RESEARCH STATEMENT

FORTE SHINKO

My research is in descriptive set theory and its connections to dynamical systems and geometric group theory.

1. INTRODUCTION

1.1. **Descriptive set theory.** Descriptive set theory is the study of "definable" subsets of Polish spaces and maps between them, for various notions of definability. Most commonly, we take "definable" to mean Borel, where a **Borel set** is an element of the σ-algebra generated by the open subsets of a Polish space, and where a **Borel function** $X \to Y$ is one under which the preimage of every open set is Borel. Other common notions of "definable" include **analytic** and **Baire-measurable**.

By restricting one's study to definable sets, it is possible to prove more structural theorems than in the general setting. For instance, the continuum hypothesis holds for Borel sets, that is to say, every Borel subset of $\mathbb{R}$ is either countable or of size continuum. This is in stark contrast to arbitrary subsets of $\mathbb{R}$, where the existence of a subset $A \subseteq \mathbb{R}$ with $\aleph_0 < |A| < |\mathbb{R}|$ is independent of ZFC. Another benefit is the absence of pathologies obtained from the Axiom of Choice and other non-constructive arguments, such as the existence of a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space, which is possible due to the Axiom of Choice but not in a definable way, in that there is no Borel basis.

1.2. **Borel equivalence relations.** Over the past 40 years, descriptive set theory has seen a wide variety of connections with areas outside of logic, such as ergodic theory, operator algebras, geometric group theory, and more recently, computer science. One important concept which has emerged in these applications is that of a **Borel equivalence relation**, that is, an equivalence relation E on a standard Borel space X such that E is a Borel subset of X^2. Many classification problems in mathematics arise as Borel equivalence relations, such as the classification of finite rank torsion-free abelian groups up to isomorphism, or the classification of unitary operators on the infinite-dimensional Hilbert space up to conjugacy.

The most important notion in Borel equivalence relations is **Borel reduction**, which lets one talk about the relative hardness of two problems, analogous to polynomial time reduction in complexity theory. Given Borel equivalence relations E and F on X and Y respectively, a Borel reduction is a Borel map $f : X \to Y$ such that $x \, E \, x'$ iff $f(x) \, F \, f(x')$. Informally, it says that if one can classify up to F-equivalence then one is also able to classify up to E-equivalence by applying f. If there is a Borel reduction from E to F, one says that E is **Borel reducible** to F, denoted $E \leq_B F$. The "simplest" class of Borel equivalence relations are the **smooth** equivalence relations, which are those Borel equivalence relations E which Borel reduce to $=_{\mathbb{R}}$, the equality relation on $\mathbb{R}$. In other words, there are concrete invariants that exactly classify the elements of X up to E-equivalence. For instance, Ornstein's isomorphism theorem says that Bernoulli shifts are classified up to isomorphism

by their Kolmogorov-Sinai entropy, and thus isomorphism of Bernoulli shifts is a smooth equivalence relation, with the Borel reduction being the map which sends a shift to its entropy.

1.3. **Countable Borel equivalence relations.** A large part of my research is focused on **countable Borel equivalence relations (CBER)**, which are Borel equivalence relations with every class countable. An important source of examples arises as follows: given a countable group Γ with a Borel action on a standard Borel space X, the **orbit equivalence relation** E_Γ^X is defined by $x \, E_\Gamma^X \, x'$ iff $\exists \gamma [x' = \gamma \cdot x]$. This is a CBER, and in fact, a theorem of Feldman and Moore shows that every CBER on X is the orbit equivalence relation of some Borel action of a countable group on X. In this way, the theory of CBERs is very intimately connected with the study of countable groups and their Borel and measurable aspects. Amenability plays an important role in the study of hyperfiniteness (defined below), and the use of property (T) groups allows us to invoke such powerful theorems as Popa's cocycle superrigidity theorems.

The Borel reducibility preorder $\leq_B$ on CBERs looks like the following:
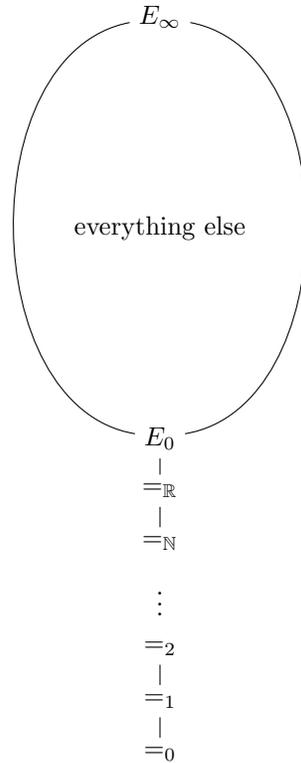
where the relations in this diagram are defined as follows (starting from the bottom):
- $=_X$ is the equality relation on the space X.
- E_0 is the **eventual equality** relation on 2^ω defined by

$$x \, E_0 \, y \iff \exists n \forall m \geq n \, [x_m = y_m].$$

- E_∞ is the orbit equivalence relation induced by the shift action of F_2 on 2^{F_2}. This is a **universal** CBER, that is, for every CBER F, we have $F \leq_B E_\infty$.

