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CAN ONE FACTOR THE CLASSICAL ADJOINT OF A GENERIC MATRIX?

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Abstract. Let k be an integral domain, n a positive integer, X a generic $n\times n$ matrix over k (i.e., the matrix (x_{ij}) over a polynomial ring $k[x_{ij}]$ in n^2 indeterminates x_{ij}), and $\mathrm{adj}(X)$ its classical adjoint. For $\mathrm{char}\, k=0$ it is shown that if n is odd, $\mathrm{adj}(X)$ is not the product of two noninvertible $n\times n$ matrices over $k[x_{ij}]$, while for n even, only one special sort of factorization occurs. Whether the corresponding results hold in positive characteristic is not known.

The operation adj on matrices arises from the $(n-1)$st exterior power functor on modules; the analogous factorization question for matrix constructions arising from other functors is raised, as are several other questions.

1. Introduction.

If A is an $n\times n$ matrix over a commutative ring, and $\mathrm{adj}(A)$ its *classical adjoint*, i.e., the $n\times n$ matrix of appropriately signed minors of A, we have the well-known factorization

$$\det(A)\, I_n \;=\; A\ \mathrm{adj}(A) \tag{1}$$

[2, p.193, (5)], [8, Prop. XIII.4.16]. Do the factors on the right in (1) have any further natural factorizations?

To make this question precise, let us fix an integral domain k, and let $k[x_{ij}]$ be a polynomial ring in n^2 indeterminates x_{ij} $(1\leq i,j\leq n)$. The matrix $X=(x_{ij})$ is called a *generic* $n\times n$ matrix over k, and we ask whether one can refine the factorization

$$\det(X)\, I_n \;=\; X\ \mathrm{adj}(X) \tag{2}$$

as a factorization of $\det(X)\, I_n$ into noninvertible $n\times n$ matrices over $k[x_{ij}]$.

Taking the determinant of (2), we see that $\det(\mathrm{adj}(X))=\det(X)^{n-1}$. Moreover, $\det(X)$ is irreducible over $k[x_{ij}]$. Indeed, $\det(X)$ is homogeneous of degree 1 in the entries of each row of X, hence any factor must be homogeneous of degree 1 or 0 in those entries. Hence if, in a factorization of $\det(X)$, one factor involves an x from some row, then the other factor cannot involve any x from that row, so the first factor must involve all the x's in that row; and the same applies to columns. It follows that in any

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factorization, one factor must involve all the indeterminates and the other factor none; hence the latter belongs to k, and must in fact be a unit thereof, since the coefficients of the monomials in $\det(X)$ are ± 1.

It follows that in (2), X cannot be factored further into noninvertible square matrices, and that if k is a field, so that $k[x_{ij}]$ is a unique factorization domain, any such factorization of the other term, say

$$\operatorname{adj}(X) \;=\; YZ, \tag{3}$$

must, up to units, satisfy

$$\det(Y) \;=\; \det(X)^d, \quad \det(Z) \;=\; \det(X)^{n-1-d}, \quad \text{where } 0<d<n-1. \tag{4}$$

One can deduce that this is also true whenever k is an integral domain, by noting that it holds over the field of fractions of k, and again handling scalars in k by looking at monomials having coefficient ± 1 in $\det(X)^{n-1}$.

Given any homomorphism φ of k-algebras, let us also use the symbol φ for the induced map on $n\times n$ matrices. Note that for each $n\times n$ matrix A over a commutative k-algebra R, there is a unique k-algebra homomorphism $\varphi_A\colon k[x_{ij}]\to R$ carrying the generic matrix X to A. This map φ_A will therefore carry a factorization (3), if one exists, to a factorization of $\operatorname{adj}(A)$, and the entries of the factor matrices $\varphi_A(Y)$ and $\varphi_A(Z)$ will be given by polynomials in the entries of A. In particular, if $R=k=$ the field of real or complex numbers, the matrices $\varphi_A(Y)$ and $\varphi_A(Z)$ will vary continuously with A.

By combining this observation with topological results from [4] and [6] we shall, in Theorem 5, exclude, for k of characteristic 0, the possibility of any factorizations (3) satisfying (4) except possibly when n is even and one of the exponents d or $n-1-d$ is 1.

