A DICHOTOMY FOR POLISH MODULES

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ABSTRACT. Let R be a ring equipped with a proper norm. We show that under suitable conditions on R, there is a natural basis under continuous linear injection for the set of Polish R-modules which are not countably generated. When R is a division ring, this basis can be taken to be a singleton.

1. INTRODUCTION

The axiom of choice allows us to construct many abstract algebraic homomorphisms between topological algebraic systems which are incredibly non-constructive. A longstanding theme in descriptive set theory is to study to what extent we can, and to what extent we provably cannot, construct such homomorphisms in a "definable" way. Here the notion of definability is context-dependent but often includes continuous, Borel, or projective maps.

A classical example of such an abstract construction, which provably cannot be constructed with "nice" sets is the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. It is well-known that such a basis cannot be Borel, or more generally, analytic. Similar phenomena show up when constructing Hamel bases for topological vector spaces, or constructing an isomorphism of the additive groups of $\mathbb{R}$ and $\mathbb{C}$.

A more recent theme in descriptive set theory is that such undefinability criteria can often be leveraged in order to gain, and hopefully utilize, additional structure. For example, Silver's theorem [Sil80] and the Glimm-Effros dichotomy [HKL90] interpret the non-reducibility of Borel equivalence relations not as a pathology but rather as the first step in the burgeoning theory of invariant descriptive set theory (see [Gao09] for background). Similarly, work starting with [KST99] studies and exploits the difference between abstract chromatic numbers and more reasonably definable (for example, continuous or Borel) chromatic numbers. A key feature in many of these theories (and all of the above examples) is the existence of dichotomy theorems, which state that either an object is simple, or there is a canonical obstruction contained inside of it. This is usually stated in terms of preorders, saying that there is a natural basis

for the preorder of objects which are not simple (recall that a **basis** for a preorder P is a subset $B \subseteq P$ such that for every $p \in P$, there is some $b \in B$ with $b \leq p$).

In this paper, we apply a descriptive set-theoretic approach to vector spaces and more generally, modules, over a locally compact Polish ring[1]. For a Polish ring R, a **Polish R-module** is a topological left R-module whose underlying topology is Polish. Given Polish R-modules M and N, we say that M **embeds** into N, denoted $M \sqsubseteq^R N$, if there is a continuous linear injection from M into N. One particularly nice aspect of Polish modules is that the notion of "definable" reduction is much simpler than in the general case. By Pettis's lemma, any Baire-measurable homomorphism between Polish modules is in fact automatically continuous (see [Kec95, 9.10]). Thus there is no loss of generality in considering continuous homomorphisms rather than a priori more general Borel homomorphisms.

Our main results give a dichotomy for Polish modules being countably generated. More precisely, we give a countable basis under $\sqsubseteq^R$ for Polish modules which are not countably generated. While these results are stated in a substantial level of generality (they are true for all left-Noetherian countable rings and many Polish division rings), we feel that the most interesting cases are over some of the most concrete rings. For example, over $\mathbb{Q}$, we show the existence of a unique (up to bi-embeddability) minimal uncountable Polish vector space $\ell^1(\mathbb{Q})$. We further show that nothing bi-embeddable with $\ell^1(\mathbb{Q})$ is locally compact, and thus that every uncountable-dimensional locally compact Polish vector space (for example, $\mathbb{R}$) is strictly more complicated than $\ell^1(\mathbb{Q})$.

Another case of particular interest is the case of $\mathbb{Z}$-modules, that is, abelian groups. We show that there is a countable basis of minimal uncountable abelian Polish groups (one for each prime number and one for characteristic 0). Furthermore, there exists a maximal abelian Polish group by [Shk99], as well as many natural but incomparable elements (for example, $\mathbb{Q}_p$ and $\mathbb{R}$ are incomparable under $\sqsubseteq^{\mathbb{Q}}$ as are $\mathbb{Q}_p$ and $\mathbb{Q}_r$ for $p \neq r$).