2. Hyperfinite Borel equivalence relations

A relatively low class (in terms of Borel reduction) of CBERs is that of the **hyperfinite** CBERs. A CBER E on a standard Borel space X is hyperfinite if it satisfies any of the following equivalent conditions:

(1) there is an increasing sequence $(E_n)_n$ of finite Borel equivalence relations on X such that $E = \bigcup_n E_n$ (an equivalence relation is **finite** if every class is finite);
(2) there is a Borel $\mathbb{Z}$-action on X such that $E = E_{\mathbb{Z}}^X$;
(3) $E \leq_B E_0$,

From the image above and the third characterization of hyperfiniteness, we see that the hyperfinite CBERs are exactly the next level of complexity after the smooth CBERs, which justifies the view that the hyperfinite CBERs have relatively low complexity.

Other examples of hyperfinite CBERs arise from boundary actions of countable groups. For instance, we have shown in [HSS20] that if Γ is a cubulated hyperbolic group. then action of Γ on its Gromov boundary $\partial\Gamma$ is hyperfinite, generalizing the classical result of Dougherty-Jackson-Kechris [DJK94, Corollary 8.2] that the tail equivalence relation E_t on 2^ω is hyperfinite. The same result for arbitrary (finitely generated) hyperbolic groups has been obtained by Marquis and Sabok in [MS20].

In a completely different direction, we have shown recently that hyperfinite CBERs arise from the cosets of a countable normal subgroup:

Theorem 1. [FSb] *If G is a Polish group and Γ is a countable normal subgroup, then G/Γ is hyperfinite (where we write G/Γ to mean E_Γ^G).*
In particular, if Γ is a countable group, then $\mathrm{Out}(\Gamma)$ is hyperfinite.

This was a surprising result since generally, a nontrivial construction taking a countable group Γ to a CBER will reflect some of the complexity of Γ. For instance, if a countable group Γ is non-amenable, then the shift action of Γ on 2^Γ is not hyperfinite, and it is an important open question as to whether this characterizes (non)-amenability.

The analogous question in the case of locally compact groups is still open. The generalization of hyperfiniteness to this setting is hypersmoothness, where a Borel equivalence relation is **hypersmooth** if it is an increasing union of smooth Borel equivalence relations.

Question 1. *Let G be a Polish group, and let H be a Borel subgroup which is Borel isomorphic to a locally compact Polish group. Is G/H hypersmooth?*

3. Topological realizations

For this section, we will mostly be concerned with **aperiodic** CBERs, that is, CBERs with every class infinite.

Although notions such as Borel reduction and Borel equivalence relations do not remember the Polish topology, one can often use the topology to prove purely Borel results. A characteristic instance of this is that if Γ is a countable group acting by homeomorphisms on a Polish space X with every equivalence class dense, then E_Γ^X is not smooth. As a corollary of this result, if Γ is a countable group acting on a compact Polish space with infinite orbits, then E_Γ^X is not smooth.

A CBER E on a Polish space X is **minimal** if every E-class is dense. An aperiodic CBER E on X has a **minimal action realization** if there is a countable group Γ and a Polish Γ-space Y such that E_Γ^Y is minimal and Borel isomorphic to E. By the fact above, if E has a minimal action realization, then it is not smooth.

The converse is still open:

Question 2. *Does every non-smooth aperiodic CBER have a minimal action realization?*

We have shown that there is a positive answer if we do not require the group action. Say that a CBER E has a **minimal realization** if there is a minimal CBER F on a Polish space Y such that E is Borel isomorphic to F. This is similar to the minimal action realization, without the requirement of a group action.

Theorem 2. [FKSV21, Theorem 3.1] *Every aperiodic CBER E has a minimal realization.*

This already has Borel consequences, such as the existence of certain marker systems and stationary measures, see [FKSV21, Section 3].

In the case of group actions, we know that every hyperfinite CBER has a minimal action realization, but other than some other special cases, the question is still wide open.

An aperiodic CBER E on X has a **compact action realization** if there is a countable group Γ and a compact Polish Γ-space Y such that E_Γ^Y is Borel isomorphic to E.