In the first preprint version of this note, I asked whether the latter case actually occurred; I and those I discussed the question with all expected a negative answer. To our surprise, an affirmative answer was obtained (for arbitrary k) by R.-O. Buchweitz and G. Leuschke [1]. Section 6 below gives a quick proof of the existence of such factorizations, inspired by working backwards from their construction in [1].

Let us note one caveat before beginning the development of Theorem 5. If we have a factorization (3), the induced factorizations $\operatorname{adj}(A)=\varphi_A(Y)\varphi_A(Z)$ will be functorial, in the sense that they will respect homomorphisms among k-algebras; but it cannot be assumed that they will have other reasonable "functoriality-like" properties, even when these hold for adj. For instance, because the construction adj is induced by dualization (matrix transpose) followed by the $(n-1)$st exterior power functor on modules, it satisfies the multiplicative relation

$$\operatorname{adj}(AB) \;=\; \operatorname{adj}(B)\operatorname{adj}(A), \tag{5}$$

but it does not follow that for Y as in (3) we will have $\varphi_{AB}(Y)=\varphi_B(Y)\,\varphi_A(Y)$. For another example, (1) applied to a matrix $U\in \mathrm{SL}_n(k)$, gives $\operatorname{adj}(U)=U^{-1}$, which combined with (5) yields $\operatorname{adj}(UAU^{-1})=U\operatorname{adj}(A)\,U^{-1}$. But again, no such property can be assumed for $\varphi_A(Y)$.

On the other hand, let us note some valid consequences of functoriality in k. If we have a factorization (3) satisfying (4) over a base ring k, we immediately get such a factorization over every ring to which k can be mapped homomorphically; hence in proving *nonexistence* of such factorizations, results for algebraically closed fields k will imply results for general commutative rings k. Moreover, since a factorization of $\mathrm{adj}(X)$ over a given k involves only finitely many elements of k, and any finitely generated field of characteristic 0 embeds in $\mathbb{C}$, restrictions on the form of factorizations with $k = \mathbb{C}$ will imply the corresponding restrictions for all fields of characteristic 0, and hence for all integral domains of characteristic 0.

(The limitation to integral domains is needed so that we can say that any factorization (3) satisfies (4) for some d. Over a ring k of the form $k_1 \times k_2$, in contrast, we can get a factorization that "looks like" $\mathrm{adj}(X) = \mathrm{adj}(X) \cdot I_n$ over k_1, but like $\mathrm{adj}(X) = I_n \cdot \mathrm{adj}(X)$ over k_2. However, if we consider only factorizations satisfying (4), the restrictions on d that we will obtain for $k = \mathbb{C}$ will hold by the above reasoning for every commutative ring k of characteristic 0.)

2. Valuations and ranks.

We shall show here for k a field of arbitrary characteristic that if there exists a factorization (3), then, for appropriate families of matrices A, the induced matrices $\varphi_A(Y)$ will have constant rank. Varying A will thus yield a continuous map between Grassmannian varieties; it is to this that we will apply topological results in the next section. Our proof of the constant-rank result begins with

Lemma 1. *Let R be a discrete valuation ring with valuation v, maximal ideal $\mathbf{m}$, and residue map $\pi: R \to R/\mathbf{m}$. If M is an $n \times n$ matrix over R such that $\pi(M)$ has nullity r (i.e., rank $n-r$), then $v(\det(M)) \geq r$.*

Proof. Left multiplication by some invertible matrix $\pi(U)$ over $R/\mathbf{m}$ turns $\pi(M)$ into a matrix whose last r rows are zero. Since $\pi(U)$ is invertible, $v(\det(U)) = 0$, so $v(\det(M)) = v(\det(UM))$, which is $\geq r$ since UM has r rows in $\mathbf{m}$. □

Corollary 2. *Let p be an irreducible element in a unique factorization domain R, v_p the corresponding valuation on R, and $\pi_p : R \to R/pR$ the residue map. Then for M a square matrix over R, $\pi_p(M)$ has rank at least $n - v_p(\det(M))$.*

