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Thoughts on Eggert's Conjecture

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To T. Y. Lam, on his 70th birthday

ABSTRACT. Eggert's Conjecture says that if R is a finite-dimensional nilpotent commutative algebra over a perfect field F of characteristic p, and $R^{(p)}$ is the image of the p-th power map on R, then $\dim_F R \geq p \dim_F R^{(p)}$. Whether this very elementary statement is true is not known.

We examine heuristic evidence for this conjecture, versions of the conjecture that are not limited to positive characteristic and/or to commutative R, consequences the conjecture would have for semigroups, and examples that give equality in the conjectured inequality.

We pose several related questions, and briefly survey the literature on the subject.

1. Introduction

If F is a field of characteristic p, and R is a commutative F-algebra, then the set $R^{(p)}$ of p-th powers of elements of R is not only closed under multiplication, but also under addition, by the well-known identity

$$(x+y)^p \;=\; x^p + y^p \quad (x, y \in R). \tag{1}$$

Hence $R^{(p)}$ is a subring of R. If, moreover, F is a *perfect* field (meaning that every element of F is a p-th power – as is true, in particular, if F is finite, or, at the other extreme, algebraically closed), then the subring $R^{(p)}$ is also closed under multiplication by elements of F :

$$a\,x^p \;=\; (a^{1/p}x)^p \in R^{(p)} \quad (a \in F,\, x \in R). \tag{2}$$

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In this situation we can ask "how big" the subalgebra $R^{(p)}$ is compared with the algebra R, say in terms of dimension over F.

If we take for R a polynomial algebra $F[x]$ over a perfect field F, we see that $R^{(p)} = F[x^p]$, so intuitively, $R^{(p)}$ has a basis consisting of one out of every p of the basis elements of R. Of course, these bases are infinite, so we can't divide the cardinality of one by that of the other. But if we form finite-dimensional truncations of this algebra, letting $R = F[x]/(x^{N+1})$ for large integers N, then we see that the dimension of $R^{(p)}$ is indeed about $1/p$ times the dimension of R. If we do similar constructions starting with polynomials in d variables, we get $R^{(p)}$ having dimension about $1/p^d$ times that of R.

Is the ratio $\dim R^{(p)}/\dim R$ always small? No; a trivial counterexample is $R = F$; a wider class of examples is given by the group algebras $R = F\,G$ of finite abelian groups G of orders relatively prime to p. In G, every element is a p-th power, hence $R^{(p)}$ contains all elements of G, hence, being closed under addition and under multiplication by members of F, it is all of R; so again $\dim R^{(p)}/\dim R = 1$.

In the above examples, the p-th power map eventually "carried things back to themselves". A way to keep this from happening is to assume the algebra R is *nilpotent*, i.e., that for some positive integer n, $R^n = 0$, where R^n denotes the space of all sums of n-fold products of members of R. This leads us to

Conjecture 1 (Eggert's Conjecture [**9**]). *If R is a finite-dimensional nilpotent commutative algebra over a perfect field F of characteristic $p > 0$, then*

$$\dim_F R \;\geq\; p \dim_F R^{(p)}. \tag{3}$$

Of course, a nonzero nilpotent algebra does not have a unit. Readers who like their algebras unital may think of the R occurring above and throughout this note as the maximal ideal of a finite-dimensional local unital F-algebra.

Let us set down some conventions.

Conventions 2. *Throughout this note, F will be a field. The symbol "*$\dim$*" will always stand for "*$\dim_F$*", i.e., dimension as an F-vector-space.*

Except where the contrary is stated (in a few brief remarks and two examples), F-algebras will be assumed associative, but not, in general, unital. (Most of the time, we will be considering commutative algebras, but we will make commutativity explicit. When we simply write "associative algebra", this will signal "not necessarily commutative".) An ideal *of an F-algebra will mean a ring-theoretic ideal which is also an F-subspace.*

If R is an F-algebra, V an F-subspace of R, and n a positive integer, then V^n will denote the F-subspace *of R spanned by all n-fold products of elements of V, while $V^{(n)}$ will denote the* set *of n-th powers of elements of V.*

Thus, if V is a subspace of a commutative algebra R over a perfect field F of characteristic p, then $V^{(p)}$ will also be a subspace of R, but for a general base-field F, or for noncommutative R, this is not so. The map $x \mapsto x^p$ on a commutative algebra R over a field of characteristic p is called the *Frobenius map*.

We remark that the unital rings $R = F[x]/(x^{N+1})$ that we discussed before we introduced the nilpotence condition generally fail to satisfy (3). Most obvious is the case $N = 0$, where $R = F$. More generally, writing $N = pk + r$ $(0 \leq r < p)$, so that the lowest and highest powers of x in the natural basis of R are x^0 and

x^{pk+r}, we find that $\dim R^{(p)}/\dim R = (k+1)/(pk+r+1)$, which is $> 1/p$ unless $r = p-1$.

The corresponding nilpotent algebras are constructed from the "nonunital polynomial algebra", i.e., the algebra of polynomials with zero constant term, which we shall write

$$[F][x] \;=\; \{\sum_{i>0} a_i x^i\} \;\subseteq\; F[x]. \tag{4}$$

When we divide this by the ideal generated by x^{N+1}, again with $N = pk + r$ $(0 \le r < p)$, we find that $\dim R^{(p)}/\dim R = k/(pk+r)$, which is always $\le 1/p$, with equality only when $r = 0$, i.e., when $p \mid n$.

As before, examples like $R = [F][x,y]/(x^M, y^N)$ give ratios $\dim R^{(p)}/\dim R$ strictly lower than $1/p$. This suggests that generation by more than one element tends to lower that ratio, and that perhaps that ratio can equal $1/p$ only for cyclic algebras. This is not the case, however. Indeed, it is easy to verify that that ratio is multiplicative with respect to tensor products,

$$\dim (R \otimes S)^{(p)}/\dim (R \otimes S) \;=\; (\dim R^{(p)}/\dim R)\,(\dim S^{(p)}/\dim S). \tag{5}$$

Hence if we tensor a nilpotent algebra R of the form $[F][x]/(x^{pk+1})$, for which we have seen that the ratio is $1/p$, with a non-nilpotent algebra for which the ratio is 1 (for instance, a group algebra $F\,G$ with $p \nmid |G|$), we get further nilpotent examples for which the ratio is $1/p$. Also, $\dim R$ and $\dim R^{(p)}$ are both additive with respect to *direct products*; so any direct product of two nilpotent algebras for each of which the ratio is $1/p$ is another such algebra. In §5 we will discover further examples in which the ratio comes out exactly $1/p$, for reasons that are less clear.

