# Generating infinite symmetric groups*

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**Abstract**

Let $S = \mathrm{Sym}(\Omega)$ be the group of all permutations of an infinite set $\Omega$. Extending an argument of Macpherson and Neumann, it is shown that if $U$ is a generating set for $S$ as a group, respectively as a monoid, then there exists a positive integer $n$ such that every element of $S$ may be written as a group word, respectively a monoid word, of length $\leq n$ in the elements of $U$.

Some related questions and recent results are noted, and a brief proof is given of a result of Ore's on commutators that is used in the proof of the above result.

## 1 Introduction, notation, and some lemmas on full moieties.

In [14, Theorem 1.1] Macpherson and Neumann show that if $\Omega$ is an infinite set, then the group $S = \mathrm{Sym}(\Omega)$ is not the union of a chain of $\leq |\Omega|$ proper subgroups. We will repeat the beautiful proof of that result, with modifications that will allow us to obtain along with it the result stated in the abstract. The present section is devoted to obtaining strengthened versions of the lemmas used in that proof.

Following the notation of [14], for $\Omega$ an infinite set, $\mathrm{Sym}(\Omega)$, generally abbreviated $S$, will denote the group of all permutations of $\Omega$, and such permutations will be written to the right of their arguments. For subsets $\Sigma \subseteq \Omega$ and $U \subseteq S$, the symbol $U_{(\Sigma)}$ will denote the set of elements of $U$ that stabilize $\Sigma$ pointwise, and $U_{\{\Sigma\}}$ the set $\{f \in U : \Sigma f = \Sigma\}$. (In [14] this notation was used only for $U$ a subgroup.) A subset $\Sigma \subseteq \Omega$ will be called *full* with respect to $U \subseteq S$ if the set of permutations of $\Sigma$ induced by members of $U_{\{\Sigma\}}$ is all of $\mathrm{Sym}(\Sigma)$. The cardinality of a set $X$ will be written $|X|$, and a subset $\Sigma \subseteq \Omega$ will be called a *moiety* if $|\Sigma| = |\Omega| = |\Omega - \Sigma|$.

Suppose $\Sigma_1$ and $\Sigma_2$ are moieties of $\Omega$ whose intersection is also a moiety, and whose union is all of $\Omega$. Then [14, Lemma 2.3] says that if $G$ is a subgroup of $S = \mathrm{Sym}(\Omega)$ such that $\Sigma_1$ and $\Sigma_2$ are both full with respect to $G$, then $G = S$. To strengthen this result, we will consider subsets $U, V \subseteq S$, closed under inverses, such that $\Sigma_1$ is full with respect to $U$ and $\Sigma_2$ with respect to $V$. By the lemma cited, $\langle U \cup V \rangle = S$; our version of this result will bound the number of factors from $U$ and $V$ needed to get the general element of $S$.

Our proof will use the following fact, first proved by Ore [18]. Much stronger results have been proved since. In §4 we will give a self-contained proof of a statement of intermediate strength.

**Lemma 1** ([18], cf. §4 below)**.** *For $\Omega$ an infinite set, every element $f \in \mathrm{Sym}(\Omega)$ can be written as a commutator, $f = g^{-1}h^{-1}gh$ $(g, h \in \mathrm{Sym}(\Omega))$.* □

Here now is the result on full moieties.

**Lemma 2** (cf. [14, Lemma 2.3])**.** *Suppose $\Sigma_1, \Sigma_2$ are moieties of $\Omega$ whose intersection is a moiety, and whose union is all of $\Omega$; and suppose $U, V$ are subsets of $S = \mathrm{Sym}(\Omega)$, each closed under inverses, such that $\Sigma_1$ is full with respect to $U$ and $\Sigma_2$ is full with respect to $V$. Then $S = (UV)^4 V \cup (VU)^4 U$.*

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Preprint versions of this paper: http://math.berkeley.edu/~gbergman/papers/Sym_Omega:1.{tex,dvi,ps} and arXiv:math.GR/0401304.

*Proof.* Note that $S_{(\Omega-(\Sigma_1\cap\Sigma_2))} \cong \mathrm{Sym}(\Sigma_1\cap\Sigma_2)$. By Lemma 1, any element $f$ of the latter group may be written as a commutator $f = g^{-1}h^{-1}gh$. Since $\Sigma_1$ is full with respect to $U$, we can find an element of $U_{\{\Sigma_1\}}$ which behaves like $g$ on $\Sigma_1\cap\Sigma_2$ and as the identity on $\Sigma_1-\Sigma_2$; likewise, we can find an element of $V_{\{\Sigma_2\}}$ which behaves like $h$ on $\Sigma_1\cap\Sigma_2$ and as the identity on $\Sigma_2-\Sigma_1$. Clearly the commutator of these elements behaves like $f$ on $\Sigma_1\cap\Sigma_2$ and as the identity on $\Omega-(\Sigma_1\cap\Sigma_2)$. Hence

