W$, this map, which we have shown to be zero, agrees with a_m at g; so $a_m(g) = 0$, as claimed. $\square$

In view of the above result, we will have what we have been aiming for, once we prove

Lemma 8. *Suppose* $a \in \mathrm{Hom}(G,\mathbb{Z})$ *is a homomorphism for which there exists a unique* $x \in X$ *satisfying* (11). *Then* a *is an integer multiple of* h_x.

Proof. The asserted conclusion is clearly equivalent to the statement that a can be factored $G \to \mathbb{Z} \to \mathbb{Z}$, where the first map is h_x, and the second is a homomorphism. Because h_x maps surjectively to $\mathbb{Z}$, this is in turn equivalent to the statement that a annihilates the kernel of h_x. That is what we shall now prove.

Let $g \in \ker(h_x)$. Since g is continuous in the fine topology on X, and $\mathbb{Z}$ is taken with the discrete topology, under which all its points are clopen, the statement that g belongs to $\ker(h_x)$ tells us that

(17) $\quad g$ is zero on some subset U of X which contains x and is clopen in the fine topology.

Note further that by (3), x has a neighborhood basis in the *coarse* topology consisting of subsets clopen in the fine topology. Since the coarse topology is first-countable, that neighborhood basis can be assumed countable. By taking successive intersections of its terms, we can assume it is decreasing, and begins with the whole space:

(18) $\quad X = V_0 \;\supseteq\; V_1 \;\supseteq\; \dots \;\supseteq\; V_n \;\supseteq\; \dots\,.$

Since X is Hausdorff, $\bigcap V_n = \{x\}$. Hence the sets

(19) $\quad V_0 - V_1, \quad V_1 - V_2, \quad \dots, \quad V_n - V_{n+1}, \quad \dots$

which are also clopen in the fine topology, give a disjoint covering of $X - \{x\}$.

Now for the tricky step. Using the U of (17), we obtain from (19) the sets

(20) $\quad U; \quad V_0 - V_1 - U, \quad V_1 - V_2 - U, \quad \dots, \quad V_n - V_{n+1} - U, \quad \dots$

which are again disjoint, and clopen in the fine topology, and which now cover all of X. Hence in view of the final paragraph of Lemma 5, $a(g)$ is determined by the values of a on the projections of g to these sets. By the choice of U in (17), g has zero projection to U. On the other hand, its projection on each set $V_n - V_{n+1} - U$ is (as a function on all of X) zero on a neighborhood of x in the coarse topology, namely

V_{n+1}. But since x is the unique point which, with a satisfies (11), the equivalence of the two conclusions of the preceding corollary show that our g is annihilated by a, as required. $\square$

The above results give

Theorem 9. *Suppose X is a set given with two topologies, called "the coarse topology" and "the fine topology", satisfying* (2)-(4)*, and that G is the subgroup of $\mathbb{Z}^X$ consisting of all elements continuous with respect to the fine topology on X and the discrete topology on $\mathbb{Z}$.*

Then $\mathrm{Hom}(G,\mathbb{Z})$ is the free abelian group on the evaluation maps h_x of (10). $\square$

Corollary 10. *Letting $[0,1]$ denote the real unit interval, and G the group of functions $[0,1]\to\mathbb{Z}$ continuous in the topology of Definition 1, the group $\mathrm{Hom}(G,\mathbb{Z})$ is free abelian on the generators h_x $(x\in[0,1])$.*

Hence, the isomorphic group G_0 of functions $[0,1]\cap\mathbb{Q}\to\mathbb{Z}$ obtained from G by restriction to $[0,1]\cap\mathbb{Q}$ has the same dual.

Hence $\mathbb{Z}^\omega$ has a subgroup whose dual is free abelian of continuum rank.