Our dichotomy theorems will hold for rings equipped with a proper norm. A **(complete, proper) norm** on an abelian group A is a function $\|\cdot\| : A \to [0, \infty)$ such that the map $(a, b) \mapsto \|a - b\|$ is a (complete, proper) metric on A (recall that a metric is proper if every closed ball is compact). A **norm** on a ring R is a norm $|\cdot|$ on $(R, +)$ such that $|rs| \leq |r||s|$ for every $r, s \in R$. A **proper normed ring** is a ring equipped with a proper norm. Every countable ring admits a proper norm (see Section 3). Given a proper normed ring R, the R-module $\ell^1(R)$ is defined as follows:

$$\ell^1(R) = \left\{ (r_k)_k \in R^{\mathbb{N}} : \sum_k \frac{|r_k|}{k!} < \infty \right\}$$

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[1]All rings will be assumed to be unital.

(here, $\frac{1}{k!}$ can be replaced with any summable sequence). Then $\|(r_k)_k\| := \sum_k \frac{|r_k|}{k!}$ is a complete separable norm on $(\ell^1(R), +)$, turning $\ell^1(R)$ into a Polish R-module.

The following theorems will be obtained as special cases of results in Section 5.

A **division ring** is a ring R such that every nonzero $r \in R$ has a two-sided inverse.

Theorem 1.1. *Let R be a proper normed division ring and let M be a Polish R-vector space. Then exactly one of the following holds:*

  (1) $\dim_R(M)$ *is countable.*
  (2) $\ell^1(R) \sqsubseteq^R M$.

This seems to be new, even when R is a finite field, in which case $\ell^1(R) = R^{\mathbb{N}}$. This also implies a special case of [Mil12, Theorem 24], which says that if $\dim_R(M)$ is uncountable, then there is a linearly independent perfect set (see Corollary 5.2).

An analogous statement holds for a large class of discrete rings. A ring is **left-Noetherian** if every increasing sequence of left ideals stabilizes.

Theorem 1.2. *Let R be a left-Noetherian discrete proper normed ring and let M be a Polish R-module. Then exactly one of the following holds:*

  (1) M is countable.
  (2) $\ell^1(S) \sqsubseteq^R M$ *for some nonzero quotient S of R.*

Note that this basis is countable since a countable left-Noetherian ring only has countably many left ideals.

For abelian Polish groups, we obtain an irreducible basis (see Theorem 4.3):

Theorem 1.3. *Let A be an uncountable abelian Polish group. Then one of the following holds:*

  (1) $\ell^1(\mathbb{Z}) \sqsubseteq^{\mathbb{Z}} A$.
  (2) $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \sqsubseteq^{\mathbb{Z}} A$ *for some prime p.*

Related statements have been shown by Solecki, see [Sol99, Proposition 1.3, Theorem 1.7].

The theorems in Section 5 will be shown for a substantially broader class of modules. In order to contextualize this, we remark that considering even very basic module homomorphisms (for example, the inclusion of $\mathbb{Q}$ into $\mathbb{R}$ as $\mathbb{Q}$-vector spaces) naturally leads us to consider the broader class of quotients of Polish modules by sufficiently definable submodules. Such quotient modules are in general not Polish (they are not necessarily even standard Borel) but are still important objects of descriptive set-theoretic interest. They play a crucial role in [BLP20] in the form of "groups with a Polish cover", and they also form some of the most classical examples of countable Borel equivalence relations (for example, the commensurability relation on the positive reals naturally comes equipped with an abelian group structure). The embedding order

on quotient modules will be defined analogously to the homomorphism reductions for Polish groups studied in [Ber14, Ber18].

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2. Polish modules

Most Polish modules which cannot be written as direct sums, even over a field. This will follow from a more general statement about Polish groups.

Given a Polish group G, a family $(H_x)_{x \in \mathbb{R}}$ of subgroups of G is

(i) **analytic** if the set $\{(g, x) \in G \times \mathbb{R} : g \in H_x\}$ is analytic;
(ii) **independent** if for every finite $F \subseteq \mathbb{R}$ and every $x \in \mathbb{R} \setminus F$, we have $H_x \cap \langle H_y \rangle_{y \in F} = 1$;
(iii) **generating** if $(H_x)_{x \in \mathbb{R}}$ generates G.

In particular, if G is the direct sum or the free product of $(H_x)_{x \in \mathbb{R}}$, then $(H_x)_x$ is an independent generating family.

