Tensor algebras, exterior algebras, and symmetric algebras

1. Basics. Throughout this note, k will be a fixed commutative ring.

Recall that if M and P are k-modules, a k-multilinear map $\mu\colon M^n \to P$ is called *alternating* if $\mu(x_1, \ldots, x_n) = 0$ whenever two successive arguments x_i, x_{i+1} are equal. From this condition one easily shows that transposing two successive arguments reverses the sign of $\mu(x_1, \ldots, x_n) = 0$, from which it follows that any permutation of the arguments multiplies the value by the sign of that permutation; hence, from the original condition we see that the value is zero if *any* two terms are equal.

We recall likewise that a k-multilinear map $M^n \to P$ is called *symmetric* if it is unchanged under transposing successive arguments; equivalently, if it is unchanged under all permutations of the arguments.

Let us now define some universal maps.

Definition 1. *Let M be a k-module.*

*By $\otimes^n M$ we shall denote the k-module with a universal k-*multilinear *map of M^n into it, written $(x_1, \ldots, x_n) \mapsto x_1 \otimes \ldots \otimes x_n$. This module is called the n-fold* tensor power *of M.*

By $\Lambda^n M$ we shall denote the k-module with a universal alternating *k-multilinear map of M^n into it, written $(x_1, \ldots, x_n) \mapsto x_1 \wedge \ldots \wedge x_n$. This module is called the n-fold* exterior power *of M.*

By $S^n M$ we shall denote the k-module with a universal symmetric *k-multilinear map of M^n into it, written $(x_1, \ldots, x_n) \mapsto x_1 \ldots x_n$. This module is called the n-fold* symmetric power *of M.*

Each of these modules can clearly be constructed directly by generators and relations. $\otimes^n M$ can also be constructed as $M \otimes_k (M \otimes_k (\ldots))$ (with n M's), while $\Lambda^n M$ and $S^n M$ can be constructed as factor-modules of $\otimes^n M$ by the submodules generated by all elements of the forms

$$x_1 \otimes \ldots \otimes y \otimes y \otimes \ldots \otimes x_n,$$

respectively

$$x_1 \otimes \ldots \otimes y \otimes z \otimes \ldots \otimes x_n \ - \ x_1 \otimes \ldots \otimes z \otimes y \otimes \ldots \otimes x_n.$$

(where each term is $x_1 \otimes \ldots \otimes x_n$ modified only in two successive places).

Let us make the convention that $\otimes^1 M$, $\Lambda^1 M$ and $S^1 M$ are all identified with M, while $\otimes^0 M$, $\Lambda^0 M$ and $S^0 M$ are all identified with k.

The universal properties defining our three constructions can be conveniently restated as follows:

A k-module homomorphism with domain $\otimes^n M$ is uniquely determined by specifying it on the elements $x_1 \otimes \ldots \otimes x_n$ for all $x_1, \ldots, x_n \in M$, in such a way that the value is a *k-multilinear* function of $x_1, \ldots, x_n$.

A k-module homomorphism with domain $\Lambda^n M$ is uniquely determined by specifying it on the elements $x_1 \wedge \ldots \wedge x_n$ for all $x_1, \ldots, x_n \in M$, so that the value is an *alternating* k-multilinear function of $x_1, \ldots, x_n$.

A k-module homomorphism with domain $S^n M$ is uniquely determined by specifying it on the elements $x_1 \ldots x_n$ for all $x_1, \ldots, x_n \in M$, so that the value is a *symmetric* k-multilinear function of $x_1, \ldots, x_n$.

Each of $\otimes^n$, Λ^n, S^n may be made into a functor. Given $f\colon M \to N$, one defines, for example, $\Lambda^n f\colon \Lambda^n M \to \Lambda^n N$ as the unique module homomorphism taking $x_1 \wedge \ldots \wedge x_n$ to $f(x_1) \wedge \ldots \wedge f(x_n)$ for all $x_1, \ldots, x_n \in M$. This homomorphism exists because the latter expression in $x_1, \ldots, x_n$ is alternating and *k*-multilinear.

Let us now show how to multiply a member of $\otimes^m M$ by a member of $\otimes^n M$ to get a member of $\otimes^{m+n} M$, and similarly for the other constructions. The universal properties of our module constructions characterize the module homomorphisms, i.e. *linear* maps, definable on them, but we now want to construct *bilinear* maps. The next Lemma allows us to translate existence and uniqueness results for the former into existence and uniqueness results for the latter. (Note that in that Lemma, the subscripts i and j index generators, while i' and j' index relations.)