2. A first try at proving Eggert's Conjecture

We have seen that for R a commutative algebra over a perfect field F of characteristic $p > 0$, the p-th power map on R over F is "almost" linear. In particular, its image is a vector subspace (in fact, a subalgebra).

Pleasantly, we can even find a vector subspace $V \subseteq R$ which that map sends bijectively to $R^{(p)}$. Namely, take any F-basis B for $R^{(p)}$, let B' be a set consisting of exactly one p-th root of each element of B, and let V be the F-subspace of R spanned by B'. Since the p-th power map sends $\sum_{x\in B'} a_x x$ to $\sum_{x\in B'} a_x^p x^p$, it hits each element of $R^{(p)}$ exactly once.

This suggests the following approach to Eggert's Conjecture. Suppose we take such a subspace V, and look at the subspaces V, V^2, ..., V^p (defined as in the last paragraph of Convention 2). Can we deduce that each of them has dimension at least that of $V^{(p)} = R^{(p)}$ (as the first and last certainly do), and conclude that their sum within R has dimension at least p times that of $R^{(p)}$?

The answer is that yes, we can show that each has dimension at least that of $R^{(p)}$, but no, except under special additional hypotheses, we cannot say that the dimension of their sum is the sum of their dimensions.

The first of these claims can be proved in a context that does not require positive characteristic, or commutativity, or nilpotence. We *will* have to assume F algebraically closed; but we will subsequently see that for commutative algebras over a perfect field of positive characteristic, the general case reduces to that case.

LEMMA 3. *Let F be an algebraically closed field, R an associative F-algebra, V a finite-dimensional subspace of R, and n a positive integer such that every nonzero element of V has nonzero n-th power. Then for all positive integers $i \leq n$ we have*

$$\dim V^i \ \geq \ \dim V. \tag{6}$$

PROOF. Let $d = \dim V$, and let $x_1, \ldots, x_d$ be a basis for V over F. Suppose, by way of contradiction, that for some $i \leq n$ we had $\dim V^i = e < d$. Then we claim that some nonzero $v \in V$ must satisfy $v^i = 0$.

Indeed, writing the general element of V as $v = a_1x_1 + \cdots + a_dx_d$ $(a_1, \ldots, a_d \in F)$, we see that the condition $v^i = 0$, expressed in terms of an e-element basis of V^i, consists of $e < d$ equations, each homogeneous of positive degree (in fact, all of the same degree, i), in d unknowns $a_1, \ldots, a_d$. But a system of homogeneous polynomial equations of positive degrees with fewer equations than unknowns over an algebraically closed field always has a nontrivial solution [**15**, p.65, Corollary 3*]; so, as claimed, there is a nonzero $v \in V$ with $v^i = 0$.

Multiplying by v^{n-i} if $i < n$, or leaving the equation unchanged if $i = n$, we see that $v^n = 0$, contradicting the hypothesis on V, and completing the proof. □

(We could even have generalized the above proof to nonassociative algebras, if we defined x^i inductively as, say, the right-bracketed product $x(x(\ldots x))$, and V^i similarly as $V(V(\ldots V))$.)

Now if F is any perfect field of characteristic p, and $n = p$ (or more generally, a power of p), and R is commutative, then the n-th power map is, up to adjustment of scalars, a linear map of F-vector-spaces, so the statement that it sends no nonzero element of V to 0 says it has trivial kernel; and this property is preserved under extension of scalars to the algebraic closure of F, as are the dimensions of the various spaces V^i. Hence, as stated earlier, in this situation Lemma 3 implies the corresponding result with "algebraically closed" weakened to "perfect".

But unfortunately, we cannot say that $\dim R \geq \sum_{i \leq p} \dim V^i$ unless we know that the sum of the V^i is direct. Here is a special case in which the latter condition clearly holds.

COROLLARY 4. *Let R be a finite-dimensional commutative algebra over a perfect field F of characteristic $p > 0$, and assume that R is graded by the positive integers, is generated by its homogeneous component R_1 of degree 1, and satisfies $(R_2)^{(p)} = 0$.*

Then $\dim R_1, \ldots, \dim R_p$ are all $\geq \dim R^{(p)}$, so $\dim R \geq p \dim R^{(p)}$.

PROOF. Since R is the direct sum of its subspaces R_i, its subalgebra $R^{(p)}$ will be the direct sum of its subspaces $(R_i)^{(p)} \subseteq R_{ip}$. Since R is generated by R_1, we have $R_{i+1} = R_i R_1$ for all i; hence $(R_{i+1})^{(p)} = (R_i)^{(p)} (R_1)^{(p)}$; hence as $(R_2)^{(p)}$ is zero, so are $(R_3)^{(p)}$, $(R_4)^{(p)}$, $\cdots$. Hence $R^{(p)} = (R_1)^{(p)}$.

Now let $d = \dim R^{(p)} \subseteq R_p$, and take a d-dimensional subspace $V \subseteq R_1$ such that the p-th power map carries V bijectively to $R^{(p)}$. By Lemma 3 and the discussion following it, we have $\dim V^i \geq d$ for $i = 1, \ldots, p$, hence

$$\dim R \ = \ \sum_{i=1}^{\infty} \dim R_i \ \geq \ \sum_{i=1}^{p} \dim V^i \ \geq \ p\,d \ = \ p \dim R^{(p)}. \tag{7}$$

□

One might hope to get a similar result for ungraded nilpotent R, by taking the filtration $R \supseteq R^2 \supseteq R^3 \supseteq \dots$, and studying the associated graded algebra, $S = \bigoplus_i S_i$ with $S_i = R^i/R^{i+1}$. This will indeed be generated by S_1; but unfortunately, $R^{(p)}$ will not in general be embedded in S_p, since an element that can be written as a p-th power of one element may be expressible in another way as a product of more than p factors (or a sum of such products), in which case it will have zero image in $S_p = R^p/R^{p+1}$. (What one can easily deduce by this approach is that $\dim R \geq p \dim(R^{(p)}/(R^{(p)} \cap R^{p+1}))$. But that is much weaker than Eggert's conjecture.)

Putting aside the question of whether we can reduce the ungraded case to the graded, let us ask whether, assuming R graded and generated by R_1, we can weaken the hypothesis $(R_2)^{(p)} = 0$ of Lemma 3. Suppose we instead assume $(R_3)^{(p)} = 0$. Thus, $R^{(p)} = (R_1)^{(p)} \oplus (R_2)^{(p)} \subseteq R_p \oplus R_{2p}$.