Proposition 2.1. *Let G be a Polish group, and let $(H_x)_{x \in \mathbb{R}}$ be an analytic independent generating family of subgroups of G. Then there are only countably many $x \in \mathbb{R}$ with H_x nontrivial, and only finitely many $x \in \mathbb{R}$ with H_x uncountable.*

Proof. Let A_n be the set of $g \in G$ which can be written in the form $h_0 h_1 \cdots h_{n-1}$ with each h_i in some H_x. Then A_n is analytic, and thus Baire-measurable. Since $G = \bigcup_n A_n$, there is some A_n which is non-meager. By Pettis's lemma, we can replace n with $2n$ and assume that A_n has non-empty interior. Thus G can be covered by countably many right translates $(A_n g_k)_k$ of A_n.

Let $X \subseteq \mathbb{R}$ be the set of $x \in \mathbb{R}$ with H_x nontrivial, and suppose that X is uncountable. For each $x \in X$, fix some nontrivial $h_x \in H_x$. Fix an equivalence relation E on X with every class of cardinality $n + 1$. Then there must be two E-classes $(x_i)_{i \leq n}$ and $(y_i)_{i \leq n}$ such that $h_{x_0} h_{x_1} \cdots h_{x_n}$ and $h_{y_0} h_{y_1} \cdots h_{y_n}$ are in the same $A_n g_k$. But then

$$h_{x_0} h_{x_1} \cdots h_{x_n} (h_{y_0} h_{y_1} \cdots h_{y_n})^{-1} \in A_n g_k (A_n g_k)^{-1} = A_{2n},$$

which is a contradiction by independence. Thus X is countable.

Now $G = \bigcup_F \langle H_x \rangle_{x \in F}$, where the union is taken over all finite $F \subseteq X$, so since X is countable, there is some F for which $H_F := \langle H_x \rangle_{x \in F}$ is non-meager, and thus open, since H_F is analytic. Then G/H_F is countable, so if $x \notin F$, then H_x is countable by independence. $\square$

In particular, this implies an unpublished result of Ben Miller showing that an uncountable-dimensional Polish vector space does not have an analytic basis.

If $M \sqsubseteq^R N$ and $N \sqsubseteq^R M$, then we say that M and N are **bi-embeddable**. Note that if M and N are R-modules, and S is a subring of R, then $M \sqsubseteq^R N$ implies $M \sqsubseteq^S N$. In particular, if M and N are $\sqsubseteq^S$-incomparable, then they are $\sqsubseteq^R$-incomparable. In general, the preorder $\sqsubseteq^R$ can contain incomparable elements. For example, $\mathbb{R}$ is $\sqsubseteq^{\mathbb{Z}}$-incomparable with the p-adic rationals $\mathbb{Q}_p$, for any prime p. To see this, we have $\mathbb{R} \not\sqsubseteq^{\mathbb{Z}} \mathbb{Q}_p$ since $\mathbb{R}$ is connected, but $\mathbb{Q}_p$ is totally disconnected. On the other hand, $\mathbb{Q}_p \not\sqsubseteq^{\mathbb{Z}} \mathbb{R}$ since $\mathbb{Q}_p$ has a nontrivial compact subgroup, but $\mathbb{R}$ does not. So $\mathbb{R}$ and $\mathbb{Q}_p$ are $\sqsubseteq^{\mathbb{Z}}$-incomparable, and thus also $\sqsubseteq^{\mathbb{Q}}$-incomparable.

For certain rings, no locally compact module embeds into $R^{\mathbb{N}}$, and thus a minimum for $\sqsubseteq^R$ cannot be locally compact:

Proposition 2.2. *Let R be a Polish ring with no nontrivial compact subgroups, and let M be a locally compact Polish R-module. If $M \sqsubseteq^R R^{\mathbb{N}}$, then M is countably generated.*

Proof. Fix a continuous linear injection $f \colon M \hookrightarrow R^{\mathbb{N}}$. Since R has no nontrivial compact subgroups, the same holds for $R^{\mathbb{N}}$, and thus for M. Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. Let $\pi_n \colon R^{\mathbb{N}} \to R^n$ denote the projection to the first n coordinates, and let $M_n = \ker(\pi_n \circ f)$, which is a closed submodule of M. Fix ε such that the closed ε-ball around $0 \in M$ is compact, and let $C = \{m \in M : \frac{\varepsilon}{2} \leq \|m\| \leq \varepsilon\}$. Then $C \cap \bigcap_n M_n = \varnothing$, so since C is compact, there is some n such that $C \cap M_n = \varnothing$. We claim that M_n is discrete. To see this, suppose that the $\frac{\varepsilon}{2}$-ball around $0 \in M$ contained some nonzero $m \in M_n$. Then the subgroup generated by m is not compact, so there is a minimal $k \in \mathbb{N}$ with $\|km\| \geq \frac{\varepsilon}{2}$, and hence $km \in C$, which is not possible. Thus M_n is countable, so if we pick preimages $(m_i)_{i<n}$ in M of the standard basis of R^n, then M is generated by $M_n \cup (m_i)_{i<n}$, and thus countably generated. $\square$