Lemma 2. *Let A be a k-module presented by generators X_i $(i \in I)$ and relations $\Sigma_i\, \alpha_{i'i} X_i = 0$ $(i' \in I')$, and B a k-module presented by generators Y_j $(j \in J)$ and relations $\Sigma_j\, \beta_{j'j} Y_j = 0$ $(j' \in J')$.*

Then for any k-module C, a k-bilinear map $A \times B \to C$ is uniquely determined by specifying the image Z_{ij} of each pair of generators (X_i, Y_j), in such a way that for each $j \in J$, $i' \in I'$ one has $\Sigma_i\, \alpha_{i'i} Z_{ij} = 0$, and for each $i \in I$, $j' \in J'$ one has $\Sigma_j\, \beta_{j'j} Z_{ij} = 0$.

(*Equivalently, the k-module $A \otimes_k B$ may be presented by the generators $X_i \otimes Y_j$ and the two sets of relations $\Sigma_i\, \alpha_{i'i} X_i \otimes Y_j = 0$ $(i' \in I', j \in J)$ and $\Sigma_j\, \beta_{j'j} X_i \otimes Y_j = 0$ $(j' \in J', i \in I)$.*)

Proof. Exercise. □

We deduce

Lemma 3. *Let M be a k-module and m, n nonnegative integers. Then there exist*

a unique k-bilinear map $\otimes^m M \times \otimes^n M \to \otimes^{m+n} M$ which carries each pair

$$(x_1 \otimes \ldots \otimes x_m,\ x_{m+1} \otimes \ldots \otimes x_{m+n}) \text{ to } x_1 \otimes \ldots \otimes x_{m+n},$$

a unique k-bilinear map $\Lambda^m M \times \Lambda^n M \to \Lambda^{m+n} M$ which carries each pair

$$(x_1 \wedge \ldots \wedge x_m,\ x_{m+1} \wedge \ldots \wedge x_{m+n}) \text{ to } x_1 \wedge \ldots \wedge x_{m+n}, \text{ and}$$

a unique k-bilinear map $S^m M \times S^n M \to S^{m+n} M$ which carries each pair

$$(x_1 \ldots x_m,\ x_{m+1} \ldots x_{m+n}) \text{ to } x_1 \ldots x_{m+n}.$$

Proof. We are describing these maps in terms of their actions on pairs of generators of the two modules in question, so by Lemma 2, it suffices to verify that these assignments satisfy the equations corresponding to the defining relations for the two modules. These equations are equivalent to saying that the assignments should be multilinear (respectively alternating multilinear, respectively symmetric multilinear) in $x_1, \ldots, x_m$, and also in $x_{m+1}, \ldots, x_{m+n}$, which is clearly true in each case. □

Let us now form the *k*-modules

$$k\langle M\rangle = \oplus_n\, (\otimes^n M),$$

$$\Lambda M = \oplus_n\, (\Lambda^n M),$$

$$S M = \oplus_n\, (S^n M).$$

We define a bilinear operation on each of them by letting it act on pairs of elements from the *m*th and *n*th summands by the operations described in Lemma 3, and extending it to sums of elements from different summands by bilinearity. We can now assert

Theorem 4. *The multiplications on* $k<M>$, ΛM *and* SM *defined above are structures of* (*associative unital*) *k-algebra.*

The algebra $k<M>$, *called the* tensor algebra *on* M, *is universal among k-algebras given with k-module homomorphisms of* M *into them.*

The algebra ΛM, *called the* exterior algebra *on* M, *is universal among k-algebras given with k-module homomorphisms of* M *into them such that the images of all elements of* M *have zero square.*

The algebra SM, *called the* symmetric algebra *on* M, *is universal among k-algebras given with k-module homomorphisms of* M *into them such that the images of elements of* M *commute with one another, and is also universal among all* commutative *k*-algebras *given with k-module homomorphisms of* M *into them.*

Proof. To verify associativity of an operation known to be k-bilinear, it suffices to verify it on 3-tuples of elements from any set of k-module generators. Thus, for instance, associativity of the multiplication of ΛM can be verified by checking it on 3-tuples $x_1 \wedge \ldots \wedge x_m$, $x_{m+1} \wedge \cdots \wedge x_{m+n}$, $x_{m+n+1} \wedge \cdots \wedge x_{m+n+p}$. But it is immediate from the definition that the product, formed either way, is $x_1 \wedge \ldots \wedge x_{m+n+p}$. The associativity of the other two constructions is proved the same way.