Question 3. *Does every non-smooth aperiodic CBER have a compact action realization?*

We can prove various cases here, including the hyperfinite CBERs, free parts of Bernoulli shifts, and universal compressible CBERs (e.g. arithmetic equivalence on $2^\mathbb{N}$), but the general case is again wide open, and will be addressed in future research.

Clinton Conley asked the weaker question of whether every E has a K_σ **realization**, i.e. whether it is isomorphic to a K_σ CBER, where a K_σ set is one which is a countable union of compact sets. We have answered this question in the positive:

Theorem 3. [FKSV21, Theorem 3.9.1]. *Every aperiodic CBER has a K_σ-realization.*

A natural setting for realizations is in the context of subshifts. Given a group Γ and a Polish space X, a **subshift** is a closed Γ-invariant subspace of X^Γ, where the Γ-action on X^Γ is the shift action. If K is a subshift, let E_K denote its orbit equivalence relation.

We can realize a universal CBER as a subshift:

Theorem 4. [FKSV21, Corollary 3.6.6] *There is a minimal subshift K of 2^{F_3} such that E_K is a universal CBER.*

A CBER E on X is **compressible** if there is a family $(f_C)_C$, indexed by the E-classes C, of proper injections $f_C : C \to C$, such that the function $X \to X$ defined by $x \mapsto f_{[x]_E}(x)$ is Borel. By Nadkarni's theorem, this is equivalent to the non-existence of an invariant Borel probability measure for E. We characterize amenable groups in terms of compressible subshifts:

Theorem 5. [FKSV21, Theorem 3.7.1] *Let Γ be a countable group. Then Γ is non-amenable iff there is a subshift K of 2^Γ such that E_K is compressible.*

A natural consideration in this context is the space of all subshifts. For a Polish space X, let $\mathrm{Sh}(X)$ denote the standard Borel space of subshifts of X^{F_∞}. If X is compact, then $\mathrm{Sh}(X)$ has a compact Polish topology compatible with the Borel structure. Every CBER arises as E_F for some $F \in \mathrm{Sh}(\mathbb{R}^\mathbb{N})$, so we think of $\mathrm{Sh}(\mathbb{R}^\mathbb{N})$ as a universal space of CBERs. Similarly, every CBER with a compact action realization arises as E_K for some $K \in \mathrm{Sh}([0,1]^\mathbb{N})$, so we think of $\mathrm{Sh}(\mathbb{R}^\mathbb{N})$ as a universal space of CBERs with a compact action realization.

We compute the descriptive and topological complexity of various classes of subshifts. A CBER E on X is measure-hyperfinite if for every Borel probability measure μ on X, there is a μ-conull subset Y of X such that the restriction $E{\restriction}Y$ of E to Y is hyperfinite. A subshift K is **smooth**

(resp. **hyperfinite**, **measure-hyperfinite**) if its orbit equivalence relation E_K is. A subshift K is **free** if the action $\Gamma \curvearrowright K$ is free. The most important results are summarized in the following table:

Φ	$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : K \text{ has property } \Phi\}$	
smooth	meager	$\mathbf{\Pi}_1^1$-complete
hyperfinite	?	$\mathbf{\Sigma}_2^1$
measure-hyperfinite	comeager	$\mathbf{\Pi}_1^1$-complete
free measure-hyperfinite	comeager	G_δ

It is a major open question to determine that the descriptive complexity for hyperfiniteness is not just $\mathbf{\Sigma}_2^1$, but in fact $\mathbf{\Sigma}_2^1$-complete, and we suspect that the set of hyperfinite subshifts is comeager. It is surprising that although the set of measure-hyperfinite subshifts is $\mathbf{\Pi}_1^1$-complete, the set of *free* measure-hyperfinite subshifts is G_δ. This uses the nontrivial equivalence between measure-amenability and topological amenability, see [FKSV21, Appendix A].

The fact that the smooth subshifts are $\mathbf{\Pi}_1^1$-complete has the following implication for compact action realizations:

Theorem 6. [FKSV21, Theorem 3.8.12] *For every $x \in 2^{\mathbb{N}}$, there is a non-smooth aperiodic subshift $F \in \mathrm{Sh}(\mathbb{R}^{\mathbb{N}})$, such that for every $K \in \mathrm{Sh}([0,1]^{\mathbb{N}})$, there is no $\Delta_1^1(F,x)$ isomorphism of E_F with E_K.*

This means that even if Question 3 has a positive answer, that is, even if every non-smooth aperiodic CBER has a compact action realization, there is no "effective" way to obtain this realization.

