Math 55: Discrete Mathematics
Solutions for the Final Exam

UC Berkeley, Spring 2009

1. (a) There are **2** non-isomorphic unrooted trees with 4 vertices: the 4-chain and the tree with one trivalent vertex and three pendant vertices.
   (b) There are **4** non-isomorphic rooted trees with 4 vertices, since we can pick a root in two distinct ways from each of the two trees in (a).
   [# 12 in §10.1, page 694]

2. (a) There are $\binom{5}{3} = 10$ ways to put three balls into one box and one ball each into the other two boxes, and there are $15 = 5 \cdot 3$ ways to put one ball into one box and two balls each into the other two boxes. Hence the total number is $10 + 15 = \mathbf{25}$.
   (b) We must place one ball into each of the three boxes. Afterwards, we are left with two unlabeled balls to be placed into three labeled boxes. The number of ways to do this is the number of 2-combinations from a set with 3 elements, so the answer is $\binom{2+3-1}{2} = \binom{4}{2} = \mathbf{6}$.
   (c) There are **2** ways to do this: three balls into one box and one each into the others, or one ball into one box and two each into the others.

3. The three strings of length at most one, namely, `0`, `1` and the empty string, are palindromes. If x is a palindrome then $\mathtt{0}x\mathtt{0}$ and $\mathtt{1}x\mathtt{1}$ are palindromes. These are all the palindromes. *[# 38 in §4.3, page 309]*

4. This poset is isomorphic to the one shown in Figure 8 in §8.6 on page 575, with $a = 1, b = 2, c = 3, d = 12, e = 18, f = 36$. As argued, it is **not a lattice** because $b = 2$ and $c = 3$ have no least upper bound.

5. Each such path can be described by a bit string of m zeros and n ones, where a zero represents a move to the right and a one represents an upward move. Hence the number of paths equals the number of bit

strings of length $m+n$ having m zeros. That number is the binomial coefficient $\binom{\mathbf{m+n}}{\mathbf{m}}$. *[# 33 in §5.4, page 369]*

6. (a) This is derived in Table 2 in §7.6 on page 512:

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \quad = \quad \frac{\mathbf{53}}{\mathbf{144}} \quad = \quad 0.368055.$$

(b) The expected number of hats that are returned correctly equals **1**, regardless of how many people check their hats at the opera. This is derived from linearity of expectation in Example 6 of §6.4 on page 430.

7. From the identifies $(-38)\cdot 2 + 1\cdot 77 = 1$, $(-3)7 + 1\cdot 22 = 1$ and $(-5)\cdot 11 + 4\cdot 14 = 1$, we see that $77 \equiv 1 \pmod 2$, $22 \equiv 1 \pmod 7$, and 4 is the inverse to 14 modulo 11. By the method used in the proof of the Chinese Remainder Theorem, $x = 0\cdot 77 + 3\cdot 22 + 6\cdot 4\cdot 14 = \mathbf{402}$ is one solution, and all solutions have the form $x = 402 + 154n$ for some integer n. Taking $n = -2$ gives the smallest positive solution $x = \mathbf{94}$.

8. This statement is **false**. We can **disprove** it by giving a counterexample. The following four graphs all have vertex set $V = \{1,2,3,4\}$. Take $G_1 = H_1 = (V, \{\{1,2\}\})$, $G_2 = (V, \{\{2,3\}))$, and $H_2 = (V, \{\{3,4\}\})$. All four graphs are isomorphic. However, $G_1 \cup G_2 = (V, \{\{1,2\},\{2,3\}\})$ is connected while $H_1 \cup H_2 = (V, \{\{1,2\},\{3,4\}\})$ is disconnected, so they are not isomorphic. *[# 20 in §9.Supp on page 679]*

9. We apply Bayes' Theorem. Let E be the event that a **6** is observed, and let D_i be the event that the ith dice is chosen, for $i = 1,2,3$. Each die is equally likely, $p(D_1) = p(D_2) = p(D_3) = 1/3$. We also know that $p(E|D_1) = p(E|D_2) = 1/6$ while $p(E|D_3) = 1/2$. Bayes' Theorem says

$$p(D_3|E) \quad = \quad \frac{p(E|D_3)p(D_3)}{p(E|D_1)p(D_1) + p(E|D_2)p(D_2) + p(E|D_3)p(D_3)}$$

So, the answer is $p(E|D_1) = \frac{(1/2)(1/3)}{(1/6)(1/3)+(1/6)(1/3)+(1/2)(1/3)} = \mathbf{3/5}$.

10. For an n-element set this appears in # 45 of §8.1 on page 529. We get
(a) $2^{\binom{4}{2}} = 2^6 = \mathbf{64}$
(b) $2^{\binom{3}{2}} = 2^3 = \mathbf{8}$
(c) $2^9 - 2\cdot 2^{3\cdot 2} = 512 - 128 = \mathbf{384}$.