(1) $\qquad S_{(\Omega-(\Sigma_1\cap\Sigma_2))} \subseteq UVUV.$

Now $|\Sigma_1\cap\Sigma_2| = |\Omega| = |\Sigma_2-\Sigma_1|$, where the first equality holds because $\Sigma_1\cap\Sigma_2$ is a moiety and the second because $\Sigma_1 = \Omega-(\Sigma_2-\Sigma_1)$ is one. Hence $\mathrm{Sym}(\Sigma_2)$ contains an element interchanging the subsets $\Sigma_1\cap\Sigma_2$ and $\Sigma_2-\Sigma_1$; hence $V$ has an element which behaves that way on $\Sigma_2$ (and in an unspecified manner on $\Sigma_1-\Sigma_2$). Conjugating (1) by such an element, we get

(2) $\qquad S_{(\Sigma_1)} \subseteq VUVUVV.$

Since the assumptions on $\Sigma_1$ and $\Sigma_2$ are symmetric, we also have the corresponding formula for $S_{(\Sigma_2)}$, with $U$ and $V$ interchanged.

Now suppose we are given $f\in S$, which we wish to write as a product of elements of $U$ and $V$.

We shall see, roughly, that a product of one element from $U$ and one element of $V$ suffices to distribute the elements of $\Omega$ between $\Sigma_1$ and its complement exactly as $f$ does. An application of an element of $U$ will then put the elements that have been moved into $\Sigma_1$ in exactly the desired places, and a final application of (2) will administer the *coup de grâce*.

The details: Note first that the set $(\Sigma_1\cap\Sigma_2)f^{-1}$ must either contain $|\Omega|$ elements of $\Sigma_1$ or $|\Omega|$ elements of $\Sigma_2$; without loss of generality assume the former. (This is the reason for the word "roughly" in the preceding paragraph. In the contrary case, the roles stated there for $\Sigma_1$ and $\Sigma_2$, and likewise for $U$ and $V$, will be reversed.) In particular, $\Sigma_1 f^{-1}$ contains $|\Omega|$ elements of $\Sigma_1$. Hence we can find a permutation $a\in U_{\{\Sigma_1\}}$ which maps all elements of $\Sigma_1$ which are not in $\Sigma_1 f^{-1}$ (if any) into $\Sigma_1\cap\Sigma_2$, and which also maps into that set $|\Omega|$ elements of $\Sigma_1$ which *are* in $\Sigma_1 f^{-1}$. These conditions, and the fact that $a\in U_{\{\Sigma_1\}}$ takes $\Omega-\Sigma_1$ to itself, together imply that $a$ maps all elements of $(\Omega-\Sigma_1)f^{-1}$ (both those in $\Sigma_1$ and those in $\Omega-\Sigma_1$) into $\Sigma_2$, and also takes $|\Omega|$ elements of $\Sigma_1 f^{-1}$ there. We can now choose $b\in V_{\{\Sigma_2\}}$ which maps into $\Omega-\Sigma_1$ the images under $a$ of all elements of $(\Omega-\Sigma_1)f^{-1}$ and nothing else; i.e., such that $(\Omega-\Sigma_1)f^{-1}ab = \Omega-\Sigma_1$. Taking complements, we have $\Sigma_1 f^{-1}ab = \Sigma_1$, so as $\Sigma_1$ is full with respect to $U$, we can find $c\in U_{\{\Sigma_1\}}$ which agrees on $\Sigma_1$ with the inverse of $f^{-1}ab$; i.e., such that $f^{-1}abc\in S_{(\Sigma_1)}$. Now (2) applied to the inverse of the latter element gives us $(abc)^{-1}f\in VUVUVV$, so $f\in (UVU)(VUVUVV) = (UV)^4V$. As noted earlier, the roles of $U$ and $V$ may be the opposite of those we have assumed, giving the alternative possibility $f\in (VU)^4U$. ◻

The result from [14] that we have just strengthened was used there to show that if a subgroup $G\leq S$ has a full moiety, then there exists $x\in S$ such that $\langle G\cup\{x\}\rangle = S$. The version proved above yields the more precise statement:

**Lemma 3** (cf. [14, Lemma 2.4])**.** *If a subset $U\subseteq S$ closed under inverses has a full moiety, then there exists $x\in S$ of order 2 such that $(Ux)^7U^2x\cup(xU)^7xU^2 = S$.*

*Proof.* Given a full moiety $\Sigma_1$ for $U$, choose any moiety $\Sigma_2\subseteq\Omega$ such that $\Sigma_1\cap\Sigma_2$ is a moiety and $\Sigma_1\cup\Sigma_2=\Omega$. Since $\Omega-\Sigma_1$ and $\Omega-\Sigma_2$ are disjoint and both have the cardinality of $\Omega$, we can find an element $x$ of order 2 which interchanges those two sets, and hence also interchanges their complements, $\Sigma_1$ and $\Sigma_2$. The fact that $\Sigma_1$ is a full moiety for $U$ makes $\Sigma_2=\Sigma_1x$ a full moiety for $x^{-1}Ux = xUx$. Setting $xUx=V$, we may apply the preceding lemma. The expression $(UV)^4V$ becomes $(UxUx)^4xUx = (Ux)^8xUx = (Ux)^7UxxUx = (Ux)^7U^2x$, while the other term is the conjugate of this by $x$, namely $(xU)^7xU^2$. ◻