4. Lifts of Borel actions

Let E be a countable Borel equivalence relation on X. A **Borel permutation** of X/E is a bijection $f : X/E \to X/E$ such that the set $\{(x, x') \in X^2 : f([x]_E) = [x']_E\}$ is Borel. Let $\mathrm{Sym}_B(X/E)$ denote the group of Borel permutations of X/E. We are concerned with the problem of lifting Borel permutations to Borel isomorphisms on the space X. More precisely, a **Borel automorphism** of E is a Borel isomorphism $T : X \to X$ such that $x \mathrel{E} x'$ iff $f(x) \mathrel{E} f(x')$. Let $\mathrm{Aut}_B(E)$ denote the group of Borel automorphisms of E. Every $T \in \mathrm{Aut}_B(E)$ induces a Borel permutation of X/E sending $[x]_E$ to $[T(x)]_E$, and a Borel permutation of X/E induced by a Borel automorphism of E is called **outer**. The **outer automorphism group** of E, denoted $\mathrm{Out}_B(E)$, is the subgroup of $f \in \mathrm{Sym}_B(X/E)$ which are outer.

For a countable group Γ, an action $\Gamma \curvearrowright X/E$ by Borel permutations is called a **Borel action**, which is equivalently a homomorphism $\Gamma \to \mathrm{Sym}_B(X/E)$. We say a Borel action **lifts** if it factors through the map $\mathrm{Aut}_B(E) \to \mathrm{Sym}_B(X/E)$ described above. We show that for compressible CBERs, every action lifts:

Theorem 7. [FKS, Theorem 3.5] *Let Γ be a countable group and let E be a compressible CBER. Then every Borel action $\Gamma \curvearrowright X/E$ lifts.*

If E is not compressible, there can be elements of $\mathrm{Sym}_B(X/E)$ which are not outer, and thus there are Borel actions on X/E which do not lift. For this reason, it is interesting to restrict the setting to that of outer actions. An **outer action** of a countable group Γ is a Borel action $\Gamma \curvearrowright X/E$ by outer permutations, in other words, a homomorphism $\Gamma \to \mathrm{Out}_B(E)$. A common situation where this arises is the following: if $\Gamma \curvearrowright X$ is a Borel action of a countable group on a standard Borel space, and $N \triangleleft \Gamma$ is a normal subgroup, then the action $\Gamma \curvearrowright X$ descends to an outer action $\Gamma \to \mathrm{Out}_B(E_N^X)$.

Let $\mathcal{G}$ be the class of countable groups for which every outer action $\Gamma \to \operatorname{Out}_B(E)$ lifts for every CBER E. We have shown that $\mathcal{G}$ contains a wide variety of groups:

Theorem 8. [FKS, Section 7]

    (1) $\mathcal{G}$ contains all amenable groups.
    (2) $\mathcal{G}$ contains all amalgamated products of finite groups.
    (3) $\mathcal{G}$ is closed under subgroups.
    (4) $\mathcal{G}$ is closed under free products.

The first point generalizes a result of Feldman, Sutherland and Zimmer in the measurable setting, see [FSZ89, Theorem 3.4]. An interesting feature of the proof of the second point is that it uses Tarski's theory of cardinal algebras (see [Tar49]), which is starting to see applications in the study of countable Borel equivalence relations, for example in [Shi21], where we apply cardinal algebras to the study of invariant measures (see also [KM16], [Che18]).

We also have an upper bound on this class of groups.

Theorem 9. [FKS, Proposition 4.11] *Every group Γ in $\mathcal{G}$ is treeable, that is, there is a free pmp action of Γ on a standard probability space which is treeable (a Borel equivalence relation E on X is* **treeable** *if there is a Borel forest T on X whose connected components are exactly the E-classes).*

There are many examples of groups which are known to not be treeable, for instance, every property (T) group, and every product $\Gamma \times \Delta$, where Γ is infinite and Δ is non-amenable.

We hope to characterize the class of groups in $\mathcal{G}$:

Question 4. *Which groups are in $\mathcal{G}$? Is it exactly the class of treeable groups?*

5. Dichotomies for Polish modules

Many of the cornerstone results in descriptive set theory are dichotomy theorems, which state that either an object satisfies some countability condition, or otherwise, there is a canonical obstruction to uncountability. The fundamental example is Cantor's perfect set theorem, which states that a Polish space is either countable, or it contains a copy of the Cantor space. A more modern example is the G_0-dichotomy of Kechris, Solecki and Todorčević, which states that a Borel graph G either has countable Borel chromatic number, or there is a Borel homomorphism from a certain graph G_0 with uncountable Borel chromatic number to G.