In addition to our subspace $V \subseteq R_1$ which is mapped bijectively to $(R_1)^{(p)}$ by the p-th power map, we can now choose a subspace $W \subseteq R_2$ that is mapped bijectively to $(R_2)^{(p)}$. Letting $d_1 = \dim (R_1)^{(p)} = \dim V$ and $d_2 = \dim (R_2)^{(p)} = \dim W$, we can deduce from Lemma 3 that $\dim R_1$, $\dim R_2$, $\dots$, $\dim R_p$ are all $\geq d_1$ and that $\dim R_2$, $\dim R_4$, $\dots$, $\dim R_{2p}$ are all $\geq d_2$. The trouble is, these two lists overlap in $\{R_2, R_4, \dots, R_{2\lfloor p/2 \rfloor}\}$, while we know nothing about the sizes of the R_i for odd i between $p+1$ and $2p$. If we could prove that they, like the R_i for even i in that range, all had dimensions at least d_2, we would be in good shape: With R_i at least d_1-dimensional for $i = 1, \dots, p$ and at least d_2-dimensional for $i = p+1, \dots, 2p$, we would have total dimension at least $p \dim(R_1)^{(p)} + p \dim(R_2)^{(p)} = p \dim R^{(p)}$.

One might imagine that since $\dim R_i$ is at least $\dim R_2^{(p)}$ for all even $i \leq 2p$, those dimensions could not perversely come out smaller for i odd. However, the following example, though involving a noncommutative ring, challenges this intuition.

Example 5. *For any positive integer d and any field F, there exists an associative graded F-algebra R, generated by R_1, such that the dimension of the component R_n is $2d$ for every odd $n > 2$, but is $d^2 + 1$ for every even $n > 2$.*

Construction. Let R be presented by $d+1$ generators x, y, z_1, $\dots$, z_{d-1} of degree 1, subject to the relations saying that $xx = yy = 0$, and that every 3-letter word in the generators that does *not* contain the substring xy is likewise 0. It is easy to verify that the nonzero words of length > 2 are precisely those strings consisting of a "core" $(xy)^m$ for some $m \geq 1$, possibly preceded by an arbitrary letter other than x, and/or followed by an arbitrary letter other than y. One can deduce that for $m \geq 1$, the nonzero words of odd length $2m+1$ are of two forms, $(xy)^m\, a$ and $a\,(xy)^m$ for some letter a, and that for each of these forms there are d choices for a, giving $2d$ words altogether; while for words of even length $2m+2$ there are also two forms, $a\,(xy)^m\, b$ and $(xy)^{m+1}$, leading to $d^2 + 1$ words. □

Even for commutative R, we can get a certain amount of irregular behavior:

Example 6. *For any field F there exists a commutative graded F-algebra R, generated by R_1, such that the dimensions of R_1, R_2, R_3, R_4 are respectively* 4, 3, 4, 3.

CONSTRUCTION. First, let S be the commutative algebra presented by generators x, y, z_1, z_2 in degree 1, and relations saying that z_1 and z_2 have zero product with all four generators. We see that for all $n > 1$ we have $\dim S_n = n+1$, as in the polynomial ring $[F][x, y]$, so $S_1, \ldots, S_4$ have dimensions 4, 3, 4, 5. If we now impose an arbitrary pair of independent relations homogeneous of degree 4, we get a graded algebra R whose dimension in that degree is 3 rather than 5, without changing the dimensions in lower degrees. □

As we shall note in §6, much of the work towards proving Eggert's Conjecture in the literature has involved showing that such misbehavior in the sequence of dimensions is, in fact, restricted.

(Incidentally, if we take F in Example 6 to be perfect of characteristic 3, and divide out by R_4, we do not get a counterexample to Eggert's Conjecture; rather, $(R_1)^{(3)}$ turns out to be a proper subspace of R_3.)

3. Relations with semigroups

The examples we began with in §1 were "essentially" semigroup algebras of abelian semigroups.

To make this precise, recall that a *zero* element in a semigroup S means an element z (necessarily unique) such that $sz = zs = z$ for all $s \in S$. If S is a semigroup with zero, and F a field, then the *contracted semigroup algebra* of S, denoted $F_0 S$, is the F-algebra with basis $S - \{z\}$, and multiplication which agrees on this basis with the multiplication of S whenever the latter gives nonzero values, while when the product of two elements of $S - \{z\}$ is z in S, it is taken to be 0 in this algebra [**7**, §5.2, p.160]. So, for example, the algebra $[F][x]/(x^{N+1})$ of §1 is the contracted semigroup algebra of the semigroup-with-zero presented as such by one generator x, and the one relation $x^{N+1} = z$. (Calling this a presentation as a semigroup-with-zero means that we also assume the relations making the products of all elements with z equal to z.)

Above (following [**7**]) I have written z rather than 0 in S, so as to be able to talk clearly about the relationship between the zero element of S and that of $F_0 S$. But since these are identified in the construction of the latter algebra, we shall, for the remainder of this section, write 0 for both, as noted in

CONVENTIONS 7. *In this section, semigroups with zero will be written multiplicatively, and their zero elements written* 0.

If X is a subset of a semigroup S (with or without zero) and n a positive integer, then X^n will denote the set of all n-fold products of elements of X, while $X^{(n)}$ will denote the set of all n-th powers of elements of X. A semigroup S with zero will be called nilpotent *if $S^n = \{0\}$ for some positive integer n.*

Clearly, $F_0 S$ is nilpotent as an algebra if and only if S is nilpotent as a semigroup.

If we could prove Eggert's Conjecture, I claim that we could deduce

CONJECTURE 8 (semigroup version of Eggert's Conjecture). *If S is a finite nilpotent commutative semigroup with zero, then for every positive integer n,*

$$\text{card}(S - \{0\}) \;\geq\; n \;\text{card}(S^{(n)} - \{0\}). \tag{8}$$

Let us prove the asserted implication:

LEMMA 9. *If Conjecture 1 is true, then so is Conjecture 8.*

PROOF. Observe that for any two positive integers n_1 and n_2, and any semigroup S, we have $(S^{(n_1)})^{(n_2)} = S^{(n_1\,n_2)}$. Hence, given n_1 and n_2, if (8) holds for all semigroups S whenever n is taken to be n_1 or n_2, then it is also true for all S whenever n is taken to be $n_1 n_2$. Indeed, in that situation we have

$$\text{(9)} \qquad \operatorname{card}(S-\{0\}) \;\geq\; n_1 \operatorname{card}(S^{(n_1)}-\{0\}) \;\geq\; n_1 n_2 \operatorname{card}(S^{(n_1\,n_2)}-\{0\}).$$

Since (8) is trivial for $n = 1$, it will therefore suffice to establish (8) when n is a prime p. In that case, let F be any perfect field of characteristic p. From (1) and (2) we see that $(F_0\,S)^{(p)} = F_0(S^{(p)})$, and by construction, $\dim_F F_0\,S = \operatorname{card}(S-\{0\})$. Applying Conjecture 1 to $F_0\,S$, we thus get (8) for $n = p$, as required. □

A strange proof, since to obtain the result for an n with k distinct prime factors, we must work successively with semigroup algebras over k different fields!