We conclude this section with a diagonal argument, using nothing but the definition of full moiety and basic set theory, which we extract virtually unchanged from the proof of [14, Theorem 1.1]:

**Lemma 4.** *Let $\Omega$ be an infinite set, let $S=\mathrm{Sym}(\Omega)$, and let $(U_i)_{i\in I}$ be any family of subsets of $S$ such that $\bigcup_I U_i = S$ and $|I|\leq|\Omega|$. Then $\Omega$ contains a full moiety with respect to at least one of the $U_i$.*

*Proof.* Since $|\Omega|$ is infinite and $I \leq |\Omega|$, we can write $\Omega$ as a union of disjoint moieties $\Sigma_i$ $(i \in I)$. If there are no full moieties with respect to $U_i$ for any $i$, then in particular, for each $i$ the set $\Sigma_i$ is non-full with respect to $U_i$, so we can choose $f_i \in \mathrm{Sym}(\Sigma_i)$ which is not the restriction to $\Sigma_i$ of a member of $U_i$. Now let $f \in \mathrm{Sym}(\Omega)$ be the permutation whose restriction to each $\Sigma_i$ is $f_i$. Then $f$ cannot belong to any of the $U_i$, contradicting the assumption $\bigcup_I U_i = S$, and completing the proof. $\square$

# 2 Chains of subsets of $\mathrm{Sym}(\Omega)$.

Let us begin by recovering [14, Theorem 1.1]. Our statement will be the contrapositive of that in [14].

I also include parenthetically the corresponding statement with chains of submonoids in place of chains of subgroups. As we will see, this follows trivially from the result on chains of subgroups; but it took me a long time to discover that trivial argument.

**Theorem 5** ([14, Theorem 1.1])**.** *If $\Omega$ is an infinite set and $(G_i)_{i\in I}$ a chain of subgroups (or more generally, submonoids) of $S = \mathrm{Sym}(\Omega)$, with $\bigcup_{i\in I} G_i = S$ and $|I| \leq |\Omega|$, then $G_i = S$ for some $i \in I$.*

*Proof.* Lemma 4 shows that $\Omega$ has a full moiety with respect to some $G_i$, hence assuming the $G_i$ are subgroups, Lemma 3 shows that $S = \langle G_i \cup \{x\}\rangle$ for some $x \in S$. Since the $G_j$ form a chain with union $S$, there is some $j \geq i$ with $x \in G_j$, hence $G_j \supseteq G_i \cup \{x\}$, hence $G_j = S$.

If the $G_i$ are merely submonoids, we apply the result of the preceding paragraph to $(G_i \cap G_i^{-1})_{i\in I}$, which is clearly a chain of sub*groups* with union $S$. $\square$

Now for our new result.

**Theorem 6.** *Suppose $\Omega$ is an infinite set, and $U$ a generating set for $\mathrm{Sym}(\Omega)$ as a group (respectively, as a monoid). Then there exists a positive integer $n$ such that every element of $\mathrm{Sym}(\Omega)$ is represented by a group word (respectively, a monoid word) of length $\leq n$ in the elements of $U$.* $\square$

*Proof.* Here it suffices to prove the monoid case, since the group words in the elements of $U$ are just the monoid words in the elements of $U \cup U^{-1}$.

So assume $U$ generates $S$ as a monoid. For $i = 1, 2, \ldots,$ let $U_i = (U \cup \{1\})^i \cap (U^{-1} \cup \{1\})^i$. By assumption the sets $(U \cup \{1\})^i$ have union $S$, hence so do their inverses, $(U^{-1} \cup \{1\})^i$, hence so do the intersections $U_i$. Since $\aleph_0 \leq |\Omega|$, Lemma 4 says that $\Omega$ has a full moiety with respect to some $U_i$. By Lemma 3 there exists $x \in S$ such that $S = (U_i x)^7 U_i^2 x \cup (x U_i)^7 x U_i^2$, which we see is contained in $(U_i \cup \{x\})^{17}$. Taking a $j \geq i$ such that $x \in U_j$, we get $(U \cup \{1\})^{17j} = ((U \cup \{1\})^j)^{17} \supseteq (U_i \cup \{x\})^{17} = S$. $\square$

One may ask whether for a given set $\Omega$, there is some single $n$ as in Theorem 6 that works for every generating set $U$. To see that this is not so, let $\Omega = \mathbb{Q}/\mathbb{Z}$, and let us give this set the natural metric, of diameter $1/2$, under which the distance between two cosets of $\mathbb{Z}$ is the minimum of the distances between their members, as real numbers. (In other words, let us use the metric on $\Omega = \mathbb{Q}/\mathbb{Z}$ induced by the arc-length metric on $\mathbb{R}/\mathbb{Z}$.) Fixing an integer $n$, let $U$ denote the set of permutations of $\Omega$ which move all elements by distances $< 1/(2n)$. Clearly, $U^n \neq \mathrm{Sym}(\Omega)$. However, I claim that $U$ is a generating set for $\mathrm{Sym}(\Omega)$ as a group (and indeed, since it is closed under inverses, as a monoid).