Proof. Localize at pR, and apply the preceding lemma in contrapositive form. □

Using this we can prove

Lemma 3. *Let $X = (x_{ij})$ be a generic $n \times n$ matrix over a field k, and suppose $\mathrm{adj}(X)$ admits a factorization (3) satisfying (4) for some d. Let A be any $n \times n$ matrix over k which has the eigenvalue 0 with multiplicity exactly 1, and let φ_A denote the homomorphism $k[x_{ij}] \to k$ taking X to A. Then*

$$\mathrm{rank}(\varphi_A(Y)) = n-d, \qquad \mathrm{rank}(\varphi_A(Z)) = d+1,$$

$$\mathrm{rank}(\varphi_A(XY)) = n-1-d, \qquad \mathrm{rank}(\varphi_A(ZX)) = d.$$

Proof. Let $k[t]$ be a polynomial ring in one indeterminate, and v_t the valuation on this ring induced by the element t. From the hypothesis on A, we see that $\det(tI_n + A)$, i.e., the characteristic polynomial of $-A$ in the indeterminate t, has constant term 0 but nonzero coefficient of t, so $v_t(\det(tI_n + A)) = 1$. Writing $\psi : k[x_{ij}] \to k[t]$ for $\varphi_{(tI_n+A)}$, the k-algebra homomorphism taking X to $tI_n + A$, we observe that

$$v_t(\det(\psi(Y))) \;=\; v_t(\psi(\det(Y))) \;=\; v_t(\psi(\det(X)^d)) \;=\; v_t(\det(tI_n + A)^d) \;=\; d. \qquad (6)$$

Letting $\pi_t\colon k[t] \to k$ take t to 0, we have $\pi_t\psi = \varphi_A$, hence taking $M = \psi(Y)$ in Corollary 2 so that $\pi_t(M) = \varphi_A(Y)$, and using (6) to evaluate $v_t(\det(M))$, we get $\operatorname{rank}(\varphi_A(Y)) \geq n - d$. Similarly, $\operatorname{rank}(\varphi_A(Z)) \geq n - (n-1-d) = d + 1$.

On the other hand, note that $A\ \varphi_A(Y)\ \varphi_A(Z) = \varphi_A(X\,Y Z) = \varphi_A(\det(X)\, I_n) = \det(A)\, I_n = 0$, so the nullities of A, of $\varphi_A(Y)$ and of $\varphi_A(Z)$ must add up to at least n, i.e., their ranks can sum to at most $2n$. Since the rank of the first is $n-1$ and we have just shown those of the other two to be at least $n - d$ and $d + 1$, these must be their exact values, giving the first two equalities. The other two are seen similarly. (In obtaining the last one, we use (2) in the form $\det(X)\, I_n = \operatorname{adj}(X)\, X$, easily deduced from the form given.) $\square$

Remark: The above hypothesis that the eigenvalue 0 have multiplicity 1 is stronger than saying that A has rank $n-1$. For example, the matrix consisting of a single $n\times n$ Jordan block with eigenvalue 0 has rank $n-1$, but its eigenvalue 0 has multiplicity n.

We will formulate our next result in algebraic-geometric terms. For base field $\mathbb{R}$ or $\mathbb{C}$, this formulation will imply the corresponding topological statement, which is what we will use in the next section; but as we will discuss in §5, the algebraic-geometric statement has the potential of also yielding results in positive characteristic.

For $0 \leq d \leq n$, let $\mathrm{Gr}_k(d, n)$ denote the Grassmannian variety over k whose K-valued points, for a field K over k, correspond to d-dimensional subspaces $V_d \subseteq K^n$. On the other hand, let $\mathrm{CGr}_k(d, n)$ (for "complemented Grassmannian") denote the variety whose K-valued points correspond to pairs (V_d, V'_{n-d}) consisting of a d-dimensional subspace V_d and an $(n-d)$-dimensional subspace V'_{n-d} such that $K^n = V_d \oplus V'_{n-d}$.

The variety $\mathrm{Gr}_k(d, n)$ is projective; in particular $\mathrm{Gr}_k(1, n)$ is $(n-1)$-dimensional projective space. On the other hand, $\mathrm{CGr}_k(d, n)$ is affine, since it can be identified with the variety of idempotent $n\times n$ matrices of rank d.

