UC Berkeley, Spring 2017

Math 170: Optimization, Midterm Exam

Prof. Sturmfels, March 2

Name: ______________________________

SID: ______________________________

Grading

1	/ 10
2	/ 10
3	/ 10
4	/ 10
5	/ 10
6	/ 10
	/ 60

Instructions:

- Open book: You may use the textbook and your own notes.
- The exam time is 80 minutes. Do all 6 problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- Answers in complete sentences are encouraged.
- If you need more space, use reverse side or scratch pages.
  Indicate clearly where to find your answers.

1. (10 points) A simplex tableau with $m = 6$ rows and $n = 8$ columns represents a basic feasible solution to a linear programming problem (of primal form, as in Chapter 3). In particular, the tableau contains an identity matrix of format 6×6.

(a) What dimensions are possible for the polyhedron of all feasible solutions?

(b) Starting at that tableau, is it possible for the simplex algorithm to cycle?

Solution:

(a) The polyhedron of feasible solutions is non-empty, and it lies in a $2-$dimensional affine space, namely the space defined by 6 independent equations in 8 unknowns. Hence its dimension is either **0**, **1**, or **2**.

(b) Cycling is not possible when $n - m \leq 2$. We showed this in Exercise 3.10.

2. (10 points) For each integer $d \in \{0, 1, 2, 3\}$, give an example of a 3-dimensional polyhedron in $\mathbb{R}^3$ whose recession cone has dimension d. Drawing pictures is fine.

Solution:

If a polyhedron is a cone then it equals its recession cone, so it suffices to find polyhedral cones in $\mathbb{R}^3$ whose dimensions d are $d = 0, 1, 2, 3$.

The following four cones have dimensions $3, 2, 1, 0$ as desired:

$\{(x, y, z) \in \mathbb{R}^3 \,:\, x \geq 0,\ y \geq 0,\ z \geq 0\}$

$\{(x, y, z) \in \mathbb{R}^3 \,:\, x \geq 0,\ y \geq 0,\ z = 0\}$

$\{(x, y, z) \in \mathbb{R}^3 \,:\, x \geq 0,\ y = 0,\ z = 0\}$

$\{(x, y, z) \in \mathbb{R}^3 \,:\, x = 0,\ y = 0,\ z = 0\}$

3. (10 points) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$, $c^T = \begin{pmatrix} 3\,, 2\,, 1\,, 2\,, 3 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Consider the corresponding linear programming problem in standard form:

$$\text{minimize } \; c^T x \; \text{ subject to } \; Ax = b, \; x \geq 0,$$

(a) Determine an optimal solution and the optimal cost.

(b) Is the optimal solution unique?

(c) How many basic solutions are there?

(d) How many basic feasible solutions are there?

(a) One optimal solution is $x^* = (0, 0, 1, 1, 0)^T$ and the optimal cost is $c^T x^* = 3$.

(b) No, it is not unique. Another optimal solution is $x^{**} = (0, 0, 3/2, 0, 1/2)$.

(c) Any two columns of A are linearly independent, giving $\binom{5}{2} = 10$ basic solutions.

(d) The feasible bases are obtained by combing any of the first three columns with either of the last two columns. These are the pairs of columns that have b in their positive span. Hence there are 6 basic feasible solutions.

4. (10 points) Consider the linear program in Problem 3, but with the right hand side $b = \binom{2}{7}$ replaced by an arbitrary vector $b = \binom{b_1}{b_2}$ in $\mathbb{R}^2$.

(a) Write down the dual program to the linear program.

(b) Draw the set of dual feasible solutions, and indicate which ones are basic.

(c) Determine the set of all of $b \in \mathbb{R}^2$ such that the dual program is bounded.

Solution:

(a) The dual program equals

$$\text{Maximize } b_1 p_1 + b_2 p_2 \text{ subject to}$$

$$p_1 + p_2 \leq 3,\ p_1 + 2p_2 \leq 2,\ p_1 + 3p_2 \leq 1,\ p_1 + 4p_2 \leq 2,\ p_1 + 5p_2 \leq 3.$$

(b)

(c) The dual program is bounded if and only if the primal primal is feasible if and only if b is in the cone spanned by the columns of A This happens if and only if

$$b_1 \ \leq \ b_2 \ \leq \ 5b_1.$$

5. (10 points) Consider the following optimization problem involving absolute values:

$$\begin{array}{ll} \text{minimize} & 2a + 3|b - 10] \\ \text{subject to} & |a+2| + |b| \leq 5. \end{array}$$

Reformulate it as a linear problem.

Solution:

This is Exercise 1.4 on page 34 in the textbook.

The given constraints can be formulated as follows:

$$a + b + 2 \leq 5 \text{ and } a - b + 2 \leq 5 \text{ and } -a + b - 2 \leq 5 \text{ and } -a - b - 2 \leq 5. \qquad (1)$$

We introduce new variable c to represent the optimal cost. With this, the given optimization problem is equivalent to the linear programming problem

$$\text{minimize } c \text{ subject to } \quad (1) \quad \text{and} \quad 2a + 3b - 30 \leq c \;\text{ and }\; 2a - 3b + 30 \leq c.$$

6. (10 points) Give an example of a pair (primal and dual) of linear programming problems, both of which have multiple optimal solutions.

Solution:

This is Exercise 4.16, and it was assigned as a homework problem.

The following example was presented in class on Tuesday, February 21:

$$m = 2,\, n = 3,\; A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix},\; b = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\; c = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

(Scratch Paper, Page 1)

(Scratch Paper, Page 2)

(Scratch Paper, Page 3)