Note first that if $\Sigma$ is the image in $\Omega$ of any interval of length $< 1/(2n)$ in $\mathbb{Q}$, then $U$ contains $S_{(\Omega-\Sigma)}$, the group of permutations that act arbitrarily on $\Sigma$ and fix all elements outside it. Now we can cover $\Omega$ with a finite number of successive overlapping sets $\Sigma_1, \Sigma_2, \ldots, \Sigma_r$ of this sort, and then use Lemma 2 to conclude inductively that $\langle U\rangle \supseteq S_{(\Omega-(\Sigma_1\cup\ldots\cup\Sigma_i))}$ for $i = 1, \ldots, r$, hence that $\langle U\rangle \supseteq S_{(\Omega-(\Sigma_1\cup\ldots\cup\Sigma_r))} = S_{(\varnothing)} = S$, as claimed.

# 3 Questions, examples, remarks, and related literature.

The statement of [14, Theorem 1.1] = Theorem 5 above, unlike that of Theorem 6, depends on the cardinal $|\Omega|$. Let us, for the purposes of this section, weaken that statement to one that holds independent of this cardinal, by using the obvious lower bound $\aleph_0 \leq |\Omega|$. Then that theorem can be looked at as saying, for every infinite set $\Omega$, that $\mathrm{Sym}(\Omega)$ belongs to the class of groups $G$ satisfying

(3) If $G$ is written as the union of a chain of subgroups $G_0 \leq G_1 \leq \ldots$ indexed by $\omega$, then for some $n$, $G_n = G$.

Theorem 6, similarly, says that $\mathrm{Sym}(\Omega)$ belongs to the class of groups $G$ satisfying

(4) If $G$ is generated as a group by a subset $U$, then for some $n$, every element of $G$ is represented by a group word of length $\leq n$ in the elements of $U$.

Clearly (3) also holds for all finitely generated groups, and (4) for all finite groups. Thus, in a strange way, the groups $\mathrm{Sym}(\Omega)$ resemble finite groups.

Condition (3) on a non-finitely-generated group is commonly expressed by saying that $G$ has *uncountable cofinality,* and is known to hold in many cases. Some works on cofinalities of groups are [5], [9], [13], [19], [22] and [21]; see also other papers cited in [9]. Whether or not $G$ is finitely generated, (3) is equivalent to the condition that the fixed point set construction on $G$-sets commutes with direct limits over countable index sets [1, end of §2]. In fact, it was the wish to give in [1] an example of a non-finitely-generated group having this property that led me to read [14], eventually resulting in this note. The analog of condition (3) for modules has been studied under many different names; cf. [11, p.895, top paragraph].

In response to a question posed in an earlier version of this note, Droste and Göbel [9] have obtained a general technique for showing that certain sorts of structures containing many isomorphic copies of themselves have automorphism groups satisfying (3) and (4); in particular, they find that (3) and (4) hold for the groups of all self-homeomorphisms of the Cantor set, of the rational numbers, and of the irrational numbers, and for the group of Borel automorphisms of the real numbers. Droste and Holland [10] obtain the same conclusions for the automorphism group of any doubly homogeneous totally ordered set, and Tolstykh [23], [24] proves (4), and, insofar as it was not already known, (3), for the automorphism groups of infinite-dimensional vector spaces and various sorts of relatively free groups. In the final remark of [23] he suggests an approach to showing that the full automorphism groups of free objects in other well-behaved varieties of algebras have this property. Mesyan [15] obtains analogous results for endomorphism rings of infinite direct sums and products of a module, and Cornulier [6, Proposition 4.4] does the same for the Boolean ring of subsets of an infinite set; he also shows in [6, Theorem 3.1] that any $\omega_1$-existentially closed group satisfies (3) and (4).

Let us note some general facts about conditions (3) and (4). It is not hard to see that both properties are preserved under taking homomorphic images, and that (3) is also preserved under group extensions and under passing to groups finitely generated over $G$. To get some similar results for (4), we will want

**Lemma 7.** *Let $H < G$ be groups and $U$ a generating set for $G$. Suppose that for some $n \geq 0$, every right coset of $H$ in $G$ contains a group word of length $\leq n$ in the elements of $U$. Then the set of elements of $H$ that can be written as words of length $\leq 2n+1$ in the elements of $U$ generates $H$.*

*Proof.* Let $V$ be a set of right coset representatives for $H$ in $G$ consisting of words of length $\leq n$ in $U$, with the coset $H$ represented by the element $1$, and let $r : G \to V$ be the retraction collapsing each coset to its representative. Let $W$ denote the set of elements of $H$ that can be written as words of length $\leq 2n+1$ in the elements of $U$.