Proposition 4. *Suppose* $\operatorname{adj}(X)$ *admits a factorization* (3) *satisfying* (4) *for some* d. *Then there exists a morphism of algebraic varieties* $\mathrm{CGr}_k(1, n) \to \mathrm{Gr}_k(d, n)$ *which takes every pair* $(V_1,\, V'_{n-1})$ *to a subspace of its second component* V'_{n-1}, *and likewise a morphism* $\mathrm{CGr}_k(1, n) \to \mathrm{Gr}_k(n-1-d, n)$ *with the same property.*

Proof. Given a K-valued point $a = (V_1, V'_{n-1})$ of $\mathrm{CGr}_k(1, n)$, let E_a denote the idempotent matrix over K that projects K^n onto V'_{n-1} along V_1. This has eigenvalue 0 with multiplicity 1, hence by Lemma 3, the matrix $E_a\ \varphi_{E_a}(Y)$ has rank $n-1-d$; so its column space, a subspace of the column space V'_{n-1} of E_a, has that rank. This construction can be seen to give a morphism of varieties, the second of the morphisms whose existence we were to prove.

To get the first, note that taking the transpose of equation (3) and applying it to the transpose of the matrix X gives a factorization $\mathrm{adj}(X) = Z'Y'$ with $\det(Y') = \det(Y)$ and $\det(Z') = \det(Z)$. Applying the preceding result to this factorization gives the desired morphism. $\square$

3. The hairy sphere raises its unkempt head.

If $n \leq 2$, the condition $0 < d < n-1$ of (4) cannot be satisfied, so the first case where a factorization (3) might be possible is when $n = 3$, $d = 1$. Suppose we had such a factorization for $k = \mathbb{R}$. Every point p of the real unit sphere S^2 determines a point $(\mathbb{R}p, (\mathbb{R}p)^{\perp})$ of $\mathrm{CGr}_{\mathbb{R}}(1,3)$, so applying to this either of the morphisms of Proposition 4, we would get a continuous map $S^2 \to \mathrm{Gr}_{\mathbb{R}}(1,3)$ that takes each $p \in S^2$ to a 1-dimensional subspace of $(\mathbb{R}p)^{\perp}$; in other words, of the tangent space to S^2 at p. This would constitute a "combing of a hairy sphere", which is known to be impossible [7, Theorem 16.5], [5, p.282], so no such factorization exists.

(The "hairy sphere" result as generally formulated asserts, for even m, the nonexistence of a nowhere zero tangent vector field on S^m. What the above construction gives is a map taking each $p \in S^2$ to a point of the projective plane $\mathrm{Gr}_{\mathbb{R}}(1,3)$ representing an unoriented tangent direction at p. But by simple connectedness of S^2, we could lift this to a map to the universal covering space of that projective plane, S^2, which we could interpret as a tangent vector field of everywhere unit length, getting the desired contradiction.)

4. The general result.

For general n and base-field $\mathbb{C}$, we can use in place of the "hairy sphere theorem" some results proved in [4] and [6]. As we did with $\mathrm{CGr}_{\mathbb{R}}(1,3)$ and $\mathrm{Gr}_{\mathbb{R}}(1,3)$ in the preceding section, in the proof of the next theorem we shall regard varieties $\mathrm{Gr}_{\mathbb{C}}(d,n)$ and $\mathrm{CGr}_{\mathbb{C}}(d,n)$ as topological manifolds (consisting of the $\mathbb{C}$-valued points of the algebraic varieties), and so be able speak of continuous maps between them.

Theorem 5. *Suppose k is an integral domain of characteristic 0. Then if n is odd, there is no factorization (3) of $\mathrm{adj}(X)$ into noninvertible matrices, while if n is even, any such factorization has one of the exponents in (4) equal to 1, i.e., has $d = 1$ or $d = n-2$.*

Proof. As noted in the next-to-last paragraph of §1, it will suffice to prove this result for $k = \mathbb{C}$. Let us put a Hermitian inner product on $\mathbb{C}^n$; then $L \mapsto (L, L^{\perp})$ is a continuous map $\mathrm{Gr}_{\mathbb{C}}(1,n) \to \mathrm{CGr}_{\mathbb{C}}(1,n)$. If we have a factorization (3), Proposition 4 gives a continuous map $\mathrm{CGr}_{\mathbb{C}}(1,n) \to \mathrm{Gr}_{\mathbb{C}}(d,n)$ taking (V_1, V'_{n-1}) to a subspace of its second component. Composing, we get a continuous function $\mathrm{Gr}_{\mathbb{C}}(1,n) \to \mathrm{Gr}_{\mathbb{C}}(d,n)$ taking each 1-dimensional subspace $L \subseteq \mathbb{C}^n$ to a d-dimensional subspace L' of $L^{\perp}$.