So much for what we *could* prove if we knew Eggert's Conjecture. What can we conclude about semigroups using what we *have* proved? By the same trick of passing to contracted semigroup algebras, Lemma 3 yields

COROLLARY 10 (to Lemma 3). *Let S be a commutative semigroup with zero, let p be a prime, and let X be a finite subset of S such that the p-th power map is one-to-one on X, and takes no nonzero element of X to 0. Then*

$$\text{(10)} \qquad \operatorname{card}(X^i-\{0\}) \;\geq\; \operatorname{card}(X-\{0\}) \qquad \textit{for } \; 1 \leq i \leq p. \qquad \square$$

Note that even though Lemma 3 was proved for not necessarily commutative R and for exponentiation by an arbitrary integer n, we have to assume in Corollary 10 that S is commutative and p a prime, in order to call on (1) and conclude that $(F_0\,X)^{(p)} = F_0\,(X^{(p)})$,

(Incidentally, the same proof gives us the corresponding result for semigroups S without zero, with (10) simplified by removal of the two "$-\{0\}$"s. However, this result is an immediate consequence of the present form of Corollary 10, since given any semigroup S and subset $X \subseteq S$, we can apply that corollary to X within the semigroup with zero $S \cup \{0\}$; and in that case, the symbols "$-\{0\}$" in (10) have no effect, and may be dropped. Inversely, a proof of Corollary 10 *from* the version for semigroups without zero is possible, though not as straightforward.)

I see no way of proving the analog of Corollary 10 with a general integer n replacing the prime p. (One *can* get it for prime-power values, by noting that (1) and hence Lemma 3 work for exponentiation by p^k. I have not so stated those results only for simplicity of presentation.) We make this

QUESTION 11. *Let S be a commutative semigroup with zero, let n be a positive integer, and let X be a finite subset of S such that the n-th power map is one-to-one on X, and takes no nonzero element of X to 0. Must* $\operatorname{card}(X^i-\{0\}) \geq \operatorname{card}(X-\{0\})$ *for $1 \leq i \leq n$?*

4. Some plausible and some impossible generalizations

The hypothesis of Corollary 10 concerns $\operatorname{card}(X^{(p)}-\{0\})$, while the conclusion is about $\operatorname{card}(X^i-\{0\})$. It is natural to ask whether we can make the hypothesis and the conclusion more parallel, either by replacing X^i by $X^{(i)}$ in the latter (in which case the inequality in the analog of (10) would become equality, since

$X^{(i)} - \{0\}$ can't be larger than $X - \{0\}$), or by replacing $X^{(p)}$ by X^p in the former.

But both of these generalizations are false, as shown by the next two examples.

EXAMPLE 12. *For any prime $p > 2$, and any i with $1 < i < p$, there exists a commutative semigroup S with zero, and a subset X such that the p-th power map is one-to-one on X and does not take any nonzero element of X to 0, but such that* $\mathrm{card}(X^{(i)} - \{0\}) < \mathrm{card}(X - \{0\})$.

CONSTRUCTION. Given p and i, form the direct product of the nilpotent semigroup $\{x, x^2, \dots, x^p, 0\}$ and the cyclic group $\{1, y, \dots, y^{i-1}\}$ of order i, and let X be the subset $\{x\} \times \{1, y, \dots, y^{i-1}\}$. Then the p-th power map from X to $X^{(p)}$ (which is also X^p) is bijective, the common cardinality of these sets being i; but $X^{(i)} = \{x^i\} \times \{1\}$ has cardinality 1. To make this construction a semigroup with zero, we may identify the ideal $\{0\} \times \{1, y, \dots, y^{i-1}\}$ to a single element. □

EXAMPLE 13. *For any prime $p > 2$ there exist a commutative semigroup S with zero, and a subset $X \subseteq S$, such that* $\mathrm{card}(X - \{0\}) = \mathrm{card}(X^p - \{0\})$, *but such that for all i with $1 < i < p$,* $\mathrm{card}(X^i - \{0\}) < \mathrm{card}(X - \{0\})$.

CONSTRUCTION. Let S be the abelian semigroup with zero presented by $p+1$ generators, x, y, $z_1, \dots, z_{p-1}$, and relations saying that each z_i has zero product with every generator (including itself). Thus, S consists of the elements of the free abelian semigroup on x and y, together with the p elements $0, z_1, \dots, z_{p-1}$.

Let X be our generating set $\{x, y, z_1, \dots, z_{p-1}\}$. Then we see that for every $i > 1$, the set $X^i - \{0\}$ has $i+1$ elements, x^i, $x^{i-1}y, \dots, y^i$. Hence $\mathrm{card}(X^p - \{0\}) = p + 1 = \mathrm{card}(X - \{0\})$; but for $1 < i < p$, $\mathrm{card}(X^i - \{0\}) = i + 1 < p + 1$. (We can make this semigroup finite by setting every member of X^{p+1} equal to 0.) □

In the above examples, the case $p = 2$ was excluded because in that case, there are no i with $1 < i < p$. However, one has the corresponding constructions with any prime power $p^r > 2$ in place of p, including powers of 2, as long as one adds to the statement corresponding to Example 12 the condition that i be relatively prime to p.

From the construction of Example 12, we can also obtain a counterexample to a statement which, if it were true, would, with the help of Lemma 3, lead to an easy affirmative answer to Question 11:

EXAMPLE 14. *There exists a commutative semigroup S with zero, a finite subset $X \subseteq S$, and an integer $n > 0$, such that the n-th power map is one-to-one on X and does not take any nonzero element of X to 0, but such that for some field F, the n-th power map on the span FX of X in F_0S does take some nonzero element to 0.*

CONSTRUCTION. Let us first note that though we assumed in Example 12 that p was a prime to emphasize the relationship with Corollary 10, all we needed was that p and i be relatively prime. For the present example, let us repeat that construction with any integer $n > 2$ (possibly, but not necessarily, prime) in place of the p of that construction, while using a prime $p < n$, not dividing n, in place of our earlier i. Thus, the n-th power map is one-to-one on X, but the p-th power map is not.

Now let F be any algebraically closed field of characteristic p. Then on the subspace $FX \subseteq F_0S$, the p-th power map is (up to adjustment of scalars) an F-linear map to the space $FX^{(p)}$ of smaller dimension; hence it has nontrivial kernel. (For the particular construction used in Example 12, that kernel contains $x - xy$.) But any element annihilated by the p-th power map necessarily also has n-th power 0. □

The use of a field F of positive characteristic in the above construction suggests the following question, an affirmative answer to which would indeed, with Lemma 3, imply an affirmative answer to Question 11.