We do not know anything about the preorder $\sqsubseteq^R$ restricted to locally compact modules, including the existence of a minimum or maximum element.

If M_0 and M_1 are Polish R-modules with Baire-measurable submodules N_0 and N_1 respectively, we write $M_0/N_0 \sqsubseteq^R M_1/N_1$ if there is a continuous linear map $M_0 \to M_1$ which descends to an injection $M_0/N_0 \hookrightarrow M_1/N_1$. This map is a Borel reduction of $E_{N_0}^{M_0}$ to $E_{N_1}^{M_1}$, where $E_{N_i}^{M_i}$ is the coset equivalence relation of N_i in M_i (see [Gao09] for background on Borel reductions). In particular, we have $\mathbb{R}/\mathbb{Q} \not\sqsubseteq^{\mathbb{Q}} \mathbb{R}$, since $E_{\mathbb{Q}}^{\mathbb{R}}$ is not smooth. We also have $\mathbb{R} \not\sqsubseteq^{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$, since any nontrivial continuous linear map $\mathbb{R} \to \mathbb{R}$ is surjective, and thus $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are $\sqsubseteq^{\mathbb{Q}}$-incomparable.

3. Proper normed rings

Every proper normed ring is locally compact and Polish. There are many examples of proper normed rings:

- The usual norms on $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are proper.
- The p-adic norm on $\mathbb{Q}_p$ is proper.
- Every countable ring R admits a proper norm as follows. Let $w\colon R \to \mathbb{N}$ be a finite-to-one function such that $w(0) = 0$, $w(r) \geq 2$ if $r \neq 0$, and $w(r) = w(-r)$. We extend w to every term t in the language $(+, \cdot) \cup R$ by $w(r+s) = w(r)+w(s)$ and $w(r \cdot s) = w(r)w(s)$. Then let $|r|$ be the minimum of $w(t)$ over all terms t representing r.
- Let R be a proper normed ring. If $S \leq R$ is a closed subring, then there is a proper norm on S obtained by restricting the norm on R. If $I \lhd R$ is a closed two-sided ideal, then there is a proper norm on R/I given by $|r + I| = \min_{s \in r+I} |s|$.

In general, we do not know if every locally compact Polish ring admits a compatible proper norm.

Given a closed two-sided ideal $I \lhd R$, there is a natural quotient map $\ell^1(R) \twoheadrightarrow \ell^1(R/I)$ with kernel $\ell^1(I) := \ell^1(R) \cap I^{\mathbb{N}}$.

If R is finite proper normed ring, then $\ell^1(R) = R^{\mathbb{N}}$, which in particular is homeomorphic to Cantor space. For infinite discrete rings, there is also a unique homeomorphism type. Recall that **complete Erdős space** is the space of square-summable sequences of irrational numbers with the ℓ^2-norm topology.

Proposition 3.1. *Let R be an infinite discrete proper normed ring. Then $\ell^1(R)$ is homeomorphic to complete Erdős space.*

To show this, we will use a characterization due to Dijkstra and van Mill [DvM09, Theorem 1.1]. A topological space is **zero-dimensional** if it is nonempty and it has a basis of clopen sets.

Theorem 3.2 (Dijkstra-van Mill). *Let X be a separable metrizable space. Then X is homeomorphic to complete Erdős space iff there is a zero-dimensional metrizable topology τ on X coarser than the original topology such that every point in X has a neighbourhood basis (for the original topology) consisting of closed nowhere dense Polish subspaces of (X, τ).*

Proof of Proposition 3.1. We check the condition from Theorem 3.2. Let τ be the product topology on $R^{\mathbb{N}}$, which is zero-dimensional and metrizable. It is enough to show that every closed ball is a closed nowhere dense Polish subspace of $(\ell^1(R), \tau)$. By translation, it suffices to consider balls of the form $B = \{m \in \ell^1(R) : \|m\| \leq \varepsilon\}$. Note that B is closed in $R^{\mathbb{N}}$. Thus (B, τ) is Polish, and B is closed in $(\ell^1(R), \tau)$.