This gives a d-dimensional subbundle of the tangent bundle on $(n-1)$-dimensional complex projective space, which by [6, Theorem 1.1(ii)] is possible if and only if n is even and $d = 1$ or $n-2$. Alternatively we may note that for L' as in the preceding paragraph, the map $L \mapsto L \oplus L'$ takes each 1-dimensional subspace L of $\mathbb{C}^n$ to a $(d+1)$-dimensional subspace containing L, which, by [4, Theorem 1.5(a)] can only happen if n is even and $d+1 = 2$ or $n-1$, i.e., again $d = 1$ or $n-2$. $\square$

5. The question in positive characteristic.

I do not know whether Theorem 5 remains true if the characteristic 0 hypothesis is deleted. One could hope to prove such a result using algebraic geometry in place of our topological arguments.

Now the analog of [4, Theorem 1.5(a)] with continuous maps of topological spaces replaced by morphisms of algebraic varieties over a general algebraically closed field indeed holds [*ibid.,* Theorem 1.5(b)]. However, the map $L \mapsto (L, L^{\perp})$ that we also called on is not a morphism of algebraic varieties, so we cannot use it as before to yoke the abovementioned result together with Proposition 4. (That map was constructed from a Hermitian inner product, which is not bilinear but sesquilinear. A genuine bilinear form on $\mathbb{C}^n$ cannot be positive definite, so that $L^{\perp}$, defined using such a form, will not always be complementary to L. And if one tries to retreat to the case $k = \mathbb{R}$ and use a real inner product, this will not keep its positive definiteness at non-real points, hence still does not lead to a morphism of varieties $\mathrm{Gr}_k(1,n) \to \mathrm{CGr}_k(1,n)$.) Indeed, there can be no nontrivial morphism of algebraic varieties $\mathrm{Gr}_k(1,n) \to \mathrm{CGr}_k(1,n)$, because $\mathrm{Gr}_k(1,n)$ is projective while $\mathrm{CGr}_k(1,n)$ is affine.

What we may hope for, instead, is an analog of [4, Theorem 1.5(b)] applying directly to morphisms $\mathrm{CGr}_k(1,n) \to \mathrm{Gr}_k(d,n)$. We note, however, that [4, Theorem 1.5(b)] has only one exceptional case for n even, the case $d = n-1$, in place of the two cases $d = 2$ and $d = n-1$ of [4, Theorem 1.5(a)]. Yet the example of the next section shows that both of the latter cases occur; so the desired result on maps $\mathrm{CGr}_k(1,n) \to \mathrm{Gr}_k(d,n)$ would have to have a weaker conclusion than that of [4, Theorem 1.5(b)].

6. A factorization when n is even.

We shall now see that factorizations of the sort allowed by Theorem 5 when n is even do occur. Our argument is inspired by the construction of Buchweitz and Leuschke [1].

Lemma 6. *Let R be a commutative integral domain, n a positive integer, and X an $n \times n$ matrix over R having determinant* 0. *Then*

(i) $\mathrm{rank}(\mathrm{adj}(X)) \leq 1$.

(ii) *For any alternating $n \times n$ matrix A over R, one has* $\mathrm{adj}(X)\, A\, \mathrm{adj}(X)^{\mathrm{T}} = 0$.

Proof. (i) If $\mathrm{rank}(X) = n-1$, this follows from the equation $X\, \mathrm{adj}(X) = \det(X) I_n = 0$. If $\mathrm{rank}(X) < n-1$, then all $(n-1) \times (n-1)$ minors of X are zero, so $\mathrm{adj}(X) = 0$.

(ii) Since A is alternating, every row r of $\mathrm{adj}(X)$ satisfies $r A r^{\mathrm{T}} = 0$. But by (i), the rows of $\mathrm{adj}(X)$ are pairwise linearly dependent, so for any two rows r, r' of $\mathrm{adj}(X)$ we see (using the fact that R has no zero-divisors) that $r A r'^{\mathrm{T}} = 0$. ☐

Our desired factorization is now given by part (iii) of

Theorem 7. *Let R be a commutative ring, n a positive integer, X an $n \times n$ matrix over R, and A any alternating $n \times n$ matrix over R. Then*

(i) *All entries of* $\mathrm{adj}(X)\, A\, \mathrm{adj}(X)^{\mathrm{T}}$ *are divisible by* $\det(X)$.