Thus, our three objects are indeed a k-algebra with a k-module homomorphism of M into it (namely, the inclusion of $\otimes^1 M$), a k-algebra with a k-module homomorphism of M into it such that the images of all elements of M have square zero, and a k-algebra with a k-module homomorphism of M into it such that the images of all elements of M commute with one another, and in this last case it follows that the algebra is commutative. To complete the proof of the Theorem, we need to show that these algebras are universal for these properties.

Consider first any k-algebra R with a k-module homomorphism $f: M \to R$. We see that for each n, the map $M^n \to R$ given by $(x_1, \ldots, x_n) \mapsto f(x_1) \ldots f(x_n)$ is k-multilinear, hence by the universal property defining $\otimes^n M$, it induces a k-module homomorphism $\otimes^n M \to R$. This family of maps yields a module homomorphism $k<M> \to R$. When we consider the behavior of this map on the k-module generators $x_1 \otimes \ldots \otimes x_n$ of $k<M>$, we find that the way we have defined multiplication in this algebra makes the map respect multiplication of these generators. From this and the fact that it is k-linear, it follows that it respects multiplication on all elements, establishing the asserted universal property.

Likewise, given a k-algebra R with a k-module homomorphism $f: M \to R$ such that for every $x \in M$, $f(x)^2 = 0$, the map $M^n \to R$ taking $(x_1, \ldots, x_n)$ to $f(x_1) \ldots f(x_n)$ is k-multilinear and alternating. Thus it extends to a map on $\Lambda^n M$, and the proof is completed as before.

In the case of the universal property of the symmetric algebra, we similarly see that $(x_1, \ldots, x_n) \mapsto f(x_1) \ldots f(x_n)$ is a *symmetric* k-multilinear map, and complete the proof in the same way. From the observation that SM is commutative and the above universal property, it follows that it is also universal among commutative k-algebras with maps of M into them. □

It follows, either from the explicit description or by the universal property, that each of the above constructions gives a *functor* from k-modules to k-algebras.

What do these three sorts of algebras look like? Let us get explicit descriptions in the case where M is a free module.

Theorem 5. *Let* M *be a free k-module on a basis* X. *Then:*

$k<M>$ *is the free associative k-algebra* $k<X>$, *equivalently, the semigroup algebra on the free semigroup* $<X>$. *It has as a k-module basis the set of all products* $x_1 \otimes \ldots \otimes x_n$ (*or, using ordinary* *multiplicative notation, as is common when this ring is regarded as a free algebra or a semigroup algebra, the set of products* $x_1 \ldots x_n$) *for* $x_1, \ldots, x_n \in X$. *Thus, if* X *is a finite set* $\{x_1, \ldots, x_r\}$, *then for each* n, $\dim_k(\otimes^n M) = r^n$.

ΛM *may be presented by the generating set* X *and the relations* $x \wedge x = 0$, $x \wedge y + y \wedge x = 0$ $(x, y \in X)$. *If a total ordering ''*$\leq$*'' is chosen on* X, *then a* k*-module basis for* ΛM *is given by those products* $x_1 \wedge \ldots \wedge x_n$ *with* $x_1 < \ldots < x_n \in X$. *In particular, if* X *is a finite set* $\{x_1, \ldots, x_r\}$, *then a basis is given by those products* $x_{i_1} \wedge \ldots \wedge x_{i_n}$ *with* $1 \leq i_1 < \ldots < i_n \leq r$, *hence for each* n, $\dim_k(\Lambda^n M) = \binom{r}{n}$.

SM *may be presented by the generating set* X, *and relations* $xy = yx$ $(x, y \in X)$, *and is the* (*commutative*) *polynomial algebra* $k[X]$, *equivalently, the free commutative* k*-algebra on* X, *equivalently, the semigroup algebra on the free commutative semigroup on* X. *If a total ordering ''*$\leq$*'' is chosen on* X, *then a* k*-module basis for* SM *is given by those products* $x_1 \ldots x_n$ *with* $x_1 \leq \ldots \leq x_n \in X$. *If* X *is a finite set* $\{x_1, \ldots, x_r\}$, *then the elements of this basis can be written* $x_1^{i_1} \ldots x_r^{i_r}$ *with* $i_1, \ldots, i_r \geq 0$, *and for each* n, $\dim_k(S^n M) = \binom{r+n-1}{n}$.