For any $v \in V$ and $u \in U \cup U^{-1}$, note that $vu = (vu\ r(vu)^{-1}) r(vu)$. Since $r(vu)$ by definition lies in the same right coset as $vu$, the factor $vu\ r(vu)^{-1}$ lies in $H$, and since $v$ and $r(vu)$, as members of $V$, each have length $\leq n$ in the elements of $U$, that factor has length $\leq 2n+1$, and so lies in $W$. Thus, $V(U \cup U^{-1}) \subseteq WV$. It follows that $\bigcup_i W^i V$ is closed under right multiplication by $U \cup U^{-1}$, hence equals the whole group $G$. We now intersect the equation $\bigcup_i W^i V = G$ with $H$. This has the effect of discarding elements on the left-hand side having right factors from $V$ other than $1$, and so gives $\bigcup_i W^i = H$, completing the proof. $\square$

We can now show that (4) is preserved under group extensions. Given a short exact sequence $1 \to H \to G \to E \to 1$ where $H$ and $E$ satisfy (4), and a generating set $U$ for $G$, the fact that $E$ satisfies (4) yields an $n$ as in the hypothesis of Lemma 7. The conclusion of that lemma, combined with the fact that $H$ satisfies (4), shows that all elements of $H$ can be written as words of length $\leq m$ in the elements of $U$ for some $m$. It follows that all elements of $G$ can be written as words of length $\leq n+m$. A similar application of that lemma shows that (4) is preserved under passing to overgroups in which $G$ has finite index.

Clearly, a countable group satisfies (3) if and only if it is finitely generated, while an infinite group that is finitely generated can never satisfy (4). In particular, (4) is not preserved under passing to groups finitely generated over $G$. Neither property is preserved under passing to normal subgroups, since for $\Omega$ a countably infinite set, the subgroup of $\mathrm{Sym}(\Omega)$ consisting of permutations that move only finitely many elements is normal, but satisfies neither (3) nor (4).

In response to a question posed in an earlier version of this preprint, Khélif has announced in [12] that (3), and also the conjunction of (3) and (4), are preserved under passing to subgroups of finite index, but that (4) alone is not.

A question that, to my knowledge, is still unanswered is

**Question 8.** *Are there any countably infinite groups that satisfy* (4)*?*

One can show that every non-finitely-generated abelian group can be mapped surjectively either onto a group $Z_{p^\infty}$, or onto an infinite direct sum $Z_{p_1} \oplus Z_{p_2} \oplus \ldots$ for (not necessarily distinct) prime numbers $p_1, p_2, \ldots$ . Neither of the latter sorts of groups satisfy (3) or (4), so no non-finitely-generated abelian group has either of these properties.

It is shown in [13] that the direct product of any family of copies of a nonabelian finite simple group satisfies (3). In [6, §4] this is strengthened to say that the direct product of any family of copies of a finite perfect group satisfies (3) and (4), while [9, Lemma 3.5] shows that the same is true of the direct product of any family of copies of $\mathrm{Sym}(\omega)$. These results suggest the general question of when (3) and (4) are inherited by products. They are both inherited by finite products, by our observations on group extensions. That they are not always inherited by infinite products is shown by any infinite direct product of nontrivial finite abelian groups: the factors, as finite groups, satisfy both conditions, but the product, a non-finitely-generated abelian group, satisfies neither. The same technique can be used to show the failure of these conditions for certain direct products of perfect groups, using the fact that an infinite product of perfect groups may be non-perfect. Khélif has announced the existence of a direct product of groups such that each factor satisfies (4) and the product is perfect but which still does not satisfy (4).

We saw in the proof of Theorem 5 that (3) implies the corresponding condition with "subgroups" replaced by "submonoids". The same implication for the negations of these conditions is clear, so the group and monoid versions of (3) are equivalent. For (4), the proof of Theorem 6 showed that the statement for generation as a monoid implied the statement for generation as a group. In this case, we can also get the converse *if* we assume (3). For suppose the group $G$ satisfies (3) and (4), and that $U$ is a generating set for $G$ as a monoid. As in the proof of Theorem 6, let $U_i = (U \cup \{1\})^i \cap (U^{-1} \cup \{1\})^i$ $(i \in \omega)$. These sets form a chain with union $G$, hence so do the subgroups $\langle U_i \rangle$, hence by (3), some $\langle U_j \rangle$ is equal to $G$; so by (4), there is an $n$ such that all elements of $G$ are group words of length $\leq n$ in elements of $U_j$. By construction, $U_j$ is closed under inverses, so these group words reduce to monoid words, so $G = (U_j)^n \subseteq ((U \cup \{1\})^j)^n = (U \cup \{1\})^{jn}$, as claimed. However, the results announced in [12] on subgroups of finite index imply that there are groups satisfying (4) but not (3), leading to

**Question 9.** *Among groups satisfying* (4) *but not* (3)*, are there any which do not satisfy the analog of* (4) *obtained by replacing "generated as a group" and "group word" with "generated as a monoid" and "monoid word"? Any which* do *satisfy that condition?*

We saw at the end of §2 that though $\mathrm{Sym}(\Omega)$ satisfies (4), there is no single $n$ that works for all generating sets $U$. On the other hand, Shelah [20] constructs an uncountable group $G$ in which every generating set $U$ satisfies $U^{240} = G$.