QUESTION 15. *Suppose X is a finite subset of a commutative semigroup S with zero, n a positive integer such that the n-th power map is one-to-one on X and does not take any nonzero element of X to 0, and F a field of characteristic 0. Must every nonzero element of the span FX of X in F_0S have nonzero n-th power?*

In a different direction, Lemma 3 leads us to wonder whether there may be generalizations of Eggert's Conjecture independent of the characteristic.

As a first try, we might consider a nilpotent commutative algebra R over any field F, and for arbitrary $n > 0$, ask whether $\dim(\mathrm{span}(R^{(n)}))/\dim R \leq 1/n$, where $\mathrm{span}(R^{(n)})$ denotes the F-subspace of R spanned by $R^{(n)}$. But this is nowhere near true. Indeed, I claim that

(11) If the characteristic of F is either 0 or $> n$, then $\mathrm{span}(R^{(n)}) = R^n$.

For it is not hard to verify that for any $x_1, \ldots, x_n \in R$,

$$\sum_{S \subseteq \{1,\ldots,n\}} (-1)^{\mathrm{card}(S)} \, (\sum_{i \in S} x_i)^n \;=\; (-1)^n \, n! \; x_1 \ldots x_n. \tag{12}$$

(Every monomial of degree n in $x_1, \ldots, x_n$ other than $x_1 \ldots x_n$ fails to involve some x_m, hence the sets indexing summands of (12) in which that monomial appears can be paired off, $S \leftrightarrow S \cup \{m\}$, one of even and one of odd cardinality. Hence the coefficients of every such monomial cancel, leaving only the multiple of the monomial $x_1 \ldots x_n$ coming from $S = \{1, \ldots, n\}$.) Under the assumption on the characteristic of F in (11), $n!$ is invertible, so (12) shows that $x_1 \ldots x_n \in \mathrm{span}(R^{(n)})$, proving (11). Now taking $R = [F][x]/(x^{N+1})$ for $N \geq n$, we see that $\mathrm{span}(R^{(n)}) = R^n$ has basis $\{x^n, x^{n+1}, \ldots, x^N\}$; so $\dim(\mathrm{span}(R^{(n)}))/\dim R = (N - n + 1)/N$, which for large N is close to 1, not to $1/n$.

However, something nearer to the spirit of Lemma 3, with a chance of having a positive answer, is

QUESTION 16. *Let R be a finite-dimensional nilpotent commutative algebra over an algebraically closed field F, let V be a subspace of R, and let n be a positive integer such that every nonzero element of V has nonzero n-th power. Must* $\dim R \geq n \dim V$ *?*

Above, V is a subspace of R, but in the absence of (1), we can't expect $V^{(n)}$ to simultaneously be one. In the next question, we turn the tables, and make the *target* of the n-th power map a subspace.

QUESTION 17. *Let R be a finite-dimensional nilpotent commutative algebra over an algebraically closed field F, let W be a subspace of R, and let n be*

a positive integer such that every element of W *is an* n*-th power in* R*. Must* $\dim R \geq n \dim W$ *?*

Let us look at the above two questions for $R = [F][x]/(x^{N+1})$. Note that an element $r \neq 0$ of this algebra has $r^n \neq 0$ if and only if the lowest-degree term of r has degree $\leq N/n$, while a necessary condition for r to be an n-th power (which is also sufficient if n is not divisible by the characteristic of F) is that its lowest-degree term have degree divisible by n. Now for each of these properties, there are, in general, large-dimensional *affine* subspaces of $[F][x]/(x^{N+1})$ all of whose elements have that property. E.g., if $n \leq N$, the $(N-1)$-dimensional affine space of elements of the form $x +$ (higher degree terms) consists of elements whose n-th powers are nonzero, and for F of characteristic not divisible by n, the $(N-n)$-dimensional affine space of elements of the form $x^n +$(higher degree terms) consists of n-th powers. In each of these cases, if we fix n and let $N \to \infty$, the ratio of the dimension of our affine subspace to that of our algebra approaches 1. But these affine subspaces are not vector subspaces! If U is a vector subspace of $R = [F][x]/(x^{N+1})$, and if for each x^m which appears as the lowest degree term of a member of U, we choose a $w_m \in U$ with that lowest degree term, it is not hard to see that these elements form a basis of U. It is easily deduced from the above discussion that if U consists of n-th powers, or consists of elements which, if nonzero, have nonzero n-th power, then U has dimension $\leq N/n$. So for this R, Questions 16 and 17 both have affirmative answers.

Can those two questions be made the $m = 1$ and $m = n$ cases of a question statable for all $1 \leq m \leq n$? Yes. The formulation is less elegant than for those two cases, but I include it for completeness.

Question 18. *Let* R *be a finite-dimensional nilpotent commutative algebra over an algebraically closed field* F*, let* U *be a subspace of* R*, and let* $1 \leq m \leq n$ *be integers such that every nonzero element of* U *has an* m*-th root in* R *whose* n*-th power is nonzero. Must* $\dim R \geq n \dim U$ *?*

(Again, we easily obtain an affirmative answer for $R = [F][x]/(x^{N+1})$, essentially as in the cases $m = 1$ and $m = n$.)

Early on, in thinking about Eggert's Conjecture, I convinced myself that the noncommutative analog was false. But the analog I considered was based on replacing $R^{(p)}$ by $\mathrm{span}(R^{(p)})$ so that one could talk about its dimension. However, the generalizations considered in Questions 16-18 are also plausible for noncommutative rings.

I also assumed in Questions 16-18 that F was algebraically closed, because that hypothesis was essential to the proof of Lemma 3, and is the condition under which solution-sets of algebraic equations behave nicely. However, I don't have examples showing that the results asked for are false without it. So let us be bold, and ask

Question 19. *Does the generalization of Conjecture 8, or an affirmative answer to any of Questions 11, 15, 16, 17 or 18, hold if the commutativity hypothesis is dropped, and/or, in the case of Question 16, if the assumption that* F *be algebraically closed is dropped (or perhaps weakened to "*F *is infinite")?*

(*For Question 17 one can similarly drop the assumption that* F *be algebraically closed; but then one would want to change the hypothesis that every element of* W *have an* n*-th root to the condition, equivalent thereto in the algebraically closed*

case, that every 1*-dimensional subspace of* W *contain a nonzero* n*-th power, since the original hypothesis would be unreasonably strong over non-algebraically-closed* F*. One can likewise make the analogous generalization of Question 18.*)

If we go further, and drop not only the characteristic p assumption and the algebraic closedness of F, but also the *associativity* of R, then there is an easy counterexample to the analog of Eggert's Conjecture.