Proof. The universal property of $k\langle M\rangle$ combined with the universal property of the free module M on X imply that $k\langle M\rangle$ has the universal property of the free algebra $k\langle X\rangle$; and the semigroup algebra on the free semigroup also has this universal property, hence $k\langle M\rangle$ has for basis the free semigroup on X. The other two algebras can now be obtained by imposing additional relations on this free algebra, and the indicated bases are easily established using the diamond lemma [**1**, §1]. □

Remark: The constructions of $\otimes^n M$ and $k\langle M\rangle$ can be carried out and their universal properties verified in the more general context where k is a not-necessarily-commutative ring and M is a k-bimodule. (In this context, $k\langle M\rangle$ is called the ''tensor k-ring on M''.)

The analogs of the exterior and symmetric algebras can, of course, be formally defined in this context as well, but these turn out to be degenerate, with all symmetric or exterior powers of a module beyond the first reducing to cases of the constructions for modules over a commutative ring developed above. For instance, suppose M is a bimodule over a ring R, and $x, y \in M$, $a \in R$. Then in the ''symmetric square'' of M we see that $axy = ayx = (ay)x = x(ay) = xay$; thus the distinction between ax and xa has been lost in this bimodule. Moreover, if b is another element of R, we see using the above computation that $(ab)xy = x(ab)y = (ax)by = b(ax)y = (ba)xy$, so left multiplication by ab and ba are the same; i.e., the left R-module structure of $S^2 M$ is induced by a module structure over the abelianization $R/[R:R]$ of R. From such observations one can show that for $n \geq 2$, $S^n M$ can be identified with the nth symmetric power of the $R/[R:R]$-module obtained by dividing M by the sub-bimodule spanned by all elements $ax - xa$ $(a \in R,\ x \in M)$, and the corresponding observations apply to $\Lambda^n M$.

2. Exterior powers and determinants. Note that if, as in the preceding Theorem, M is a free k-module on r generators $x_1, \ldots, x_r$, then $\Lambda^r M$ will be a free k-module on one generator $x_1 \wedge \ldots \wedge x_r$. Hence if $f: M \to M$ is a module endomorphism, the induced endomorphism $\Lambda^r f: \Lambda^r M \to \Lambda^r M$ will take $x_1 \wedge \ldots \wedge x_r$ to $d(f)\, x_1 \wedge \ldots \wedge x_r$ for some constant $d(f) \in k$. Let us note the properties these constants $d(f)$ must have.

Since specifying f is equivalent to specifying $f(x_1), \ldots, f(x_r)$, the element $d(f)$ can be thought of as a function of this r-tuple of elements of M. As such it can be evaluated by forming the product $f(x_1) \wedge \ldots \wedge f(x_r)$ in $\Lambda^r M$, and writing this as a multiple of $x_1 \wedge \ldots \wedge x_r$; the coefficient is $d(f)$. It is easy to see from this formulation that $d(f)$ is an alternating multilinear functional of $f(x_1), \ldots, f(x_r)$, which takes the identity map to 1. We thus recognize d as the determinant function! In fact, this can be taken as the *definition* of determinant, and the standard properties of that construction derived elegantly from this definition.

One such property is that every alternating r-linear functional on M is a multiple of the determinant. Let us, more generally, prove such a result for alternating r-linear functions from M into *any* k-module. Let $a: M^r \to N$ be such a function. Then by the universal property of $\Lambda^r M$ we can write $a(y_1, \ldots, y_r) = \alpha(y_1 \wedge \ldots \wedge y_r)$ for some k-module homomorphism $\alpha: \Lambda^n M \to N$. Let $\alpha(x_1 \wedge \ldots \wedge x_r) = u$. Take any $y_1, \ldots, y_r \in M$, and let f denote the unique endomorphism of M that carries the elements $x_1, \ldots, x_r$ to $y_1, \ldots, y_r$ respectively. Then $a(y_1, \ldots, y_r) = a(f(x_1), \ldots, f(x_r)) = \alpha(f(x_1) \wedge \ldots \wedge f(x_r)) = \alpha(d(f)\, x_1 \wedge \ldots \wedge x_r) = d(f)u$. In other words, the general alternating map $a: M^r \to N$ may be represented as the alternating map d, times some element of N.

If f, g are two endomorphisms of M, then from the functoriality of Λ^r we see that $d(fg) = d(f)d(g)$.

Note that for $f: M \to M$, the induced map $\Lambda^n f$ multiplies *every* element of $\Lambda^n M$ by the scalar $d(f)$, and that this property uniquely determines that scalar. Hence, though we needed the existence of a basis of r elements to show the existence of such a scalar, the value of the scalar can be characterized independent of choice of basis. This shows why the determinant of an endomorphism of a free module makes sense independent of basis.