We record two easily reformulations of the conjunction of (3) and (4).

**Lemma 10.** *For any group $G$, the following conditions are equivalent:*

(i) *$G$ satisfies both* (3) *and* (4).

(ii) *If $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_i \subseteq \ldots$ is a chain of subsets of $G$, indexed by the natural numbers and having union $G$, such that $U_i = (U_i)^{-1}$ for all $i$, and such that for all $i, j$ there exists some $k$ with $U_i U_j \subseteq U_k$, then some $U_i$ is equal to $G$.*

(iii) *If $L$ is a natural-number-valued function on $G$ such that for all $g, h \in G$ one has $L(g^{-1}) = L(g)$ and $L(gh) \leq L(g) + L(h)$, then $L$ is bounded above.* □

Functions $L$ as in (iii) above are studied in geometric group theory. (Further conditions are generally assumed, in particular, that the number of $g \in G$ with $L(g) \leq n$ is finite and grows at most exponentially in $n$. Under this assumption, $L$ is known to be approximable by the function giving the length of $g$ with respect to some finite generating set of an overgroup of $G$ [17]. Of course, such a finiteness condition cannot be satisfied when $G$ is an uncountable group such as we are considering.) In [6, §2], condition (iii) above is translated as saying that every isometric action of $G$ on a metric space has bounded orbits, and it is deduced that isometric actions of $G$ on certain sorts of metric spaces must in fact have fixed points.

A property of finite groups $G$ that does *not* hold for the groups $\mathrm{Sym}(\Omega)$ is

(5) Every subset of $U \subseteq G$ which generates $G$ as a group generates $G$ as a monoid.

To see the failure of (5) in $\mathrm{Sym}(\Omega)$ for countable $\Omega$, we may take for $U$ a submonoid $M$ as in the next result.

**Lemma 11.** *Let $\Omega$ be any countable totally ordered set without least or greatest element, and let $M$ be the monoid $\{g \in \mathrm{Sym}(\Omega) : (\forall \alpha \in \Omega)\ \alpha g \geq \alpha\}$. Then $\mathrm{Sym}(\Omega) = M M^{-1} M$.*

*Proof.* Given $f \in \mathrm{Sym}(\Omega)$, I claim that we can find $g, h \in \mathrm{Sym}(\Omega)$ such that for all $\alpha \in \Omega$,

(6) $\alpha \leq \alpha g \geq \alpha h \leq \alpha f.$

From this it will follow that $g,\ h^{-1}g,\ h^{-1}f \in M$, whence $f = g(h^{-1}g)^{-1}(h^{-1}f) \in M M^{-1} M$, as desired.

To get $g$ and $h$ satisfying (6), let an enumeration of the elements of $\Omega$ be chosen. (We will not introduce a notation for this enumeration, but simply speak of "the first element with respect to our enumeration such that ...". In particular, "$\leq$" will continue to denote the given ordering of $\Omega$, not the ordering corresponding to our enumeration.) We shall now construct recursively for each $i \in \omega$ a 3-tuple $(\alpha_i, \beta_i, \gamma_i)$ of elements of $\Omega$; these will eventually be the 3-tuples $(\alpha, \alpha g, \alpha h)$ $(\alpha \in \Omega)$.

To define $(\alpha_i, \beta_i, \gamma_i)$, let us, if $i$ is divisible by 3, take for $\alpha_i$ the first element of $\Omega$ with respect to our enumeration which has not been chosen as $\alpha_j$ at any previous step (i.e., for $0 \leq j < i$), let us then take for $\beta_i$ any element of $\Omega$ which was not chosen as $\beta_j$ at a previous step and which is $\geq \alpha_i$, and for $\gamma_i$ any element not chosen as $\gamma_j$ at a previous step which is both $\leq \beta_i$ and $\leq \alpha_i f$. On the other hand, if $i \equiv 1 \pmod 3$, we start by taking for $\beta_i$ the first element of $\Omega$ with respect to our enumeration which was not chosen as $\beta_j$ at any previous step, then take for $\alpha_i$ any element not previously chosen as an $\alpha_j$ which is $\leq \beta_i$, and for $\gamma_i$ any element not previously chosen as a $\gamma_j$ which is both $\leq \beta_i$ and $\leq \alpha_i f$. Finally, when $i \equiv 2 \pmod 3$, we take for $\gamma_i$ the first element with respect to our enumeration that has not yet been used as a $\gamma_j$, for $\alpha_i$ any element not previously chosen as an $\alpha_j$ such that $\alpha_i f \geq \gamma_i$ (which is possible because there are infinitely many elements $\geq \gamma_i$), and for $\beta_i$ any element not previously chosen in that role which is both $\geq \alpha_i$ and $\geq \gamma_i$.