Example 20. *For every positive integer* d*, there exists a graded, nilpotent, commutative,* nonassociative *algebra over the field* $\mathbb{R}$ *of real numbers,* $R = R_1 \oplus R_2 \oplus R_3$*, generated by* R_1*, in which the respective dimensions of the three homogeneous components are* d*,* 1*,* d*, and in which the "cubing" operation* $r \mapsto r(rr)$ *gives a bijection from* R_1 *to* R_3*.*

Hence, writing $R^{(3)}$ *for* $\{r(rr) \mid r \in R\} = R_3$*, we have* $\dim R^{(3)}/\dim R = d/(2d+1)$*, which is* $> 1/3$ *if* $d > 1$*.*

Construction. Let W be a real inner product space of dimension d, let $A = W \oplus \mathbb{R}$, made an $\mathbb{R}$-algebra by letting elements of $\mathbb{R} \subseteq A$ act on A on either side by scalar multiplication, and letting the product of two elements of W be their inner product in $\mathbb{R}$. Note that W, W^2, W^3 are respectively W, $\mathbb{R}$, W, and that on W, the operation $w \mapsto w(ww)$ takes every element to itself times the square of its norm, hence is a bijection $W \to W$.

For the above A, let us form $A \otimes_{\mathbb{R}} [\mathbb{R}][x]/(x^4)$, which is clearly nilpotent; let V be its subspace Wx; and let R be the subalgebra generated by V; namely, $(Wx) \oplus (\mathbb{R}\, x^2) \oplus (Wx^3)$. Then the asserted properties are clear. □

The parenthetical comment following Lemma 3 shows, however, that over an algebraically closed base field F, there is no example with the corresponding properties.

If in Example 20 we let B be an orthonormal basis of W, then on closing $Bx \subseteq R$ under the multiplication of R (but not under addition or scalar multiplication), we get a $2d + 2$-element structure (a "nonassociative semigroup", often called a "magma") which is a counterexample to the nonassociative analogs of Conjecture 8, Corollary 10 and Question 11.

I will end this section by recording, for completeness, a positive-characteristic version of Example 20 (though the characteristic will not be the exponent whose behavior the example involves). Before stating it, let us recall that a nonassociative algebra is called *power-associative* if every 1-generator subalgebra is associative; equivalently, if the closure of every singleton $\{x\}$ under the multiplication (intuitively, the set of "powers" of x) is in fact a semigroup. Let us call a graded nonassociative algebra *homogeneous-power-associative* if the subalgebra generated by every homogeneous element is associative. Example 20 above is easily seen to be homogeneous-power-associative. The same property in the next example will allow us to avoid having to specify the bracketing of the power operation we refer to.

Example 21. *For every prime* p*, there exists a graded, nilpotent, commutative, nonassociative, but homogeneous-power-associative algebra* $R = R_1 \oplus \cdots \oplus R_{p+1}$ *over a non-perfect field* F *of characteristic* p*, such that* R *is generated by* R_1*, the* $p+1$*-st power operation gives a surjection* $R_1 \to R_{p+1}$ *taking no nonzero element to zero, and* $\dim R_i = p$ *for* $i < p$ *and for* $i = p + 1$*, but* $\dim R_p = 1$*.*

Hence, $\dim R^{(p+1)}/\dim R = p/(p^2+1) > 1/(p+1)$*.*

SKETCH OF CONSTRUCTION. Given p, let F be any field of characteristic p having a proper purely inseparable extension $F' = F(u^{1/p})$, such that every element of F' has a $p+1$-st root in F'. (We can get such F and F' starting with any algebraically closed field k of characteristic p, and any subgroup G of the additive group $\mathbb{Q}$ of rational numbers which is $p+1$-divisible but not p-divisible. Note that $p^{-1}G \subseteq \mathbb{Q}$ will have the form $G + p^{-1}h\mathbb{Z}$ for any $h \in G - pG$. Take a group isomorphic to G but written multiplicatively, t^G, and its overgroup $t^{p^{-1}G}$, and let F and F' be the Mal'cev-Neumann power series fields $k((t^G))$ and $k((t^{p^{-1}G}))$ [**8**, §2.4], [**6**]; and let $u \in F$ be the element t^h. The asserted properties are easily verified.)

Let us now form the (commutative, associative) truncated polynomial algebra $[F'][x]/(x^{p+2})$, graded by degree in x, and let R be the F-subspace of this algebra consisting of those elements for which the coefficient of x^p lies in the subfield F of F' (all other coefficients being unrestricted). We make R a graded nonassociative F-algebra by using the multiplication of $[F'][x]/(x^{p+2})$ on all pairs of homogeneous components *except* those having degrees summing to p, while defining the multiplication when the degrees sum to p by fixing an F-linear retraction $\psi : F' \to F$, and taking the product of $a\,x^i$ and $b\,x^{p-i}$ $(0 < i < p,\ a, b \in F')$ to be $\psi(ab)\,x^p$.

We claim that R is homogeneous-power-associative; in fact, that powers of homogeneous elements of R, however bracketed, agree with the values of these same powers in the associative algebra $[F'][x]/(x^{p+2})$. Note first that the evaluations of powers of elements homogeneous of degrees other than 1 never pass through R_p, so they certainly come out as in $[F'][x]/(x^{p+2})$. For an element $a\,x$ of degree 1 $(a \in F')$, the same reasoning holds for powers less than the p-th. In the case of the p-th power, the last stage in the evaluation of any bracketing of $(a\,x)^p$ takes the form $(a\,x)^i \cdot (a\,x)^{p-i} = \psi(a^i\,a^{p-i})\,x^p$; but $a^i\,a^{p-i} = a^p \in F$, which is fixed by ψ, so the result again comes out as in $[F'][x]/(x^{p+2})$. Knowing this, it is easy to verify likewise that all computations of the $p+1$-st power of $a\,x \in R_1$ agree with its value in $[F'][x]/(x^{p+2})$.

The other asserted properties are now straightforward. In particular the $p+1$-st power map $R_1 \to R_{p+1}$ is surjective, and sends no nonzero element to 0, because these statements are true in $[F'][x]/(x^{p+2})$ (surjectivity holding by our assumption on $p+1$-st roots in F'). □

5. Some attempts at counterexamples to Eggert's Conjecture for semigroups

Since Eggert's Conjecture implies the semigroup-theoretic Conjecture 8, a counterexample to the latter would disprove the former. We saw in §1 that for certain sorts of truncated polynomial algebras over a field F of characteristic p, the ratio $\dim R^{(p)}/\dim R$ was exactly $1/p$; i.e., as high as Eggert's Conjecture allows. Those algebras are contracted semigroup algebras $F_0\,S$, where S is a semigroup with zero presented by one generator x and one relation $x^{pk+1} = 0$; so these semigroups have equality in Conjecture 8. It is natural to try to see whether, by some modification of this semigroup construction, we can push the ratio $\mathrm{card}(S^{(p)} - \{0\})/\mathrm{card}(S - \{0\})$ just a little above $1/p$.