(ii) $\mathrm{adj}(X)\, A$ *is right divisible by* X^{T} *and* $A\, \mathrm{adj}(X)$ *is left divisible by* X^{T}.

(iii) (*Buchweitz and Leuschke* [1]) *If A is invertible* (*so that n is necessarily even*) *then* $\mathrm{adj}(X)$ *is right divisible by* $X^{\mathrm{T}} A$ *and left divisible by* $A X^{\mathrm{T}}$.

Proof. Clearly (i) and (ii) reduce to the case where R is a polynomial ring over the integers, X a matrix of distinct indeterminates, and A an alternating matrix having distinct indeterminates for its above-diagonal entries. In this case, R is a UFD and $\det(X)$ an irreducible element, so that $R/(\det(X))$ is an integral domain. Applying Lemma 6(ii) to the image of the element $\operatorname{adj}(X)\ A\ \operatorname{adj}(X)^{\mathrm{T}}$ in this domain, we get (i).

To get (ii), let us rewrite (i) (still in the case where R is a polynomial ring) as

$$\operatorname{adj}(X)\ A\ \operatorname{adj}(X)^{\mathrm{T}} \;=\; Y\,(\det(X)\,I_n)$$

for some matrix Y over R. Making the substitution $\det(X)\,I_n = X^{\mathrm{T}}\operatorname{adj}(X)^{\mathrm{T}}$ in the right-hand side of this equation, we can right-cancel $\operatorname{adj}(X)^{\mathrm{T}}$ (since it has nonzero determinant and R is a domain), getting $\operatorname{adj}(X)\,A = Y X^{\mathrm{T}}$, the desired right divisibility relation. The left divisibility statement follows by symmetry.

For A invertible, (iii) follows from (ii) by putting A^{-1} in place of A. (Note that in the resulting factorization, the factor $X^{\mathrm{T}}A$ or $A\,X^{\mathrm{T}}$ has, up to units, determinant $\det(X)$, hence the other factor has determinant $\det(X)^{n-1}$. The latter factor is constructed explicitly in [1], in terms of determinantal minors.) □

Let us record an interesting way of looking at (i) above.

Corollary 8. *Under the general hypothesis of Theorem 7, if R is an integral domain and X is nonsingular, then the matrix $X^{-1}A\,(X^{\mathrm{T}})^{-1}$ over the field of fractions of R has all its entries in $\det(X)^{-1}R$. (I.e., these entries, which one would* a priori *expect to write using denominator $\det(X)^2$, can in fact be written with denominator $\det(X)$.)*

Proof. Multiply the statement of Theorem 7(i) by $(\det X)^{-2}$, recalling that by (2), $\det(X)^{-1}\operatorname{adj}(X) = X^{-1}$, and hence that $\det(X)^{-1}\operatorname{adj}(X^{\mathrm{T}}) = (X^{\mathrm{T}})^{-1}$. □

Can we push the factorizations of Theorem 7(iii) still further? Suppose A and A' are two invertible alternating matrices over R, and we write the factorizations given by that result as

$$(A X^{\mathrm{T}})\,Z \;=\; \operatorname{adj}(X) \;=\; Y'\,(X^{\mathrm{T}}A'). \tag{7}$$

Might Z and Y' themselves admit nontrivial factorizations?

A look at Theorem 5 quickly eliminates all possibilities except that Z might have a right factor whose determinant (up to units) is $\det(X)$ and/or that Y' might have a left factor with this property. Buchweitz and Leuschke inform me, however, that they can show that no such factorizations occur.

Nonetheless, their result [1, Corollary 2.4] shows that in a different sense, the two factorizations of (7) have a "common refinement"; that sense being that there exist a scalar $r \in R$ and a matrix W over R such that

$$\operatorname{adj}(X) \;=\; A(rX^{\mathrm{T}} + X^{\mathrm{T}}WX^{\mathrm{T}})A', \tag{8}$$

an equation which we see includes both the left divisibility of $\operatorname{adj}(X)$ by AX^{T} and its right divisibility by $X^{\mathrm{T}}A'$.

(Cf. [3, Proposition 7.3(i), p.118]. Nothing like the hypothesis of that result is satisfied here. However that result presents a sequence of ways in which a noncommutative ring expression can have two factorizations; and (8) is an instance of the $n = 2$ term of that sequence.)