Given a linear map $f: M \to N$ between free k-modules of finite rank, how will the induced map Λf act on the general element of ΛM? Let $x_1, \ldots, x_r$ be a basis of M, $y_1, \ldots, y_s$ a basis of N, and $((f_{ij}))$ the matrix of f (considered to act on the right) with respect to these bases. A typical member of our basis of ΛM has the form

$$p \;=\; x_{i_1} \wedge \ldots \wedge x_{i_m} \in \Lambda^m M \qquad (1 \leq i_1 < \ldots < i_m \leq r),$$

and a typical member of our basis of ΛN the form

$$q \;=\; y_{j_1} \wedge \ldots \wedge y_{j_n} \in \Lambda^n N \qquad (1 \leq j_1 < \ldots < j_n \leq s).$$

I claim that the coefficient of q in $(\Lambda f)(p)$ is zero unless $m = n$, and that in this case, it is the determinant of the $n \times n$ submatrix of $((f_{ij}))$ gotten by intersecting the $i_1, \ldots, i_n$th rows of that matrix with the $j_1, \ldots, j_n$th columns. The first part of the claim follows from the fact that Λf is put together out of maps $\Lambda^n f$, taking $\Lambda^n M$ to $\Lambda^n N$, but has no components carrying $\Lambda^m M$ to $\Lambda^n N$ for $m \neq n$. The second can be seen by composing f on the one side with the inclusion in M of the free submodule M' spanned by $x_{i_1}, \ldots, x_{i_n}$, and on the other side with the projection of N onto the free submodule N' spanned by $y_{j_1}, \ldots, y_{j_n}$. It is easy to see that on application of Λ^n, the first of these maps sends the single generator $p' = x_{i_1} \wedge \ldots \wedge x_{i_n}$ of $\Lambda^n M'$ to the basis element p of $\Lambda^n M$ (also formally $x_{i_1} \wedge \ldots \wedge x_{i_n}$, but here distinguished symbolically because it represents an element of a different algebra), and that the second sends the basis element q of $\Lambda^n M$ to the generator $q' = y_{j_1} \wedge \ldots \wedge y_{j_n}$ of $\Lambda^n N'$, and all other basis elements to 0. Now the composite f' of f with these maps on the indicated sides is represented by the $n \times n$ submatrix of $((f_{ij}))$ in question, hence $\Lambda^n f'$ sends p' to q' times the determinant of that matrix, hence that determinant is the coefficient of q in $(\Lambda f)(p)$, as desired

If M is a free module with basis $x_1, \ldots, x_r$, note that $\Lambda^{r-1} M$ is also free of rank r: for each $i \leq r$ we get a basis vector of this module by *leaving out* the ith factor of $x_1 \wedge \ldots \wedge x_r$. Let f be an endomorphism of M. Then the coefficients in the action of $\Lambda^{r-1} f$ with respect to this basis are essentially the entries of the *classical adjoint* of the matrix of f (Hungerford [**2** p. 353]; written $\widetilde{A}$ but not named in Lang [**3** p.518]). Precisely, the classical adjoint is the *transpose* of a matrix having these entries, with certain changes of sign. Let us now derive in our present context the result that the product of a square matrix with its classical adjoint is the identity times the determinant of the matrix. We will find it easiest to keep track of what should map into what if we start with a linear map $f: M \to N$, where M and N are possibly distinct free k-modules of the same rank r, and only at the end specialize to the case where they are the same.

Let us note that every element $x \in M$ induces a map $x \wedge -: \Lambda^{r-1} M \to \Lambda^r M$. If we let x run over our basis of M, then, up to sign, these maps run over the obvious basis of $\mathrm{Hom}(\Lambda^{r-1} M, \Lambda^r M)$. To get the correct signs, let us take a slightly modified basis of $\Lambda^{r-1} M$ consisting of the elements

$$\begin{gathered} x_2 \wedge \ldots \wedge x_r, \\ -x_1 \wedge x_3 \wedge \ldots \wedge x_r, \\ \cdots \\ \pm\, x_1 \wedge \ldots \wedge x_{r-1}, \end{gathered}$$

using a $+$ or $-$ according to whether, on left multiplying by the appropriate x_i, this has to pass through an even or an odd number of factors to reach its proper place, i.e., according as i is odd or even. We can now say that with respect to our given basis of M, the above basis of $\Lambda^{r-1} M$, and the singleton basis $\{x_1, \ldots, x_r\}$ of $\Lambda^r M$, the ''multiplication'' map $M \to \mathrm{Hom}(\Lambda^{r-1} M, \Lambda^r M)$ is represented by the identity matrix.

