Term Paper
Almost-toric Projective Hypersurfaces

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May 7, 2014

Abstract

Almost-toric projective hypersurfaces could be parameterized by $n+2$ monomials in n variables multiplying univariate polynomials in an extra variable. In this paper we give a combinatorial description of the Newton polytope (actually it's a polygon) of the defining polynomial of almost-toric hypersurfaces.

1 Introduction

1.1 Toric varieties

Toric varieties form an important and rich class of examples in algebraic geometry, which often provide a testing ground for theorems. The geometry of a toric variety is fully determined by the combinatorics of its associated fan, which often makes computations far more tractable. We adopt the definition and notations in [2].

In this paper we assume that K is an algebraically closed field. We treat almost-toric hypersurfaces, so we first consider a toric variety with codimension 2 as follows:

Let A be a $n \times (n+2)$ full-rank matrix with integer entries. Then the column vectors of A admit a parameterization of an affine toric variety as follows. Suppose

$$A = \begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_{n+1} \end{bmatrix}.$$

We have a map from any n-tuple $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ in $(K^*)^n$ to Laurent monomials:

$$\Phi_A : \mathbf{t} \to (\mathbf{t}^{\mathbf{a_0}}, \mathbf{t}^{\mathbf{a_1}}, \cdots, \mathbf{t}^{\mathbf{a_{n+1}}}),$$

where $\mathbf{t}^{\mathbf{a_i}} = t_1^{a_{1,i}} t_2^{a_{2,i}} \cdots t_n^{a_{n,i}}$ for $i = 0, 1, \cdots, n+1$.

Then

$$\hat{X} = \mathrm{cl}(\mathrm{Im}\Phi_A) = \mathrm{cl}(\{(\mathbf{t}^{\mathbf{a_0}}, \mathbf{t}^{\mathbf{a_1}}, \cdots, \mathbf{t}^{\mathbf{a_{n+1}}}) | \mathbf{t} \in (K^*)^{n+2}\})$$

is an affine toric variety.

If all columns of A have the same sum of their components, then those Laurent monomials are homogeneous, then

$$X = \mathrm{cl}(\{(\mathbf{t}^{\mathbf{a_0}} : \mathbf{t}^{\mathbf{a_1}} : \cdots : \mathbf{t}^{\mathbf{a_{n+1}}}) \in (\mathbb{P}^*)^{n+1} | \mathbf{t} \in (K^*)^n\})$$

is a projective toric variety. In this parameterization, the properties, for example, their degrees, have been well studied.(e.g. see [6][Chapter 13])

1.2 Almost Toric Hypersurfaces

We can generalize the parameterization by adding one variable x, which means the projective coordinates are the product of monomials in n variables and a univariate polynomial in x. To be specific, let

$$f_0(x), f_1(x), \cdots, f_{n+1}(x)$$

be $n+2$ polynomials in $K[x]$. Let

$$Y = \mathrm{cl}(\{(f_0(x) : f_1(x) : \cdots : f_{n+1}(x)) \in \mathbb{P}^{n+1} | x \in K\})$$

be another projective variety, then we define Z as the *Hadamard product* of X and Y. The definition is

$$Z = \mathrm{cl}(\{(\mathbf{t}^{\mathbf{a_0}} f_0(x) : \mathbf{t}^{\mathbf{a_1}} f_1(x) : \cdots : \mathbf{t}^{\mathbf{a_{n+1}}} f_{n+1}(x)) \in \mathbb{P}^{n+1} | \mathbf{t} \in (K^*)^n, x \in K^*\}).$$

Then Z has codimension 1, and it's called *almost toric* hypersurface. We know that the ideal of Z is principal. So we would like to know its combinatorial properties, for example, its Newton polytope $Newt(Z)$.

Proposition 1.1. *$Newt(Z)$ is a polygon in K^{n+2}.*

Proof. Because the defining polynomial is the zero polynomial in variables $t_1, t_2, \cdots, t_n, x$, so each term has the same degree of $t_1, t_2, \cdots, t_n$, then each edge of $Newt(Z)$ is in the kernel of A, which is a 2-dimensional linear subspace, so $Newt(Z)$ is a polygon. □

In order to describe this polygon, it's enough to fix one vertex and the vectors corresponding to each edge. In the next section we present a main theorem to describe those vectors. We need the following constructions.

