LUSIN-NOVIKOV VIA σ-IDEALS

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Abstract. We present an elementary proof of the Lusin-Novikov theorem which in particular does not mention analytic sets.

> At the end of the day, we are all trying to find a Cantor set.
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> Անուշ Ծերունյան

A **partial section** of a function $f : X \to Y$ is a subset $A \subseteq X$ such that $f{\restriction}A$ is injective.

Lusin-Novikov theorem. *Let $f : X \to Y$ be a continuous map of Polish spaces. Then exactly one of the following holds:*

(1) X can be covered by countably many Borel partial sections of f.
(2) A fiber of f contains a Cantor set.

Let $\mathcal{I}$ be the σ-ideal on X generated by the Borel partial sections of f.

Let $\mathbb{P}$ be the collection of finite disjoint families of closed subsets of X. We say that $\mathcal{F} \in \mathbb{P}$ is **null** if there is a Borel cover $(B_F)_{F\in\mathcal{F}}$ of Y such that each $F \cap f^{-1}(B_F)$ is in $\mathcal{I}$. Note that a singleton family $\{F\}$ is null iff $F \in \mathcal{I}$.

Lemma 1. *Let $\mathcal{F} \sqcup \{U\} \in \mathbb{P}$, and suppose that $(U)^2 = \bigcup_{n\in\mathbb{N}} V_n \times W_n$ with each $\mathcal{F} \sqcup \{V_n, W_n\}$ null. Then $\mathcal{F} \sqcup \{U\}$ is null.*

Recall that $(A)^2 := \{(x, x') \in A^2 : x \neq x'\}$.

Proof. For every $n \in \mathbb{N}$, fix a Borel cover $(B_F)_{F\in\mathcal{F}} \cup \{B_{V_n}, B_{W_n}\}$ of Y witnessing that $\mathcal{F} \sqcup \{V_n, W_n\}$ is null; we can use the same B_F for every n, since if each n had its own B^n_F, we could replace them with $B_F := \bigcup_n B^n_F$. By passing to subsets, we can assume that each cover is a partition. Set $B_U = Y \setminus \bigsqcup_{F\in\mathcal{F}} B_F$. Note that $B_U = B_{V_n} \sqcup B_{W_n}$ for every n. It remains to show that $U \cap f^{-1}(B_U) \in \mathcal{I}$, so it suffices to show that

$$U \cap f^{-1}(B_U) \setminus \left(\bigcup_n \big(V_n \cap f^{-1}(B_{V_n})\big) \sqcup \big(W_n \cap f^{-1}(B_{W_n})\big) \right) \tag{*}$$

is a partial section. Let $x_V, x_W \in U \cap f^{-1}(B_U)$ be distinct. Then there is some n for which $x_V \in V_n$ and $x_W \in W_n$. If $f(x_V) = f(x_W)$, then since this is in B_U, we have either $f(x_V) \in B_{V_n}$ or $f(x_W) \in B_{W_n}$, so either $x_V \in V_n \cap f^{-1}(B_{V_n})$ or $x_W \in W_n \cap f^{-1}(B_{V_n})$, and thus at most one of x_V and x_W can be in the set (*). So it is a partial section. □

Lemma 2. *Let $\mathcal{F} \in \mathbb{P}$, and suppose that there is a Borel cover $(A_n)_{n\in\mathbb{N}}$ of Y such that for every $n \in \mathbb{N}$, the family $(F \cap f^{-1}(A_n))_{F\in\mathcal{F}}$ is null. Then $\mathcal{F}$ is null.*

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Proof. For each $n \in \mathbb{N}$, fix a Borel cover $(B_F^n)_{F\in\mathcal{F}}$ of Y witnessing that $(F \cap f^{-1}(A_n))_{F\in\mathcal{F}}$ is null. For each $F \in \mathcal{F}$, let $B_F = \bigcup_n (A_n \cap B_F^n)$. Then the Borel cover $(B_F)_{F\in\mathcal{F}}$ of Y witnesses that $\mathcal{F}$ is null. □

We proceed with the proof of Lusin-Novikov. Fix compatible Polish metrics on X and Y. We make the following observations.

(1) Given $\varepsilon > 0$ and a closed $U \subseteq X$, we can write $(U)^2 = \bigcup_n V_n \times W_n$ for some closed sets V_n and W_n of diameter $\leq \varepsilon$.
(2) Given $\varepsilon > 0$, we can write Y as a countable union of closed sets of diameter $\leq \varepsilon$.

Now suppose that the first condition of Lusin-Novikov fails, i.e. that $\{X\}$ is non-null. We will construct non-empty closed subsets $(F_s)_{s\in 2^{<\omega}}$ of X such that the following conditions hold:

(i) If $s \succ t$, then $F_s \subseteq F_t$.
(ii) If $s \perp t$, then F_s and F_t are disjoint.
(iii) $\operatorname{diam}(F_s) \leq 2^{-|s|}$ for every nonempty $s \in 2^{<\omega}$.
(iv) $\operatorname{diam}\left(f\left(\bigcup_{|s|=n} F_s\right)\right) \leq 1/n$ for every $n > 0$.

To ensure that the recursive construction works, we will require that for each $n \in \mathbb{N}$, the family $(F_s)_{|s|=n}$ is non-null.

To start, set $F_\varnothing = X$.

Suppose we have constructed the non-null family $(F_s)_{|s|=n}$. By applying Lemma 1 and the first observation, we obtain closed disjoint subsets $G_{0^n\frown 0}$ and $G_{0^n\frown 1}$ of F_{0^n} of diameter $\leq 2^{-(n+1)}$, such that the family $(F_s)_{|s|=n \text{ and } s\neq 0^n} \cup \{G_{0^n\frown 0}, G_{0^n\frown 1}\}$ is non-null. By applying Lemma 1 and the first observation $2^n - 1$ more times (once for each F_s), we obtain a non-null family $(G_s)_{|s|=n+1}$ satisfying the first three conditions. Then by applying Lemma 2 and the second observation, we obtain a non-null family $(F_s)_{|s|=n+1}$ satisfying all four conditions. Note that the sets are nonempty since the family is non-null.

This concludes the construction.

Now we are done: the decreasing intersection $\bigcap_n \bigcup_{|s|=n} F_s$ is a Cantor set, and it lies in a single fiber, since its image has diameter 0:

$$\begin{aligned}
\operatorname{diam}\left(f\left(\bigcap_n \bigcup_{|s|=n} F_s\right)\right) &= \operatorname{diam}\left(\bigcap_n f\left(\bigcup_{|s|=n} F_s\right)\right) \\
&\leq \inf_n \operatorname{diam}\left(f\left(\bigcup_{|s|=n} F_s\right)\right) \\
&\leq \inf_n 1/n \\
&= 0
\end{aligned}$$