Now consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Lambda^{r-1} M, \Lambda^r M) & & \mathrm{Hom}(\Lambda^{r-1} N, \Lambda^r N) \\ {\scriptstyle (\Lambda^r f)\circ -}\downarrow & & \downarrow{\scriptstyle -\circ(\Lambda^{r-1} f)} \\ & \mathrm{Hom}(\Lambda^{r-1} M,\ \Lambda^r N), & \end{array}$$

where the upper two vertical arrows are those just described, while the lower ones represent post- and pre-composition with $\Lambda^r f$ and with $\Lambda^{r-1} f$ respectively. The diagram commutes: If we start with an element $x \in M$ we get

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \wedge - & & f(x) \wedge - \\ \downarrow & & \downarrow \\ (\Lambda^r f)(x \wedge -) & = & f(x) \wedge (\Lambda^{r-1} f)(-), \end{array}$$

with equality at the bottom because Λf is a ring homomorphism.

Now suppose $M = N$, and we fix a basis of this module, so that f, the top arrow of this diagram, an endomorphism of M, is represented by a certain matrix with respect to our basis. The upper vertical arrows are the same map, which, as we have noted, is represented in the appropriate basis by the identity matrix; the left hand lower vertical arrow is just the identity matrix times $\det(f)$, while the lower right-hand matrix has for entries the determinants of $(r-1)\times(r-1)$ minors of f, but with signs modified because of our choice of basis of $\Lambda^{r-1}M$, and transposed, because we are letting it act on the first (contravariant) argument of $\mathrm{Hom}(\Lambda^{r-1}M, \Lambda^{r}N)$. It is easy to verify that it is precisely the classical adjoint of f, so the commutativity of the above diagram shows that the matrix of f followed by its classical adjoint gives $(\det f)\,I$.

I do not see as direct a way to get the corresponding result for composition of these matrices in the opposite order. However, it can be obtained by applying the above result to the dual of f, and taking the transpose of both sides.

Subsequent sections of this note will, like earlier sections, look at properties of exterior algebras belonging to frameworks that also apply to tensor algebras and symmetric algebras. For further results on exterior algebras in particular (and the related construction of Clifford algebras), see Lang [**3**, Chapter XIX].

3. Geometric interpretations of exterior powers. [To be written up some time in the future. Cross-products; differential forms; closed polyhedra]

4. Graded algebras.

Definition 6. *A* graded (*or more precisely,* **Z**-graded) *k-algebra will mean a k-algebra R given with a direct sum decomposition into k-submodules*

$$R = \oplus_{n\in\mathbf{Z}} R_n$$

such that

$$1 \in R_0$$

and such that for all m, n,

$$R_m R_n \subseteq R_{m+n}.$$

If R is a graded algebra, an element is said to be homogeneous of degree n *if it belongs to the summand R_n,* homogeneous *if it belongs to one of these summands, which are called the* homogeneous components *of R.*

If R and S are graded algebras, then a homomorphism $R\to S$ as graded algebras means a homomorphism as algebras, which for each n carries R_n into S_n.

(*More generally, for any monoid G, which we will assume written multiplicatively, a G-graded k-algebra means a k-algebra R given with a direct sum decomposition into submodules*

$$R = \oplus_{g\in G} R_g$$

such that

$$1 \in R_e$$

and for all $g, h\in G$

$$R_g R_h \subseteq R_{gh}.$$

Homogeneous components and graded-algebra homomorphisms are defined in the same way for this case.)

We thus see that $k\langle M\rangle$, ΛM and SM can all be regarded as **Z**-graded k-algebras, or, indeed, as

N-graded k-algebras, where **N** is the additive monoid of nonnegative integers.

Gradings constitute an important tool in both commutative and noncommutative ring theory. We will not be able to look at that subject in this course; we have introduced the definition only so that we can define the concept of the next section, which is relevant to exterior algebras.

5. ‘‘Graded’’ or ‘‘super’’ or ‘‘topologists’’’ commutativity – a sketch. It is easy to verify that in an exterior algebra, homogeneous elements of even degree commute with everything, while homogeneous elements of odd degree anticommute with each other. In other words, for $a \in \Lambda^m M$, $b \in \Lambda^n M$,

$$ab = (-1)^{mn}\, ba.$$

This same equation turns out to hold frequently in the cohomology rings studied in topology, so that some topologists have taken to calling this ‘‘commutativity’’ as graded rings!