1.3 Plücker Matrix

Given a matrix A as in 1.1, we can define its corresponding Plücker matrix P_A.

Definition 1.2. *P_A is a $(n+2) \times (n+2)$ square matrix, with entries*

$$p_{i,j} = \begin{cases} \frac{1}{d}(-1)^{i+j} \det(A_{[i,j]}), & i < j; \\ -p_{j,i}, & i > j; \\ 0, & i = j. \end{cases}$$

where $A_{[i,j]}$ is the submatrix obtained from A by deleting the i-th and j-th columns of A.

Example 1.3. *Let*

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Then

$$P_A = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

Proposition 1.4. *P_A is skew-symmetric. The rank of P_A is 2. The entries in each row and column of P_A sum to 0.*

1.4 Valuation Matrices

Given univariate polynomials $f_0(x), f_1(x), \cdots, f_{n+1}(x)$, we consider all irreducible factors of

$$F(x) = \prod_{i=0}^{n+1} f_i(x)$$

in $K[x]$. Since K is algebraically closed, F factors into linear factors. Suppose $g_1(x), g_2(x), \cdots, g_m(x)$ are all distinct linear factors of $F(x)$, then we can define vectors

$$\mathbf{v_j} = (\operatorname{ord}_{g_j} f_0, \operatorname{ord}_{g_j} f_1, \cdots, \operatorname{ord}_{g_j} f_{n+1}) \in \mathbb{N}^{n+2}.$$

for $1 \leq j \leq m$.
Now we need to simplify the set of these vectors. We combine the pairwise linearly dependent vectors, because in the end they correspond to the same edge of the polygon.
Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_m}\}$. If two vectors in S are linearly dependent, then we delete them and add their sum to the set. We repeat this procedure. After finite steps, we end up with another set without pairwise linearly dependent vectors.

$$S' = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_l}\}.$$

Definition 1.5. *The valuation matrix of Z is*

$$W = \begin{bmatrix} \mathbf{u_1}^T & \mathbf{u_2}^T & \cdots & \mathbf{u_l}^T & (-\sum_{j=1}^{l} \mathbf{u_j})^T \end{bmatrix}.$$

The last vector represents the valuation at ∞.

Proposition 1.6. *The sum of each row in W is zero.*

Example 1.7.

$$n = 2, f_0 = x-1, f_1 = (x-1)^2(x+1), f_2 = (x+1)x^3, f_3 = (x-2)x.$$

Then the irreducible factors are $x-1, x, x+1, x-2$.

- $x-1$ *corresponds to* $(1, 2, 0, 0)$

- *x corresponds to* $(0,0,3,1)$
- *$x+1$ corresponds to* $(0,1,1,0)$
- *$x-2$ corresponds to* $(0,0,0,1)$

These vectors are pairwisely linearly independent, so the valuation matrix would be

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & -3 \\ 0 & 3 & 1 & 0 & -4 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix}.$$

2 Main Theorem

Theorem 2.1. *An almost toric projective hypersurface Z has a Plücker matrix P_A for its toric part and a valuation matrix W for its coefficients. The directed edges of the Newton polygon of Z are the nonzero column vectors of $P_A \cdot W$.*

Remark 2.2. *The zero column vectors in $P_A \cdot W$ correspond to irreducible factors whose vectors of valuation belong to the row space of A, then these factors essentially belong to the toric part, so they don't contribute to the Newton polygon. Actually they correspond to degenerate edges of the Newton polygon.*

3 Proof of The Main Theorem

3.1 Tropicalization of Z

We illustrate our approach by an example. Consider the almost-toric projective hypersurface parameterized as

$$Z_3 = \{(s^3 f_0(x) : s^2 t f_1(x) : st^2 f_2(x) : t^3 f_3(x)) \in \mathbb{P}^3 | s,t,x \in \mathbb{C}\}.$$

By the formula in [5][Proposition 4.1], the degree of Z depends on some min-type functions of multiplicities of the roots of $f_0 f_1 f_2 f_3$. This reminds us to consider *tropical* approach, to be specific the *tropicalizations* of algebraic varieties.