Clearly this construction uses each element of $\Omega$ once and only once in each position; hence the set of pairs $(\alpha_i, \beta_i)$ is the graph of a permutation $g \in \mathrm{Sym}(\Omega)$, and the set of pairs $(\alpha_i, \gamma_i)$ is the graph of a permutation $h \in \mathrm{Sym}(\Omega)$. The conditions we imposed on our choices at each step insure that (6) holds. $\square$

(In the above lemma we can drop the countability assumption, if we replace the hypothesis of no least or greatest element by the assumption that each element of $\Omega$ has $|\Omega|$ elements above it and $|\Omega|$ elements below it. The assumption that the ordering is total can also be weakened to say that it is upward and downward directed.)

It would be interesting to know what sorts of groups satisfy (5), other than those whose elements all have finite order. One such group is the infinite dihedral group.

My first attempts to prove the statements about sub*monoids* of $S$ in Theorems 5 and 6 revolved around trying to remove from Lemmas 2 and 3 the hypotheses on closure under inverses. Those hypotheses are required by our proof of Lemma 2, since inverses are needed to form commutators and conjugates. Though a different method eventually gave the monoid cases of those theorems, it would still be interesting to know the answer to

**Question 12.** *Are versions of Lemma 2 and/or 3 (possibly using longer products of $x$, $U$ and $V$) true without the hypothesis that $U$ and $V$ are closed under inverses?*

We shall note a weak result in this direction at the end of the next section.

We remark, finally, that it is easy to adapt the method of proof of Theorem 5 to give an apparently more general statement, in which the *chain* of subgroups $G_i$ indexed by a set $I$ of cardinality $\leq |\Omega|$ is replaced by any *directed system* of subgroups $G_i$ indexed by such a set, again having $S$ as union. However, that result in fact follows easily from the theorem as stated. For given such a directed system, let $\{G_i : i \in \kappa\}$ be a subset thereof whose union generates $S$, having least cardinality $\kappa$ among such subsets, and indexed by that cardinal. Then $\{\langle \bigcup_{i<j} G_i \rangle : j < \kappa\}$ forms a chain of *proper* subgroups of $S$, which, unless $\kappa$ is finite, will have union $S$. This would contradict Theorem 5; so $\kappa$ is finite, and the finite family $\{G_i : i \in \kappa\}$ will be majorized by a member of the original directed system, which thus equals $S$.

Further results on the groups $\mathrm{Sym}(\Omega)$ are obtained in [3].

*Added July 2005:* It is shown in [2, §6] that $S = \mathrm{Sym}(\Omega)$ has the property that its countable direct power $S^\omega$ is finitely generated over the diagonal image $\Delta(S) \subseteq S^\omega$, and that this property (for general finitary algebras in the sense of universal algebra, and for groups in particular) is strictly stronger than the conjunction of (3) and (4).

# 4 Appendix: Writing every element of $\mathrm{Sym}(\Omega)$ as a commutator.

In proving Lemma 2, we called on the result of [18] that every element of an infinite symmetric group is a commutator. Now a commutator is an element obtained by dividing an element by a conjugate, $(g^{-1}hg)^{-1}h$; moreover, it is clear that in a symmetric group, every element is conjugate to its inverse; so a commutator in a symmetric group can be described as a product of two elements in the same conjugacy class. Almost a decade after [18] appeared, it was shown that in an infinite symmetric group there in fact exist single conjugacy classes whose square is the whole group. In a series of papers by several authors, culminating in [16] (which describes this history), the conjugacy classes having this property were precisely characterized.

We shall give a self-contained proof of this property for one such conjugacy class in Lemma 14 below, then use the fact that $\mathrm{Sym}(\Omega)$ is the square of a conjugacy class to get a result related to Question 12.

**Definition 13.** *For $\Omega$ an infinite set, we shall call an element $f \in \mathrm{Sym}(\Omega)$* replete *if it has $|\Omega|$ orbits of each positive cardinality $\leq \aleph_0$ (including 1). For a subset $\Sigma \subseteq \Omega$ of cardinality $|\Omega|$, we shall say that $f$ is replete on $\Sigma$ if $\Sigma f = \Sigma$ and the restriction of $f$ to $\Sigma$ is a replete permutation of $\Sigma$.*

Note that a permutation of $\Omega$ that is replete on a subset $\Sigma \subseteq \Omega$ of cardinality $|\Omega|$ is necessarily replete on $\Omega$.