In recent years, the study of graded rings with this property has flourished under the name ‘‘super’’ commutativity. In this context, analogs of a large number of familiar concepts can be defined. For instance, just as in a not necessarily commutative algebra, one can define commutator brackets, which measure the failure of commutativity, and then define objects with an operation which behaves like commutator brackets, called ‘‘Lie algebras’’, so in a graded ring, one can define ‘‘super’’ commutator brackets, which act on homogeneous elements by the law $[a, b] = ab - (-1)^{\deg(a)\deg(b)}\, ba$, and define ‘‘super Lie algebras’’ as graded k-modules with an operation satisfying the identities of these brackets. And just as one may define the tensor product of two algebras, with multiplication based on the formula $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, which, if the algebras are commutative, is itself commutative and gives their coproduct in the category of commutative algebras, so one may define a modified multiplication on the module-theoretic tensor product of two graded algebras using the formula $(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb'$ for homogeneous elements a, b, a', b', and this turns out to preserve super commutativity, and to give the coproduct in the class of such algebras.

The condition of super commutativity is slightly weaker than the alternating condition imposed in the exterior algebra construction. If one takes $b = a$ in the equation $ba + ab = 0$, one gets $2a^2 = 0$, which, if 2 is invertible in the base ring k, is indeed equivalent to the alternating condition; but is not in cases such as $k = \mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$. So over a ring in which 2 is invertible, the exterior algebra construction, regarded as a functor from k-modules M to super commutative k-algebras, is left adjoint to the functor taking such an algebra to its homogeneous component of degree 1; and the functor associating to a set X the exterior algebra on the free k-module on X is left adjoint to the functor taking a super commutative k-algebra to the *set* of its elements of degree 1; i.e., this exterior algebra is a super commutative k-algebra ‘‘freely generated by an X-tuple of elements of degree 1’’. But over a general base-ring k, though such left adjoints still exist, they are somewhat larger than the exterior algebras.

Note that in defining ‘‘super commutativity’’, one only needs to know whether degrees are odd or even, not their exact values. Thus, the concept can be defined on $\mathbf{Z}/2\mathbf{Z}$-graded rings. Moreover, any $\mathbf{Z}$-graded ring can be regarded as a $\mathbf{Z}/2\mathbf{Z}$-graded ring by calling the sum of the components R_n with n even the ‘‘even’’ component and the sum of those with n odd the ‘‘odd’’ component. Thus, the concept of super commutativity is often defined for $\mathbf{Z}/2\mathbf{Z}$-graded rings, with the understanding that $\mathbf{Z}$-graded rings are automatically regarded as $\mathbf{Z}/2\mathbf{Z}$-graded in this way.

Now suppose we define a ‘‘forgetful functor’’ taking each super commutative $\mathbf{Z}/2\mathbf{Z}$-graded k-algebra R to the ordered pair of sets (R_0, R_1). This functor has a left adjoint, which takes a pair of sets (X_0, X_1) to the super commutative k-algebra ‘‘freely generated by an X_0-tuple of even elements and an X_1-tuple of odd elements’’. Assuming 2 is invertible in k, we see that when $X_0 = \varnothing$, this is the exterior algebra on X_1; when $X_1 = \varnothing$, it is the polynomial algebra on X_0, while for general X_0 and X_1, it is the tensor product of these two algebras.

6. Direct sum decompositions. In Theorem 5 we described the form our constructions take when applied to free modules. Let us consider what we get when we start with a general direct sum of modules, $M \oplus N$.

The tensor power construction is based on iterating the pairwise tensor product construction, which satisfies the laws $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$ and $A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$. It is not hard to deduce that $\otimes^n (M \oplus N)$ is the direct sum of all n-fold tensor products of strings of M's and N's (2^n summands in all).

The cases of the symmetric and exterior powers are not so obvious. Note that $\Lambda^n (M \oplus N)$, respectively $S^n (M \oplus N)$, is spanned as a k-module by products of elements of M and elements of N, and that by the anticommutativity or commutativity law, every such product may be reduced to a product in which all factors coming from M precede all factors coming from N. Within each of these two strings of factors, the commutative or alternating law can be used to get further equalities, but it does not appear that there will be any relations connecting the two strings of elements, except those arising from bilinearity. This motivates the next result, for which I will briefly sketch three methods of proof.