Let X, Y, Z be defined as in Section 1. Note that Z is the *Hadamard product* $\overline{X * Y}$ of varieties X and Y. We have the following relation of the tropicalizations of them:

Proposition 3.1. *[4][Proposition 5.5.8]*

$$trop(\overline{X * Y}) = trop(X) + trop(Y),$$

where the sum is Minkowski sum. Next we find $\mathrm{trop}(X)$ and $\mathrm{trop}(Y)$.

Lemma 3.2.

$$trop(X) = \{\sum_{i=1}^{n} r_i \mathbf{b}_i \in \mathbb{R}^n | r_i \in \mathbb{R}, i = 1, 2, \cdots, n\},$$

where $\mathbf{b}_1, \cdots, \mathbf{b}_n$ are all row vectors of matrix A.

Proof. By Proposition 3.1, it's enough to prove the case when $n = 1$. That is, if $a_0, a_1, \cdots, a_{n+1}$ are integers and

$$X = \mathrm{cl}(\{(t^{a_0} : t^{a_1} : \cdots : t^{\mathbf{a_{n+1}}}) \in (\mathbb{P}^*)^{n+1} | t \in K^*\}),$$

then

$$trop(X) = \{r \cdot (a_0, a_1, \cdots, a_{n+1}) | r \in \mathbb{R}\}.$$

While in this case the ideal $I(X)$ is generated by binomial ideals as follows:

$$I(X) = \langle x_i^{\frac{lcm(a_i, a_j)}{a_i}} - x_j^{\frac{lcm(a_i, a_j)}{a_j}} | 0 \leq i < j \leq n+1 \rangle,$$

where $lcm(a, b)$ is the least common multiple of integers a, b. Then all points in $\mathrm{trop}(X)$ has the above form. $\square$

Lemma 3.3. *Let $K = \mathbb{C}\{\{t\}\}$ be an algebraically closed field. If*

$$Y = \{(f_0(x) : f_1(x) : \cdots : f_{n+1}(x)) \in (\mathbb{P}^*)^{n+1} | x \in K\}$$

is a projective variety, then

$$trop(Y) = \bigcup_{z \in K \ is \ a \ root \ of \ \prod_{i=0}^{n+1} f_i \ or \ \infty} \{\lambda(ord_z f_0, ord_z f_1, \cdots, ord_z f_{n+1}) \in \mathbb{R}^{n+2} | \lambda \geq 0\}.$$

Proof. By [4][Theorem 3.2.5], $\mathrm{trop}(Y) = \overline{\mathrm{val}(Y)}$, where

$$\mathrm{val}(Y) = \{(\mathrm{val}(u_0), \cdots, \mathrm{val}(u_{n+1})) | (u_0, \cdots, u_{n+1}) \in Y\}.$$

Note that $A = \overline{B}$, where

$$B = \bigcup_{z \in K \text{ is a root of } \prod_{i=0}^{n+1} f_i \text{ or } \infty} \{\lambda(\mathrm{ord}_z f_0, \mathrm{ord}_z f_1, \cdots, \mathrm{ord}_z f_{n+1}) \in \mathbb{R}^{n+2} | \lambda \in \mathbb{Q}^+\}.$$

Then it's enough to show the equivalence of $\mathrm{val}(X)$ and B. First, fix any $u = \lambda(\mathrm{ord}_z f_0, \mathrm{ord}_z f_1, \cdots, \mathrm{ord}_z f_{n+1}) \in B$.
(i) If $z \neq \infty$, then for each i, we have that $f_i(x) = (x - z)^{\mathrm{ord}_z f_i} g_i(x)$, where $\mathrm{val}(g_i(z)) = 0$. Then if we plug-in $x = z + t^\lambda$,

$$f_i(x) = t^{\lambda \mathrm{ord}_z f_i} g_i(z + t^\lambda).$$

Notice that

$$t^\lambda | g_i(z + t^\lambda) - g_i(z).$$

So $\mathrm{val}(g_i(z+t^{\lambda})-g_i(z))>0$. Then

$$\mathrm{val}(g_i(z+t^{\lambda}))=0.$$

Thus

$$\mathrm{val}(f_i(x))=\lambda\mathrm{ord}_z f_i,$$

which means $u\in\mathrm{val}(Y)$.
(ii) If $z=\infty$, then $\mathrm{ord}_z f_i=-\deg(f_i)$. We plug-in $x=t^{-\lambda}$, then among all terms in $f_i(x)$, the term with least valuations is the leading term, because $\lambda>0$. Then