It remains to show that the complement of B is dense in $(\ell^1(R), \tau)$. Let U be a nonempty open subset of $(\ell^1(R), \tau)$. We can assume that there is a finite sequence $(r_k)_{k<n}$ in R such that U is the set of sequences in $\ell^1(R)$ starting with $(r_k)_{k<n}$. Since R is infinite and the norm is proper, there is some $r \in R$ with $|r| > n!\varepsilon$. Then $(r_0, \ldots, r_{n-1}, r, 0, 0, 0, \ldots) \in U \setminus B$. $\qquad\square$

4. Special cases

For a general Polish ring R, we do not know much about the preorder $\sqsubseteq^R$, including the following:

Problem 4.1. Is there a maximum Polish R-module under $\sqsubseteq^R$?

This is known for some particular rings, which we mention below.

4.1. **Principal ideal domains.** Recall that a **principal ideal domain (PID)** is an integral domain in which every ideal is generated by a single element. There is an irreducible basis for uncountable Polish modules over a PID:

Theorem 4.2. *Let R be a proper normed discrete PID and let M be a Polish R-module. Then exactly one of the following holds:*

(1) M is countable.
(2) There a prime ideal $\mathfrak{p} \lhd R$ such that $\ell^1(R/\mathfrak{p}) \sqsubseteq^R M$.

Moreover, the $\ell^1(R/\mathfrak{p})$ are $\sqsubseteq^R$-incomparable for different $\mathfrak{p}$.

Proof. Suppose that M is not countable. By Theorem 1.2, there is some proper ideal $I \lhd R$ such that $\ell^1(R/I) \sqsubseteq^R M$. Then since R is a PID, there is some prime ideal $\mathfrak{p} \lhd R$ and some nonzero $s \in R$ such that $I = \mathfrak{p}s$. Then the linear injection $R/\mathfrak{p} \hookrightarrow R/I$ defined by $r \mapsto rs$ induces a continuous linear injection $\ell^1(R/\mathfrak{p}) \hookrightarrow \ell^1(R/I)$.

It remains to show that if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals with $\ell^1(R/\mathfrak{p}) \sqsubseteq^R \ell^1(R/\mathfrak{q})$, then $\mathfrak{p} = \mathfrak{q}$. Fix a continuous linear injection $\ell^1(R/\mathfrak{p}) \hookrightarrow \ell^1(R/\mathfrak{q})$. Since $R/\mathfrak{p}$ is an integral domain, the annihilator of any nonzero element of $\ell^1(R/\mathfrak{p})$ is $\mathfrak{p}$, and similarly for $\mathfrak{q}$. Then for any nonzero $x \in \ell^1(R/\mathfrak{p})$, its image in $\ell^1(R/\mathfrak{q})$ must have the same annihilator since the map is injective, and thus $\mathfrak{p} = \mathfrak{q}$. $\qquad\square$

4.2. **Abelian groups.** Applying Theorem 4.2 with $R = \mathbb{Z}$ gives an irreducible basis for uncountable abelian groups:

Theorem 4.3. *Let A be an uncountable abelian Polish group. Then one of the following holds:*

(1) $\ell^1(\mathbb{Z}) \sqsubseteq^{\mathbb{Z}} A$.
(2) $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \sqsubseteq^{\mathbb{Z}} A$ for some prime p.