In scratchwork on such examples, it is convenient to write the infinite cyclic semigroup not as $\{x, x^2, x^3, \dots\}$, but additively, as $\{1, 2, 3, \dots\}$. Since in additive notation, 0 generally denotes an identity element, it is best to denote a "zero"

element by ∞. So in this section we shall not adopt Convention 7, but follow this additive notation. Thus, the sort of nilpotent cyclic semigroup with zero that gives equality in the statement of Conjecture 8 is

$$\{1,\ 2,\ \dots,\ N,\ \infty\},\quad \text{where } N \text{ is a multiple of } n. \tag{13}$$

For S a finite nilpotent abelian semigroup with zero, the semigroup version of Eggert's conjecture can be written as saying that the integer

$$n\ \mathrm{card}(S^{(n)} - \{\infty\})\ -\ \mathrm{card}(S - \{\infty\}) \tag{14}$$

is always ≤ 0. (We continue to write $S^{(n)}$ for what in our additive notation is now $\{n\,x \mid x \in S\}$.)

What kind of modifications can we apply to (13) in the search for variant examples? We might impose a relation; but it turns out that this won't give anything new. E.g., if for $i < j$ in $\{1, 2, \dots, N\}$ we impose on (13) the relation $i = j$, then this implies $i + 1 = j + 1$, and so forth; and this process eventually identifies some $h \leq N$ with an integer $> N$, which, in (13), equals ∞. So h and all integers $\geq h$ fall together with ∞; and if we follow up the consequences, we eventually find that every integer $\geq i$ is identified with ∞. Thus, we get a semigroup just like (13), but with $i - 1$ rather than N as the last finite value.

So let us instead pass to a subsemigroup of (13). The smallest change we can make is to drop 1, getting the subsemigroup generated by 2 and 3, which we shall now denote S. Then $\mathrm{card}(S - \{\infty\})$ has gone down by 1, pushing the value of (14) up by 1; but the integer n has ceased to belong to $S^{(n)}$, decreasing (14) by n. So in our attempt to find a counterexample, we have "lost ground", decreasing (14) from 0 to $-n + 1$.

However, now that $1 \notin S$, we can regain some ground by imposing relations. Suppose we impose the relation that identifies $N - 1$ either (a) with N or (b) with ∞. If we add any member of S (loosely speaking, any integer ≥ 2) to both sides of either relation, we get $\infty = \infty$, so no additional identifications are implied. Since we are assuming N is divisible by n, the integer $N - 1$ is not; so we have again decreased the right-hand term of (14), this time without decreasing the left-hand term; and thus brought the total value to $-n + 2$. In particular, if $n = 2$, we have returned to the value 0; but not improved on it.

I have experimented with more complicated examples of the same sort, and gotten very similar results: I have not found one that made the value of (14) positive; but surprisingly often, it was possible to arrange things so that for $n = 2$, that value was 0. Let me show a "typical" example.

We start with the additive subsemigroup of the natural numbers generated by 4 and 5. I will show it by listing an initial string of the positive integers, with the members of our subsemigroup underlined:

$$1\ 2\ 3\ \underline{4\ 5}\ 6\ 7\ \underline{8\ 9\ 10}\ 11\ \underline{12\ 13\ 14\ 15\ 16\ 17\ \dots}\ . \tag{15}$$

Assume this to be truncated at some large integer N which is a multiple of n, all larger integers being collapsed into ∞. If we combine the effects on the two terms of (14) of having dropped the six integers 1, 2, 3, 6, 7, 11 from (13), we find that, assuming $N \geq 11n$, (14) is now $6(-n + 1)$.

Now suppose we impose the relation $i = i + 1$ for some i such that i and $i + 1$ both lie in (15). Adding 4 and 5 to both sides of this equation, we get $i + 4 = i + 5 = i + 6$; adding 4 and 5 again we get $i + 8 = i + 9 = i + 10 = i + 11$.

At the next two rounds, we get strings of equalities that overlap one another; and all subsequent strings likewise overlap. So everything from $i+12$ on falls together with $N+1$ and hence with ∞; so we may as well assume

$$N+1 \;=\; i+12. \tag{16}$$

What effect has imposing the relation $i = i+1$ had on (14)? The amalgamations of the three strings of integers described decrease $\mathrm{card}(S-\{\infty\})$ by 1, 2 and 3 respectively, so in that way, we have gained ground, bringing (14) up from $6(-n+1)$ to possibly $6(-n+2)$. But have we decreased $\mathrm{card}(S^{(n)}-\{\infty\})$, and so lost ground, in the process?

If $n > 2$, then even if there has been no such loss, the value $6(-n+2)$ is negative; so let us assume $n = 2$. If we are to avoid bringing (14) below 0, we must make sure that none of the sets that were fused into single elements,

$$\{i,\, i+1\}, \quad \{i+4,\, i+5,\, i+6\}, \quad \{i+8,\, i+9,\, i+10,\, i+11\}, \tag{17}$$

contained more than one member of $S^{(2)}$. For the first of these sets, that is no problem; and for the second, the desired conclusion can be achieved by taking i odd, so that of the three elements of that set, only $i+5$ is even. For the last it is more difficult – the set will contain two even values, and if i is large, these will both belong to $S^{(2)}$.

However, suppose we take i not so large; say we choose it so that the smaller of the two even values in that set is the largest even integer that does *not* belong to $S^{(2)}$. That is 22, since 11 is the largest integer not in (15). Then the above considerations show that we do get a semigroup for which (14) is zero.

The above choice of i makes $i+9$ (the smallest even value in the last subset in (17)) equal to 22 (the largest even integer not in $S^{(2)}$), so $i = 13$, so by (16), $N+1 = 25$.

Let us write down formally the contracted semigroup algebras of the two easier examples described earlier, and of the above example.