We have shown a family of dichotomy theorems for vector spaces, and more generally, modules over certain nice classes of rings. If a Polish $\mathbb{Q}$-vector space V is uncountable, then it has dimension $2^{\aleph_0}$, so it is uniquely determined up to isomorphism. However, the existence of a basis uses the Axiom of Choice, and indeed, it turns out that many of these Polish $\mathbb{Q}$-vector spaces are not Borel isomorphic. For Polish $\mathbb{Q}$-vector spaces V and W, write $V \sqsubseteq W$ if there is an injective Borel homomorphism from V to W. We construct a Polish $\mathbb{Q}$-vector space $\ell^1(\mathbb{Q})$, and show that it is the obstruction to countability:

Theorem 10. [FSa] *Let F be a countable field, and let V be a Polish $\mathbb{Q}$-vector space. Then exactly one of the following holds:*

    (1) V is countable.
    (2) $\ell^1(\mathbb{Q}) \sqsubseteq V$.

There are still many properties of the embedding order which we do not yet understand. For instance, the above theorem shows that $\ell^1(\mathbb{Q}) \sqsubseteq \mathbb{R}$, but we do not know if there is anything in between:

Question 5. *Is there a $\mathbb{Q}$-vector space V such that $\ell^1(\mathbb{Q}) \sqsubset V \sqsubset \mathbb{R}$?*

Another natural question is the existence of a maximum element for the embedding order. In the case of $\mathbb{R}$ and $\mathbb{C}$, a positive answer is known due to Kalton [Kal77], and in the case of $\mathbb{Z}$, a positive answer is known due to Shkarin [Shk99]. In general, the answer is not known:

Question 6. *Let R be a Polish ring. Is there a Polish R-module M such that for every Polish R-module N, we have $N \sqsubseteq M$?*

6. FUTURE DIRECTIONS

In future work, I would also like to look more into problems in Borel combinatorics, including the very exciting recent developments which connect it with the LOCAL model for distributed algorithms. I would also like to look more into Borel and analytic equivalence relations which are not necessarily countable, since there are many techniques and areas explored there which do not come up in the countable context, such as turbulence for Polish group actions, the classification of classes of countable structures up to isomorphism, and the use of forcing.

REFERENCES

[Che18]   Ruiyuan Chen. Decompositions and measures on countable Borel equivalence relations. *arXiv:1801.02767*, 2018.
[DJK94]   R. Dougherty, S. Jackson, and A. S. Kechris. The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.*, 341(1):193–225, 1994.
[FKS]   Joshua Frisch, Alexander S. Kechris, and Forte Shinko. Lifts of Borel actions on quotient spaces. *arXiv:2011.01395, submitted*.
[FKSV21] Joshua Frisch, Alexander S. Kechris, Forte Shinko, and Zoltán Vidnyánszky. Realizations of countable Borel equivalence relations. *arXiv:2109.12486*, 2021.
[FSa]   Joshua Frisch and Forte Shinko. A dichotomy for Polish modules. *arXiv:2009.05855, to appear in Israel J. Math*.
[FSb]   Joshua Frisch and Forte Shinko. Quotients by countable subgroups are hyperfinite. *arXiv:1909.08716, to appear in in Groups Geom. Dyn*.
[FSZ89]   J. Feldman, C. E. Sutherland, and R. J. Zimmer. Subrelations of ergodic equivalence relations. *Ergodic Theory Dynam. Systems*, 9(2):239–269, 1989.
[HSS20]   Jingyin Huang, Marcin Sabok, and Forte Shinko. Hyperfiniteness of boundary actions of cubulated hyperbolic groups. *Ergodic Theory Dynam. Systems*, 40(9):2453–2466, 2020.
[Kal77]   N. J. Kalton. Universal spaces and universal bases in metric linear spaces. *Studia Math.*, 61(2):161–191, 1977.
[KM16]   Alexander S. Kechris and Henry L. Macdonald. Borel equivalence relations and cardinal algebras. *Fund. Math.*, 235(2):183–198, 2016.
[MS20]   Timothée Marquis and Marcin Sabok. Hyperfiniteness of boundary actions of hyperbolic groups. *Math. Ann.*, 377(3-4):1129–1153, 2020.
[Shi21]   Forte Shinko. Equidecomposition in cardinal algebras. *Fund. Math.*, 253(2):197–204, 2021.
[Shk99]   S. A. Shkarin. On universal abelian topological groups. *Mat. Sb.*, 190(7):127–144, 1999.
[Tar49]   Alfred Tarski. *Cardinal Algebras*. Oxford University Press, 1949.