Theorem 7. *Let M and N be k-modules. Then*

$$\Lambda^p (M \oplus N) \cong \oplus_{m+n=p} (\Lambda^m M) \otimes (\Lambda^n N),$$

via the map

$$x_1 \wedge \ldots \wedge x_m \wedge y_1 \wedge \ldots \wedge y_n \leftarrow (x_1 \wedge \ldots \wedge x_m) \otimes (y_1 \wedge \ldots \wedge y_n),$$

and likewise

$$S^p (M \oplus N) \cong \oplus_{m+n=p} (S^m M) \otimes (S^n N),$$

via the map

$$x_1 \ldots x_m y_1 \ldots y_n \leftarrow (x_1 \ldots x_m) \otimes (y_1 \ldots y_n).$$

Sketch of 3 proofs. I will discuss the exterior-algebra case. The symmetric-algebra case is similar.

Method 1. The idea is to construct a k-multilinear alternating map from $(M \oplus N)^p$ to the indicated direct sum, and verify that it has the universal property of p-fold exterior multiplication. The map is defined so that a string consisting of m elements from M and n elements from N in some order goes to the element of $(\Lambda^m M) \otimes (\Lambda^n N)$ gotten by rearranging them so that all members of M precede all members of N, multiplying them together in the respective exterior powers, and putting on a + or – sign according to whether the ‘‘rearrangement’’ involved an odd or an even number of transpositions. This is then extended by multilinearity to terms consisting of sums of elements of M and elements of N. Once this operation is defined precisely, it is straightforward to establish that it has the desired universal property.

Method 2. Use the version of the diamond lemma in [**1**, §6] which deals with presentations of rings in terms of generating (bi)modules instead of elements. Present $\Lambda(M \oplus N)$ in terms of the modules M and N, get a normal form for this algebra, and extract $\Lambda^p (M \oplus N)$ as the pth power therein of the generating module $M \oplus N$.

Method 3. It is easy to verify by universal properties that $\Lambda(M \oplus N)$ is the coproduct, in the category of ‘‘super commutative’’ graded k-algebras, of ΛM and ΛN. (Key point: an easy computation shows that in a super commutative algebra, a sum of odd-degree elements with zero square again has zero square.) Now use the description of coproducts in this category as tensor products, referred to in the preceding section. The degree-p component of $(\Lambda M)\otimes(\Lambda N)$ is $\bigoplus_{m+n=p}(\Lambda^m M)\otimes(\Lambda^n N)$, as desired. □

7. Functorial notes. Given a k-module M and a nonnegative integer n, the universal property of each of $\otimes^n M$, $\Lambda^n M$, $S^n M$ can be considered as saying that the module is a *representing object* for a certain set-valued functor. E.g., $\Lambda^n M$ represents the functor which takes every module P to the set of alternating k-multilinear maps $M^n \to P$. Now it frequently happens that when, for each object X of a category $\mathbf{C}$, we can construct an object $F(X)$ of a category $\mathbf{D}$, representing a covariant functor $\mathbf{D}\to\mathbf{Set}$ defined in terms of X, then these objects fit together to form a functor $F:\mathbf{C}\to\mathbf{D}$, which is a left adjoint to some functor $U:\mathbf{D}\to\mathbf{C}$. In the present case each of our three module constructions does indeed give a functor k-**Mod** $\to$ k-**Mod**, but we have an exception to the second general observation: none of these functors is a left adjoint. Indeed, left adjoint functors respect coproducts, but the observations of the preceding section show that for $n>1$, none of these functors takes direct sums to direct sums.

Curiously, however, when we put these modules together to form k-algebras, one of the resulting constructions, the tensor algebra, does, as we have seen, become a left adjoint – the left adjoint to the forgetful functor from k-algebras to k-modules. The other two are still not adjoints when regarded as functors to general k-algebras, but as we have also seen, the symmetric algebra construction becomes one when regarded as a functor to commutative k-algebras. It is not hard to show that, similarly, the exterior algebra functor, when regarded as a functor to ‘‘super-commutative’’ graded k-algebras in which odd-degree elements satisfy $a^2 = 0$, becomes left adjoint to the ‘‘module of elements of degree 1’’ functor.

On the other hand, all of our constructions of both modules and algebras can be made into left adjoints if their descriptions are adjusted somewhat. For example, if the n-fold tensor power construction is inflated so that the codomain category is not the category of k-modules, but the category of 3-tuples (M, a, P) where M and P are k-modules and a is a k-multilinear map $M^n \to P$, then it becomes a left adjoint to the forgetful functor $(M, a, P)\mapsto M$.

8. Operations relating these algebras and their duals.

[To be written up some time in the future?]

For more on the exterior algebra cf. Lang’s *Algebra*, 3rd edition, Chapter XIX.

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