Example 22. *Let F be a perfect field of characteristic 2. Then the following nilpotent algebras have equality in the inequality of Eggert's Conjecture.*

$$R \;=\; [F][x^2,\, x^3] \,/\, (x^{N-1} - x^N,\, x^{N+1},\, x^{N+2}) \qquad \textit{for every even } N > 2, \tag{18}$$

$$R \;=\; [F][x^2,\, x^3] \,/\, (x^{N-1},\, x^{N+1},\, x^{N+2}) \qquad \textit{for every even } N > 2, \tag{19}$$

$$R \;=\; [F][x^4,\, x^5] \,/\, (x^{13} - x^{14},\, x^{25}, \ldots, x^{28}). \tag{20}$$

More precisely, in both (18) *and* (19) $\dim R = N-2$, *and* $\dim R^{(2)} = (N-2)/2$, *while in* (20), $\dim R = 18$, *and* $\dim R^{(2)} = 9$. □

Many examples behave like these. A couple more are

$$[F][x^2,\, x^5]/(x^{11} - x^{12},\, x^{\geq 15}), \qquad [F][x^3,\, x^7]/(x^{13} - x^{14},\, x^{\geq 25}) \tag{21}$$

(where "$x^{\geq n}$" means "x^n and all higher powers"; though in each case, only finitely many are needed).

Perhaps Eggert's Conjecture is true, and these examples "run up against the wall" that it asserts. Or – who knows – perhaps if one pushed this sort of exploration further, to homomorphic images of semigroups generated by families of three or more integers, and starting farther from 0, one would get counterexamples.

For values of n greater than 2, I don't know any examples of this flavor that even bring (14) as high as zero. (But a class of examples of a different sort, which does, was noted in the last paragraph of §1.)

Incidentally, observe that in the semigroup-theoretic context that led to (18) and (19), we had the choice of imposing either the relation $N-1=N$ or the relation $N-1=\infty$. However, in the development that gave (20), setting a semigroup element equal to ∞ would not have done the same job as setting two such elements equal. If we set $i=\infty$, then, for example, $i+4$ and $i+5$ would each become ∞, so looking at the latter two elements, we would lose one from $S^{(2)}$ as well as one not in $S^{(2)}$. Above, we instead set $i=i+1$, and the resulting pair of equalities $i+4=i+5=i+6$ turned a family consisting of two elements not in $S^{(2)}$ and one in $S^{(2)}$ into a single element of $S^{(2)}$.

Turning back to Eggert's ring-theoretic conjecture, it might be worthwhile to experiment with imposing on subalgebras of $[F][x]$ relations "close to" those of the sort used above, but not expressible in purely semigroup-theoretic terms; for instance, $x^i+x^{i+1}+x^{i+2}=0$, or $x^i-2x^{i+1}+x^{i+2}=0$.

6. Sketch of the literature

The main positive results in the literature on Eggert's Conjecture concern two kinds of cases: where $\dim(R^{(p)})$ (or some related invariant) is quite small, and where R is graded.

N. H. Eggert [**9**], after making the conjecture, in connection with the study of groups that can appear as the group of units of a finite unital ring A (the nonunital ring R to which the conjecture would be applied being the Jacobson radical of A), proved it for $\dim(R^{(p)})\leq 2$. That result was extended to $\dim(R^{(p)})\leq 3$ by R. Bautista [**5**], both results were re-proved more simply by C. Stack [**16**], [**17**], and most recently pushed up to $\dim(R^{(p)})\leq 4$ by B. Amberg and L. Kazarin [**2**]. Amberg and Kazarin also prove in [**1**] some similar results over an arbitrary field, in the spirit of our Questions 16 and 17, and they show in [**3**] that, at least when the values $\dim(R^i/R^{i+1})$ are small, these give a nonincreasing function of i. In [**3**] they give an extensive survey of results on this subject and related group-theoretic questions.

K. R. McLean [**13**], [**14**] has obtained strong positive results in the case where R is graded and generated by its homogeneous component of degree 1. In particular, in [**13**] he proves Eggert's Conjecture in that case if $(R_3)^{(p)}=0$ (recall that in Corollary 4 we could not get beyond the case $(R_2)^{(p)}=0$), or if $R^{(p)}$ is generated by two elements. Moreover, without either assumption (but still assuming R graded and generated in degree 1), he proves that $\dim R^{(p)}/\dim R\leq 1/(p-1)$. His technique involves taking a subspace $V\subseteq R_1$ as at the start of §2 above, and constructing recursively a family of direct-sum decompositions of V, each new summand arising as a vector-space complement of the kernel of multiplication by an element obtained using the previous steps of the construction. He also shows in [**13**] that Eggert's Conjecture holds for the radicals of group algebras of finite abelian groups over perfect fields F of nonzero characteristic.

S. Kim and J. Park [**11**] prove Eggert's Conjecture when R is a commutative nilpotent *monomial algebra*, i.e., an algebra with a presentation in which all relators are monomials in the given generators.

M. Korbelář [**12**] has recently shown that Eggert's Conjecture holds whenever $R^{(p)}$ can be generated as an F-algebra by two elements. (So a counterexample in the spirit of the preceding section would require at least 3 generators.) [**12**] ends with a generalization of Eggert's conjecture, which is equivalent to the case of Question 16 above in which F is a field of positive characteristic p and $n = p$, but F is not assumed perfect.

In [**10**], a full proof of Eggert's Conjecture was claimed, but the argument was flawed. (The claim in the erratum to that paper, that the proof is at least valid for the graded case, is also incorrect.)

There is considerable variation in notation and language in these papers. E.g., what I have written $R^{(p)}$ is denoted $R^{(1)}$ in Amberg and Kazarin's papers, $R^{(p)}$ in Stack's and Korbelář's, and $R[p]$ in McLean's (modulo differences in the letter used for the algebra R). McLean, nonstandardly, takes the statement that R is graded to include the condition that it is generated by its degree 1 component.

Though I do not discuss this above, I have, also examined the behavior of the sequence of dimensions of quotients R^i/R^{i+1} for a commutative algebra R. Most of my results seem to be subsumed by those of Amberg and Kazarin, but I will record here a question which that line of thought suggested, which seems of independent interest for its simplicity. Given two subspaces V and W of a commutative algebra, let $\mathrm{Ann}_V W$ denote the subspace $\{x \in V \mid xW = \{0\}\} \subseteq V$.

QUESTION 23. *If R is a commutative algebra over a field F, V a finite-dimensional subspace of R, and n a positive integer, must*

$$\dim(V/\mathrm{Ann}_V V^n) \;\leq\; \dim V^n \;? \tag{22}$$

I believe I have proved (22) for $\dim V^n \leq 4$. The arguments become more intricate with each succeeding value 1, 2, 3, 4.

I am indebted to Cora Stack for bringing Eggert's Conjecture to my attention and providing a packet of relevant literature, to Martin Olsson for pointing me to the result in [**15**] used in the proof of Lemma 3, and to the referee for making me justify an assertion that was not as straightforward as I had thought.

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