Math 275: Introduction to Non-Linear Algebra

Bernd Sturmfels, UC Berkeley, Spring 2014
Homework # 1, due Monday, January 27

1. Prove the *Shape Lemma*, which describes the structure of the lexicographic Gröbner bases of a zero-dimensional radical ideal.

2. Let $n = 4$ and consider the ideal I generated by the power sums
$$x_1^i + x_2^i + x_3^i + x_4^i \quad \text{for } i = 1, 2, 3, 4.$$
Pick a term order and compute the reduced Gröbner basis of I. What are the standard monomials? What is the variety $V(I)$? Describe the companion matrices T_1, T_2, T_3, T_4 and show that they are nilpotent.

3. Prove *Dickson's Lemma*: Every infinite set of monomials in n variables contains two monomials such that one monomial divides the other.

4. *[CBMS 2.1]* Let $A = (a_{ij})$ be a non-singular $n \times n$-matrix whose entries are positive integers, and consider the system of equations
$$\prod_{j=1}^{n} x_j^{a_{1j}} \;=\; \prod_{j=1}^{n} x_j^{a_{2j}} \;=\; \cdots \;=\; \prod_{j=1}^{n} x_j^{a_{nj}} \;=\; 1.$$
How many complex solutions do these equations have?

5. Pick three random cubic polynomials f, g, h in $\mathbb{Q}[x, y]$, let $I = \langle f, g \rangle$, pick a term order of your choice, and compute the symmetric matrix that represents the trace form B_h. What is the signature of your B_h?

6. *[CBMS 2.7]* How many distinct vectors $(x, y, z) \in \mathbb{R}^3$ satisfy the system
$$x^3 + z = 2y^2, \;\; y^3 + x = 2z^2, \;\; z^3 + y = 2x^2\,?$$

7. *[CBMS 2.10]* Consider a polynomial system which has infinitely many complex zeros but only finitely many of them have all their coordinates distinct. How would you compute those zeros with distinct coordinates?