Proof. The first assertion follows from the above theorem, since we showed in the preceding section that the standard topology and the topology of Definition 1 satisfy (2)-(4). The one-one-ness of the restriction map $G\to\mathbb{Z}^{[0,1]\cap\mathbb{Q}}$, on which the second assertion then hangs, follows from the density statement of (6). Finally, taking any bijection between $[0,1]\cap\mathbb{Q}$ and ω, we get an isomorphism between $\mathbb{Z}^{[0,1]\cap\mathbb{Q}}$ and $\mathbb{Z}^\omega$, so our subgroup $G_0\subseteq\mathbb{Z}^{[0,1]\cap\mathbb{Q}}$ leads to an isomorphic subgroup of $\mathbb{Z}^\omega$. $\square$

5. Can we do better?

The group $\mathbb{Z}^\omega$ has continuum cardinality, $c=2^{\aleph_0}$; hence, although it has relatively few homomorphisms to $\mathbb{Z}$ (only countably many), there is no reason why it should not have subgroups admitting as many as 2^c such homomorphisms.

And, in fact, it does. It is known that the subgroup B of *bounded* functions $\omega\to\mathbb{Z}$ is free (proved in [13] assuming the continuum hypothesis, then in [11] and [2] without that assumption). Having continuum cardinality and being free, B must be free of continuum rank; so its dual $\mathrm{Hom}(B,\mathbb{Z})$ can be identified with $\mathbb{Z}^c$, and so has cardinality 2^c. Hence that dual group has rank (maximum number of linearly independent elements) also 2^c, in other words, it *contains* a subgroup free of that rank.

Perhaps unexpectedly, one can characterize an explicit family of 2^c linearly independent elements in $\mathrm{Hom}(B,\mathbb{Z})$. The set $\beta(\omega)$ of all ultrafilters on ω has cardinality 2^c [5, Corollary 7.4], and since each $g\in B$ assumes only finitely many values in $\mathbb{Z}$, every ultrafilter $\mathcal{U}\in\beta(\omega)$ gives a way of associating to each such g one of those values, the value such that the set on which it is assumed belongs to $\mathcal{U}$. For each $\mathcal{U}\in\beta(\omega)$ this gives a homomorphism $h_{\mathcal{U}}:B\to\mathbb{Z}$, and it is not hard to check that the $h_{\mathcal{U}}$ are linearly independent.

But the free group generated by the maps $h_{\mathcal{U}}$ is not the whole of $\mathrm{Hom}(B,\mathbb{Z})\cong\mathrm{Hom}(\bigoplus_c\mathbb{Z},\,\mathbb{Z})\cong\mathbb{Z}^c$, since the latter, having subgroups isomorphic to $\mathbb{Z}^\omega$, is non-free. So we may ask

Question 11. *Is there a subgroup $G\subseteq\mathbb{Z}^\omega$ whose dual is free of rank 2^c; or, at least, free of rank $>c$?*

If we had a candidate subgroup G for the above property roughly along the lines of the construction of the preceding sections, a possible difficulty with applying the methods of those sections to it is that the coarse topology on X might no longer be first-countable, as required by (2). Let us show, therefore, that the general results of that section remain true if hypotheses (2) and (3) are replaced by

(21) Under the coarse topology, X is compact; and the cardinality of the set X is less than every countably measurable cardinal (if any such cardinals exist).

(22) For every $x\in X$, there exists a family $(W_i)_{i\in I_x}$ of pairwise disjoint sets each clopen in the fine topology, such that
(i) $X-\{x\} = \bigcup_{i\in I_x}W_i$,
(ii) x has a basis of open neighborhoods U in the coarse topology, each of which is clopen in the fine topology, and has the form $U=\{x\}\cup\bigcup_{i\in J}W_i$ for some subset $J\subseteq I_x$.

For the concept of a countably measurable cardinal (in many works simply called a measurable cardinal) see [4]. Condition (21) is indeed a consequence of (2), since by [1], every compact Hausdorff first-countable topological space has cardinality $\le c$, which is far less than any countably measurable cardinal. Condition (22) by itself is stronger than (3), but the construction of (19) in the proof of Lemma 8 shows that (2)-(4)

together imply (22), so the conjunction of (21), (22) and (4) is implied by that of (2), (3) and (4). Moreover, this new set of conditions does not imply first-countability of the coarse topology (as can be verified using examples where the fine topology is discrete); so it is strictly weaker than the old one.

We can now prove

Corollary 12 (to the proofs of §4)**.** *The general results of §4 (the results through Theorem 9) remain true if the hypotheses (2)-(4) are weakened to (21), (22) and (4).*

Moreover (under these weakened hypotheses, and hence under the original hypotheses) one can delete the countability assumption from the second paragraph of Lemma 5.