The replete permutations of $\Omega$ clearly form a conjugacy class, so it will suffice to prove

**Lemma 14.** *Every permutation $f$ of an infinite set $\Omega$ is the product of two replete permutations.*

*Proof.* Given $f$, let us choose a moiety $\Sigma_0$ of $\Omega$ such that $f$ moves only finitely many elements from $\Sigma_0$ to $\Omega - \Sigma_0$ or from $\Omega - \Sigma_0$ to $\Sigma_0$. That there exists such a $\Sigma_0$ is immediate if $\Omega$ is uncountable, for we can break $\Omega$ into two families of $|\Omega|$ orbits each, and take for $\Sigma_0$ the union of one of these families. If $\Omega$ is countable, we can use the same method if $f$ has infinitely many orbits, and can also get the same conclusion in an obvious way if $f$ has more than one infinite orbit. If it has exactly one infinite orbit, $\alpha_0\langle f\rangle$, and finitely many finite orbits, then we can take $\Sigma_0 = \{\alpha_0 f^n : n \leq 0\}$; clearly $f$ moves exactly one element out of $\Sigma_0$, and none into it.

After choosing $\Sigma_0$, let us split $\Omega - \Sigma_0$ into two disjoint moieties $\Sigma_1$ and $\Sigma_2$, so that $\Sigma_1$ contains the finitely many elements of $(\Sigma_0 f \cup \Sigma_0 f^{-1}) - \Sigma_0$. I claim that if $g_0$ is any permutation of $\Sigma_0$ and $g_2$ any permutation of $\Sigma_2$, then there exists a permutation $h$ of $\Omega$ such that $fh$ agrees on $\Sigma_0$ with $g_0$, while $h$ agrees on $\Sigma_2$ with $g_2$. Indeed, this pair of conditions specifies the values of $h$ on the two disjoint sets $\Sigma_0 f$ and $\Sigma_2$ in a one-to-one fashion, and both the set on which it leaves $h$ unspecified and the set of elements that it does not specify as values for $h$ are of cardinality $|\Omega|$. Hence the former set can be mapped bijectively to the latter, and the resulting bijection will serve to complete the definition of $h$. (We have used $\Sigma_1$ as a "Hilbert's Hotel" in case $\Sigma_0 f \neq \Sigma_0$.)

Now if we take for $g_0$ and $g_2$ replete permutations of $\Sigma_0$ and $\Sigma_2$ respectively, then $h$ will be replete on $\Sigma_2$, hence replete, while $fh$ will be replete on $\Sigma_0$, hence replete. Thus $f = (fh)h^{-1}$ is a product of two replete permutations, as we wished to show. $\square$

(In [8, Theorem 3.1(a)] the same result is proved for a different conjugacy class, that of permutations with $|\Omega|$ infinite orbits and no finite orbits, also by an unexpectedly simple argument. However, it takes some work to isolate that argument from the lengthier proofs of other results that are being carried out there simultaneously. A different sort of generalization of Ore's result is obtained in [7] and papers cited there, which characterize the group words $w$ which are "universal" in infinite permutation groups, in the sense established for the word $x^{-1}y^{-1}xy$ by Ore's result.)

We shall now show that the proofs of Lemmas 2 and 3 can be adapted to the situation where $U$ and $V$ are not assumed closed under inverses if we allow ourselves to use, along with multiplication, the right conjugation operation

(7) $\qquad g^h \;=\; h^{-1}g\,h.$

**Lemma 15** (cf. Lemma 3)**.** *Let $\Omega$ be an infinite set, and $U \subseteq S = \mathrm{Sym}(\Omega)$ a subset with respect to which some moiety of $\Omega$ is full. Then there exist $x, y \in S$, with $x$ of order 2, such that*

(8) $\qquad S \;=\; (Ux)^3(y^U)^2x \;\cup\; x(Ux)^3(y^U)^2.$

*Sketch of proof.* Let $\Sigma_1$ be a full moiety for $U$, let $\Sigma_2$ be a moiety such that $\Sigma_1 \cap \Sigma_2$ is a moiety and $\Sigma_1 \cup \Sigma_2 = \Omega$, let $x \in S$ be an involution such that $\Sigma_1 x = \Sigma_2$, and let $y \in S_{(\Sigma_2)}$ be an element such that the group $S_{(\Sigma_2)} \cong \mathrm{Sym}(\Omega - \Sigma_2)$ is the square of the conjugacy class of $y$ in that group. Such a $y$ exists by Lemma 14 above, or the results in the papers cited. Combining this property of $y$ with the fact that $\Sigma_1$, and hence its subset $\Omega - \Sigma_2$, is full with respect to $U$, we conclude that $S_{(\Sigma_2)} \subseteq (y^U)^2$; hence, conjugating by $x$,

(9) $\qquad S_{(\Sigma_1)} \subseteq x(y^U)^2x.$

Now let $V = x\,Ux$. Since $\Sigma_1$ is a full moiety under $U$, $\Sigma_2$ will be a full moiety under $V$. Using the technique of proof of Lemma 2, with (9) in place of (2), we get (8). $\square$

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