By [Shk99], there is a $\sqsubseteq^{\mathbb{Z}}$-maximum abelian Polish group $A_{\max}$. So the preorder $\sqsubseteq^{\mathbb{Z}}$ on uncountable abelian Polish groups looks like the following:
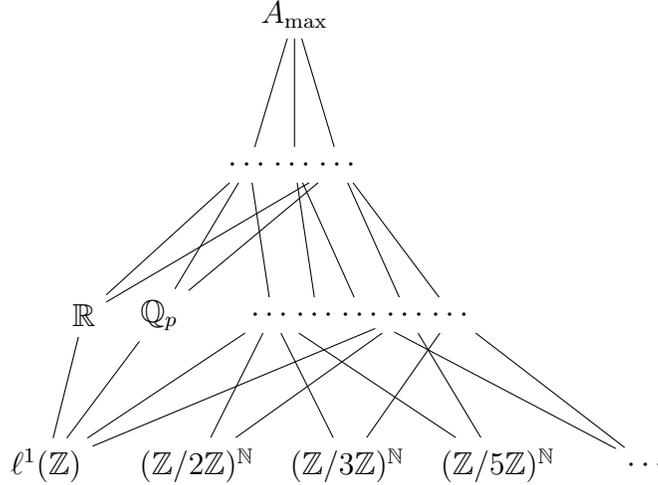
4.3. $\mathbb{Q}$-vector spaces.

Fix a proper norm on $\mathbb{Q}$. By Proposition 2.2, a $\sqsubseteq^{\mathbb{Q}}$-minimum uncountable Polish $\mathbb{Q}$-vector space cannot be locally compact. By Theorem 1.2, we have $\ell^1(\mathbb{Q}) \sqsubset^{\mathbb{Q}} \mathbb{R}$, where the strictness is due to $\ell^1(\mathbb{Q})$ being totally disconnected. However, it is open as to whether there is an intermediate vector space:

Problem 4.4. Is there a Polish $\mathbb{Q}$-vector space V such that $\ell^1(\mathbb{Q}) \sqsubset^{\mathbb{Q}} V \sqsubset^{\mathbb{Q}} \mathbb{R}$?

4.4. Real vector spaces.

We consider the order $\sqsubseteq^{\mathbb{R}}$ on uncountable-dimensional Polish $\mathbb{R}$-vector spaces. By Theorem 1.1, there is a minimum element $\ell^1(\mathbb{R})$, which is bi-embeddable with the usual space ℓ^1 of absolutely summable sequences. By Proposition 2.2, any uncountable-dimensional locally compact Polish $\mathbb{R}$-vector space must be strictly above ℓ^1. By [Kal77], there is a maximum Polish $\mathbb{R}$-vector space $V_{\max}$.

5. Proof of the main theorems

Every abelian Polish group A has a compatible complete norm defined by $\|a\| = d(a,0)$, where d is an invariant metric on A (see [BK96, 1.1.1, 1.2.2]). If $B \subseteq A$ is a Baire-measurable subgroup, then by Pettis's lemma, B is either open or meager (see [Kec95, 9.11]).

Setting $N = 0$ in the following theorem recovers Theorem 1.1.

Theorem 5.1. *Let R be a proper normed division ring, let M be a Polish R-vector space, and let $N \subseteq M$ be an F_σ vector subspace. Then exactly one of the following holds:*

(1) $\dim_R(M/N)$ *is countable.*
(2) $\ell^1(R) \sqsubseteq^R M/N.$

In most natural examples, N is F_σ, such as for $\ell^1(\mathbb{N}) \subseteq \ell^2(\mathbb{N})$. It would be interesting to prove this for more general subspaces.

Proof. Suppose that the dimension of M/N is uncountable. Then N is not open, so N is meager, i.e. we have $N = \bigcup_k F_k$ for some increasing sequence $(F_k)_k$ of closed nowhere dense sets. Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. For every k, we define $\varepsilon_k > 0$ and $m_k \in M$ such that the image of $(m_k)_k$ in M/N is linearly independent over R. We proceed by induction on k. Choose $\varepsilon_k > 0$ such that

  (i) $\varepsilon_k < \frac{1}{2}\varepsilon_i$ for every $i < k$,
  (ii) for every $(r_i)_{i<k}$ such that $\sum_{i<k} \frac{|r_i|}{i!} \le k$ and there is some $l < k$ with $r_l = 1$ and $r_i = 0$ for $i < l$, the open ε_k-ball centered at $\sum_{i<k} r_i m_i$ is disjoint from F_k.

Then choose $m_k \in M$ such that

  (i) $m_k \notin N + Rm_0 + Rm_1 + \cdots + Rm_{k-1}$,
  (ii) $\|rm_k\| < \frac{1}{2}\varepsilon_k$ whenever $\frac{|r|}{k!} \le k$.