7. Further questions.

The factorizations of Theorem 7(iii) are noncanonical: they depend on an invertible alternating matrix A. This suggests that a context in which they would have a natural meaning is that of a vector space given with a nondegenerate alternating bilinear form. It would be interesting to know whether they can be given some illuminating interpretation in that context.

Returning to the question with which we began this paper, but taking a more extravagant goal than we did, we may ask whether one can describe *all* maximal factorizations of the matrix $\det(X)I_n$ into noninvertible $n\times n$ matrices over $k[x_{ij}]$. For any n, in addition to the factorization (2), the same factorization with the order of factors reversed, and the two factorizations arising similarly from the transpose of X, there is an obvious factorization into n diagonal matrices each having determinant $\det(X)$:

$$\det(X)\,I_n \;=\; \mathrm{diag}(\det(X),1,...,1)\cdot\ \ldots\ \cdot\mathrm{diag}(1,...,1,\det(X)). \tag{9}$$

When n is odd, are (2), its three variants just noted, and (9) "essentially" all there are?

A factorization can be trivially perturbed by multiplying two successive factors on the right and the left respectively by an invertible matrix U over $k[x_{ij}]$ and its inverse. Also, because $\det(X)I_n$ is central, we can multiply the last factor in a factorization by such a matrix U on the right and the first factor by U^{-1} on the left. We may ask whether for n odd the five factorizations of the preceding paragraph form a set of representatives of the orbits of all maximal factorizations of $\det(X)I_n$ under these sorts of perturbations. (Factorizations gotten by permuting the terms of (9) are not counterexamples: they can be obtained using permutation matrices in the role of U.) For n even, we get additional factorizations from Theorem 7(iii), this time parametrized by an alternating matrix A. (It is not clear how many degrees of freedom these additional families have, modulo the equivalence relation introduced above.)

In each of the explicit factorizations noted above, one can see or show by homogeneity arguments that the degree of the product matrix $\det(X)\,I_n$ in the n^2 indeterminates is precisely the maximum of the sums of the degrees of the matrix entries that get multiplied together; i.e., that there is not too much "cancellation" in the calculation of $\det(X)\,I_n$ as a product. We can, however, easily destroy this property by interpolating invertible matrices U over $k[x_{ij}]$ having entries of high degree, and their inverses. Could there be some principle saying that any factorization of a "good" matrix over a polynomial ring is a perturbation, via interpolated matrices and their inverses, of a factorization in which the degree is well-behaved?

Turning in a different direction, let us observe that for an $n\times n$ matrix A over a commutative ring k, say representing a linear map $a\colon k^n\to k^n$, the classical adjoint $\mathrm{adj}(A)$ can be characterized as the transpose of the matrix representing the linear map $\bigwedge^{n-1}a\colon \bigwedge^{n-1}k^n\to\bigwedge^{n-1}k^n$, where $\bigwedge^{n-1}$ denotes the $(n-1)$st exterior power functor. If instead we apply to a a lower exterior power functor $\bigwedge^m$, we get an endomorphism of the module $\bigwedge^m k^n$, which is free of rank $\binom{n}{m}$. Again taking for A a generic matrix X, we may ask whether the resulting $\binom{n}{m}\times\binom{n}{m}$ matrix over $k[x_{ij}]$ can be factored into noninvertible square matrices. (This matrix, incidentally, has determinant $\det(X)^{\binom{n-1}{m-1}}$, and its product with the transpose of the matrix representing

$\bigwedge^{n-m} a$, with rows and columns appropriately indexed, is $\det(X)$ times the $\binom{n}{m}\times\binom{n}{m}$ identity matrix.)

Each of the above functors $\bigwedge^m$ is a homomorphic image of the m-fold tensor product functor $\otimes^m$. In fact, when char $k=0$, $\otimes^m$ decomposes into a direct sum of subfunctors indexed by Young diagrams, and the functor $\bigwedge^m$ corresponds to the height-m column of boxes. (The length-m row of boxes corresponds in the same way to the mth symmetric power functor.) We may thus pose for each of these summands of $\otimes^m$ (and for the various naturally arising functors in finite characteristic) the same question we have studied here for $\bigwedge^{n-1}$!

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