Sketch of proof. The first-countability condition of (2) was not used before the proof of Lemma 8, so the results proved up to that point remain true under our weakened hypotheses, and the only thing we have to prove regarding those results is that in the second paragraph of Lemma 5, we can replace the condition that the index set I be countable by the assumption from (21) that X have cardinality less than all countably measurable cardinals.

The latter assumption certainly implies that (after dropping from I any i such that U_i is empty) the cardinality of I is likewise less than all countably measurable cardinals. Our earlier proof of the desired statement now goes over if, where we previously called on Specker's result (13) that $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z})$ is free on the projections to the individual coordinates, we now call on the stronger known result that the same is true of $\mathrm{Hom}(\mathbb{Z}^Y, \mathbb{Z})$ for any set Y having cardinality less than all countably measurable cardinals (cf. [10, Theorem 94.4], [8]).

Moving on to the proof of Lemma 8, the first-countability condition was finally used there to construct the family of sets $V_n - V_{n+1}$. We claim that in our present context, the W_i of (22) can serve in the same role. Without loss of generality, let us assume all those W_i nonempty.

In view of the strengthened final statement of Lemma 5 noted above, we do not need to assume that there are only countably many W_i; the hypothesis on the cardinality of X implies the very weak bound we need. The proof of Lemma 8 also used the fact that each $V_n - V_{n+1}$ was disjoint from some neighborhood of x under the coarse topology, namely, V_{n+1}. To get the same conclusion for a given W_j $(j \in I)$, take any $w \in W_j$. By Hausdorffness of the coarse topology, x has a neighborhood U in that topology not containing w, so by (22)(ii), it has such a neighborhood of the form $U = \{x\} \cup \bigcup_{i \in J} W_i$ $(J \subseteq I_x)$. Since U does not contain $w \in W_j$, we have $j \notin J$, so U is disjoint from W_j. With these modifications, the proof of Lemma 8, and hence that of Theorem 9, go over under the present hypotheses. □

To what spaces X might one apply such a construction? The set $\beta(\omega)$ has a natural compact Hausdorff topology, under which it is the Stone-Čech compactification of the discrete space ω. However, taking X to be $\beta(\omega)$ and the dense subset to which we eventually restrict our functions to be ω does not seem a good candidate for our purposes. For in the natural topology on $\beta(\omega)$, the set of neighborhoods of any $\mathcal{U} \in \beta(\omega)$, intersected with ω, give precisely the members of $\mathcal{U}$. Since $\mathcal{U}$ is a maximal filter on ω, if we tried to "puncture" these neighborhoods further, in a way that affected their intersections with ω, we would get some neighborhoods that intersected ω in the empty set; i.e., ω would cease to be dense.

On the other hand, the less exotic space 2^c, i.e., the continuum power of the discrete space 2, also has countable dense subsets; let me sketch how to obtain one. Identify c with the set of $\{0,1\}$-valued functions on ω. Let $\mathrm{Boole}(\omega)$ be the free Boolean algebra on an ω-tuple of indeterminates $x_0, x_1, \ldots$, i.e., the set of finitary Boolean operations in countably many variables. Then to each $b(x_0, x_1, \ldots) \in \mathrm{Boole}(\omega)$ we can associate a subset of c, namely the set S_b of ω-tuples $(e_0, e_1, \ldots)$ $(e_i \in \{0,1\})$ such that $b(e_0, e_1, \ldots) = 1$. (Intuitively, the set of assignments that "satisfy" the Boolean condition given by b.) If we now think of 2^c as the power set of c, so that each S_b is a member thereof, I claim that the countable set $\{S_b \mid b \in \mathrm{Boole}(\omega)\} \subseteq 2^c$ is dense. Indeed, a basis of the topology of 2^c is given by the solution-sets of statements saying that a certain finite list of elements of c should, and another finite list should not, belong to the members of 2^c considered. Since the members of any finite family of elements of c may be distinguished by looking at finitely many coordinates, we can, for any solution-set U as above, find some $b \in \mathrm{Boole}(\omega)$ such that S_b satisfies the criterion for belonging to U.

It should be possible to "puncture" the topology on 2^c, so as to get stronger, non-compact topologies under which the above countable set remains dense. Whether one could get such a topology that satisfied (22), or some other condition from which one could prove that the group G of $\mathbb{Z}$-valued functions continuous in that topology had dual free on the set of evaluations at the points of 2^c, is not clear to me.