$$\mathrm{val}(f_i(x))=(-\lambda)\deg(f_i)=\lambda\mathrm{ord}_z f_i.$$

So $u\in\mathrm{val}(Y)$, too.
Second, we show that $\mathrm{val}(X)\subseteq B$.
Suppose $u\in\mathrm{val}(X)$, then there exists $x\in K$ such that

$$u=(\mathrm{val}(f_0(x)),\cdots,\mathrm{val}(f_{n+1}(x))).$$

We may assume that $u\neq 0$. (i) If x contains at least one term with positive exponent, then we could write $x=z+t^{\frac{p}{q}}$, where $\frac{p}{q}\in\mathbb{Q}^+$. Then $u=\frac{p}{q}(\mathrm{ord}_z f_0,\mathrm{ord}_z f_1,\cdots,\mathrm{ord}_z f_{n+1})$. Since $u\neq 0$ and all f_i are polynomials, at least one $\mathrm{ord}_z f_i>0$, then z is a root of $\prod_{i=0}^{n+1} f_i$, so $u\in B$.
(ii) If all terms of x have nonpositive exponents. Suppose the term with least degree is $ct^{-\frac{p}{q}}$, where $\frac{p}{q}\in\mathbb{Q}^+$. Then $u=\frac{p}{q}(\mathrm{ord}_\infty f_0,\mathrm{ord}_\infty f_1,\cdots,\mathrm{ord}_\infty f_{n+1})\in B$. □

Note that $\mathrm{trop}(Y)$ is exactly the union of rays generated by column vectors in W. So we have the following corollary:

Corollary 3.4.

$$trop(Z)=\{\mathbf{u}+\lambda\cdot\mathbf{v}|\mathbf{u}\in row(A),\mathbf{v}^T \textit{ is a column vector of } W,\lambda\geq 0\},$$

where $row(A)$ is the vector space spanned by all row vectors of A.

3.2 Multiplicity of $(n+1)$-dimensional Polyhedron

Next we explore the edges of $Newt(z)$.

Proposition 3.5. *The edges of $Newt(Z)$ are parallel to the nonzero column vectors in $P_A\cdot W$, repsectively.*

Proof. By [4][Proposition 3.1.10], $\mathrm{trop}(Z)$ is the support of a $(n+1)$-dimensional polyhedral fan, which is the $(n+1)$-skeleton of the normal fan of $Newt(Z)$. Since $Newt(Z)$ is a polygon, every polyhedron in the $(n+1)$-skeleton of its normal fan is a cone spanned by the orthogonal complement of the plane that contains it (which is exactly $row(A)$) and a ray on this plane that is orthogonal to the corresponding edge of the polygon. Then by Corollary 3.4, these directed edges belong to $\ker(A)$ and are orthogonal to column vectors of W, respectively. Let u be a column vector of W. Since P_A is skew-symmetric, it corresponds to an identically zero quadratic form. So we have

$$u \cdot P_A \cdot u^T = 0.$$

Hence $P_A \cdot u^T$ is orthogonal to u, which means that there exists a scalar c such that $c(P_A \cdot u^T)$ represents an edge of $Newt(Z)$. $\square$

Then it remains to show that the length of these edges are the same as the corresponding nonzero column vectors in $P_A \cdot W$. To achieve this, we need the notion of *multiplicity* of a polyhedron which is maximal in a polyhedral complex. We adopt the definition in [4][Definition 3.4.3]. Then we have the following result.

Lemma 3.6. *([4][Lemma 3.4.6]) The lattice length of any edge $e(\sigma)$ (defined as the number of lattice points on the edge minus* 1*) of $Newt(Z)$ is the multiplicity mult(σ) of the corresponding $(n+1)$-dimensional polyhedron σ in the normal fan of $Newt(Z)$.*

Now it's enough to find out $\text{mult}(\sigma)$ for each σ in the $(n+1)$-skeleton of the normal fan of $Newt(Z)$. We cannot achieve this by definition, because the definition of $\text{mult}(\sigma)$ involves the defining polynomial of Z, which is unknown to us. It turns out that we could use another tool called Sturmfels-Tevelev multiplicity formula.