We verify that this is possible. When choosing ε_k, to satisfy the second condition, note that the set of considered $(r_i)_{i<k}$ is compact, so the set of $\sum_{i<k} r_i m_i$ is also compact, and it is disjoint from N (and hence F_k) by the choice of $(m_i)_{i<k}$. Thus such an ε_k must exist. When choosing m_k, note that the first condition holds for a comeager set of m_k, since $N + Rm_0 + Rm_1 \cdots + Rm_{k-1}$ is analytic, and it is not open, since otherwise M/N would have countable dimension. The second condition holds for an open set of m_k, since the set of r with $\frac{|r|}{k!} \le k$ is compact. Thus such an m_k must exist.

We define a map $\ell^1(R) \hookrightarrow M$ by

$$(r_k)_k \mapsto \sum_k r_k m_k.$$

First we show that this is well-defined, from which linearity and continuity are immediate. Let $(r_k)_k \in \ell^1(R)$ be nonzero. By scaling, we can assume that there is some l such that $r_l = 1$ and $r_i = 0$ for $i < l$. Let $n > l$ be sufficiently large such that $\sum_k \frac{|r_k|}{k!} \le n$ and $0 \in F_n$. Then

$$\varepsilon_n \le \left\| \sum_{k<n} r_k m_k \right\|.$$

For every i, we have $\|r_{n+i}m_{n+i}\| < \frac{1}{2}\varepsilon_{n+i}$, and thus $\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$ by inductively using $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$. Thus

$$\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}\left\|\sum_{k<n}r_k m_k\right\|.$$

Thus $\sum_k r_k m_k$ is well-defined with

$$\left\|\sum_k r_k m_k\right\| < 2\left\|\sum_{k<n}r_k m_k\right\|.$$

It remains to show that the induced map $\ell^1(R) \to M/N$ is an injection. Let $(r_k)_k \in \ell^1(R)$ be nonzero. By scaling, we can assume that there is some l such that $r_l = 1$ and $r_i = 0$ for $i < l$. Suppose that $n > l$ is sufficiently large such that $\sum_k \frac{|r_k|}{k!} \le n$. Since $\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$, we have $\sum_{i\ge 0}\|r_{n+i}m_{n+i}\| < \varepsilon_n$, and so $\sum_k r_k m_k \notin F_n$. This holds for all sufficiently large n, so $\sum_k r_k m_k \notin N$. □

We recover [Mil12, Theorem 24] for proper normed division rings:

Corollary 5.2 (Miller). *Let R be a proper normed division ring, and let M be a Polish R-module. If $\dim_R(M)$ is uncountable, then there is a linearly independent perfect subset of M.*

Proof. By Theorem 1.1, we can assume that $M = \ell^1(R)$. Fix an enumeration $(q_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$. For every $x \in \mathbb{R}$, define $\chi_x \in \ell^1(R)$ by

$$(\chi_x)_n = \begin{cases} 1 & q_n < x \\ 0 & \text{otherwise} \end{cases}$$

Then $(\chi_x)_{x\in\mathbb{R}}$ is an uncountable linearly independent Borel subset of $\ell^1(R)$, so we are done by taking any perfect subset of this. □

There is an analogous generalization of Theorem 1.2.

Theorem 5.3. *Let R be a left-Noetherian discrete proper normed ring, let M be a Polish R-module, and let $N \subseteq M$ be an F_σ submodule. Then exactly one of the following holds:*

(1) *M/N is countable.*
(2) *$\ell^1(R)/\ell^1(I) \sqsubseteq^R M/N$ for some proper[2] two-sided ideal $I \triangleleft R$. In particular, there is a linear injection $\ell^1(R/I) \hookrightarrow M/N$.*

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[2]By proper, we mean a proper subset (no relation to proper norms).

Proof. Suppose that M/N is not countable. Then N is not open, and thus meager. Let $(U_k)_k$ be a descending neighborhood basis of $0 \in M$, and let $I_k = \{r \in R : rU_k \subseteq N\}$. Then $(I_k)_k$ is an increasing sequence of ideals, so since R is left-Noetherian, this sequence stabilizes at some $I = I_n$. Note that I is a proper ideal, since otherwise $U_n \subseteq N$, a contradiction to N being meager. Note also that I is a two-sided ideal, since if $r \in R$, then there is some $k > n$ with $rU_k \subseteq U_n$, and thus $IrU_k \subseteq IU_n \subseteq N$, and thus $Ir \subseteq I$. By replacing M with the submodule generated by U_n (which is analytic non-meager, and therefore open), we can assume that for every nonempty open $V \subseteq M$, we have $\{r \in R : rV \subseteq N\} = I$. Then for every $r \notin I$, the subgroup $\{m \in M : rm \in N\}$ is not open, and therefore meager. Thus more generally, if $m' \in M$, then $\{m \in M : rm \in N + m'\}$ is meager.

Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. Let $(F_k)_k$ be an increasing sequence of closed nowhere dense sets with $N = \bigcup_k F_k$. For every k, we define $\varepsilon_k > 0$ and $m_k \in M$ such that the image of $(m_k)_k$ in M/N is linearly independent over R/I. We proceed by induction on k. Choose $\varepsilon_k > 0$ such that

(i) $\varepsilon_k < \frac{1}{2}\varepsilon_i$ for every $i < k$,
(ii) for every $(r_i)_{i<k}$ with $\sum_{i<k} r_i m_i$ nonzero and $\sum_{i<k} \frac{|r_i|}{i!} \leq k$, we have $\varepsilon_k \leq \left\|\sum_{i<k} r_i m_i\right\|$,
(iii) for every $(r_i)_{i<k}$ with $\sum_{i<k} r_i m_i \notin N$ and $\sum_{i<k} \frac{|r_i|}{i!} \leq k$, the open ε_k-ball centered at $\sum_{i<k} r_i m_i$ is disjoint from F_k.

Then choose $m_k \in M$ such that

(i) $rm_k \notin N + Rm_0 + Rm_1 + \cdots + Rm_{k-1}$ for every $r \notin I$,
(ii) $\|rm_k\| < \frac{1}{2}\varepsilon_k$ whenever $\frac{|r|}{k!} \leq k$.

We verify that this is possible. When choosing ε_k, for the second and third condition, there is only a finite set of $\sum_{i<k} r_i m_i$ to consider, and for the third condition, this set is disjoint from N, and hence from F_k. Thus such an ε_k must exist. When choosing m_k, for the first condition, for a fixed $r \notin I$ and $m' \in Rm_0 + \cdots + Rm_{k-1}$, we have shown earlier that $\{rm \notin N + m'\}$ is comeager, so by quantifying over the countably many r and m', the set of m_k satisfying the first condition is comeager. The second condition holds for an open set of m_k, since the set of r with $\frac{|r|}{k!} \leq k$ is finite. Thus such an m_k must exist.

We define a map $\ell^1(R) \hookrightarrow M$ by

$$(r_k)_k \mapsto \sum_k r_k m_k.$$

First we show that this is well-defined, from which linearity and continuity are immediate. Let $(r_k)_k \in \ell^1(R)$. We can assume that there is some n such that

$\sum_{k<n} r_k m_k$ is nonzero and $\sum_{k<n} \frac{|r_k|}{k!} \le n$. Then

$$\varepsilon_n \le \left\| \sum_{k<n} r_k m_k \right\|.$$

For every i, we have $\|r_{n+i} m_{n+i}\| < \frac{1}{2}\varepsilon_{n+i}$, and thus $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$ by inductively using $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$. Thus

$$\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}} \left\| \sum_{k<n} r_k m_k \right\|.$$

Thus $\sum_k r_k m_k$ is well-defined with

$$\left\| \sum_k r_k m_k \right\| < 2 \left\| \sum_{k<n} r_k m_k \right\|.$$

It remains to show that the kernel of the induced map $\ell^1(R) \to M/N$ is $\ell^1(I)$. The kernel clearly contains $\ell^1(I)$, since $IM \subseteq N$. Now let $(r_k)_k \in \ell^1(R) \setminus \ell^1(I)$. Since the image of $(r_k)_k$ in M/N is linearly independent over R/I, if n is sufficiently large, then $\sum_{k<n} r_k m_k \notin N$ and $\sum_k \frac{|r_k|}{k!} \le n$. Since $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$, we have $\sum_{i\ge 0} \|r_{n+i} m_{n+i}\| < \varepsilon_n$, and so $\sum_k r_k m_k \notin F_n$. This holds for all sufficiently large n, so $\sum_k r_k m_k \notin N$. $\qquad\square$

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