6. Some other subgroups of $\mathbb{Z}^\omega$.

We have seen that it is not likely that one can get an affirmative answer to Question 11 by finding a group $G \subseteq \mathbb{Z}^\omega$ whose dual is spanned by evaluations of elements at members of $\beta(\omega)$. However, while hoping to do so, I came upon some curious subgroups G, which I sketch below for their own interest.

My idea was that since $\beta(\omega)$ is constructed using only the set-theoretic structure of ω, and not its order, etc., one should look at groups whose definitions likewise "treat all points of ω alike"; i.e., subgroups of $\mathbb{Z}^\omega$ invariant under the action of the full symmetric group on ω. These contrast with subgroups of the sort commonly studied. (For instance, those in [13] consist of the sequences with prescribed bounds on their growth rates.)

I will start with a class of examples that actually offers a faint hope of giving a construction with the desired sort of dual. Suppose we take any nondiscrete Hausdorff *group topology* $\mathcal{T}$ on $\mathbb{Z}$ (a topology under which the group operations are continuous; for instance, the p-adic topology for some prime p, or the topology induced by an embedding in the circle group, or one of the topologies constructed in [12]). Now let $G_\mathcal{T} \subseteq \mathbb{Z}^\omega$ consist of all g such that the set of values of g has compact closure $C(g)$ within $\mathbb{Z}$ under that topology. (For instance, if $(k_i)_{i\in\omega}$ is a sequence of integers converging under $\mathcal{T}$ to an integer k, then any $g \in \mathbb{Z}^\omega$ whose components all lie in $\{k_i \mid i \in \omega\} \cup \{k\}$ will have this property.) It is easy to see that $G_\mathcal{T}$ is a subgroup of $\mathbb{Z}^\omega$. Given $g \in G_\mathcal{T}$, every ultrafilter $\mathcal{U}$ on ω induces an ultrafilter $g(\mathcal{U})$ on $g(\omega) \subseteq C(g)$, which, by compactness of the latter set, will converge to some element $\lim_\mathcal{U} g \in C(g) \subseteq \mathbb{Z}$. For each $\mathcal{U}$, this construction $g \mapsto \lim_\mathcal{U} g$ is a homomorphism $h_\mathcal{U} : G_\mathcal{T} \to \mathbb{Z}$. Whether for some topology $\mathcal{T}$ on $\mathbb{Z}$ these homomorphisms $h_\mathcal{U}$ span $\mathrm{Hom}(G_\mathcal{T}, \mathbb{Z})$, I do not know.

A different sort of subgroup invariant under all permutations of ω is

$$\{f \in \mathbb{Z}^\omega \mid (\exists\, k \in \omega)\, (\forall\, n > 0)\ f \text{ assumes at most } k \text{ values modulo } n \text{ infinitely many times}\}. \tag{23}$$

A subgroup of (23) is

$$\{f \in \mathbb{Z}^\omega \mid (\forall\, n > 0) \text{ there are only finitely many } m \in \omega \text{ such that } n \nmid f(m)\}. \tag{24}$$

For a final class of examples, let us start with any set S of integers. Then the subgroup $G_S \subseteq \mathbb{Z}^\omega$ generated by the set $(\{0\} \cup S)^\omega$ is clearly invariant under permutations of ω. It can be described as

$$G_S \ = \ \{f \in \mathbb{Z}^\omega \mid (\exists\, k \in \omega) \text{ every value assumed by } f \text{ is the sum of at most } k \text{ terms taken from } \pm S \text{ (counting repetitions)}\}. \tag{25}$$

So, for instance, if $S = \{1\}$, then G_S is the group B of bounded functions; if S is the set of powers of 2, then G_S can be described as the set of all sequences of integers whose binary expressions (ignoring initial $\pm$ signs) have a common bound on the number of substrings "10" that they contain. If S has compact closure under some group topology $\mathcal{T}$ on $\mathbb{Z}$, then G_S will be contained in the group $G_\mathcal{T}$ discussed above. On the other hand, if S is not sufficiently sparse, e.g., if it is the set of all squares, then G_S is the full group $\mathbb{Z}^\omega$.

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