3.3 Sturmfels-Tevelev Multiplicity Formula

Sturmfels-Tevelev Multiplicity Formula ([7][Theorem 1.1]) is an important tool in tropical implicitization. We use [7][Formula (1.2)] to compute $\text{mult}(\sigma)$.
Let $g_1, g_2, \cdots, g_m$ be all linear factors of F, as in Subsection 1.4. We define a homomorphism of tori as follows:

$$\alpha : \mathbb{T}^{n+m+1} \to \mathbb{T}^{n+2}$$

with

$$\begin{aligned} &(t_1 : t_2 : \cdots : t_n : g_1(x) : g_2(x) : \cdots : g_m(x) : \frac{1}{\prod_{i=1}^m g_i(x)}) \\ \mapsto &(\mathbf{t}^{\mathbf{a_0}} f_0(x) : \mathbf{t}^{\mathbf{a_1}} f_1(x) : \cdots : \mathbf{t}^{\mathbf{a_{n+1}}} f_{n+1}(x)). \end{aligned}$$

Let $C = \text{cl}(\{(t_1 : t_2 : \cdots : t_n : g_1(x) : g_2(x) : \cdots : g_m(x) : \frac{1}{\prod_{i=1}^m g_i(x)}) \in \mathbb{P}^{n+m} | t_1, \cdots, t_n \in K^*, x \in K\})$, then C is a curve in $\mathbb{T}^{n+m+1}$ and α is a homomorphism from C to $\alpha(C) = Z$.

Suppose $g_1, \cdots, g_m, \frac{1}{\prod_{i=1}^m g_i(x)}$ corresponds to coordinates $c_1, c_2, \cdots, c_{m+1}$. Since $g_1, \cdots, g_m$ are linear polynomials, the ideal of C is generated by linear binomials in $c_1, c_2, \cdots, c_m$ and $\prod_{i=1}^{m+1} c_i - 1$. So

$$\begin{aligned} trop(C) = \{&(r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_{m+1})| \\ &r_i \in \mathbb{R} \text{ for all } 1 \leq i \leq n, s_i = 0 \text{ for all } i \neq j, \\ &s_j \geq 0 \text{ if } j \leq m, s_j \leq 0 \text{ if } j = m+1\}. \end{aligned}$$

Then the matrix of f is

$$\mathbf{A} = \begin{bmatrix} A^T & W \end{bmatrix}.$$

Now since the Lauret monomials are homogenous of degree d, so the degree of α is d. In addition the map $f : \text{trop}(C) \to \text{trop}(Z)$ is a bijection. And the multiplicity of all maximal polyhedra τ in $\text{trop}(C)$ is 1, because by definition, for $\nu \in \text{relint}(\tau)$, the ideal $\text{in}_\nu(I_C)$ is prime.
Then we choose maximal polyhedron σ of $\text{trop}(Z)$ corresponding to nonzero column vector $\mathbf{u_i}^T$ in W. Let

$$\mathbf{w} = \sum_{i=1}^{n} r_i \mathbf{b_i} + \mathbf{u_i}.$$

Then

$$\mathbf{v} = f^{-1}(\mathbf{w}) = (r_1, r_2, \cdots, r_n, 0, 0, \cdots, \underset{i}{\hat{1}} \cdots, 0).$$

By [8][Formula (1.2)], we have that

$$\text{mult}(\sigma) = m_{\mathbf{w}} = \frac{1}{d} \text{index}(\mathbb{L}_{\mathbf{w}} \cap \mathbb{Z}^{n+2} : \mathbf{A}(\mathbb{L}_{\mathbf{v}} \cap \mathbb{Z}^{n+m+1})).$$

Note that the above index is the index of the lattice generated by vectors

$$\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}, \mathbf{u_i}.$$

While it's well known that the index of a lattice is the greatest common divisor among the determinants of all maximal minors, and these determinants of $(n+1) \times (n+1)$ minors are exactly d times the components of $P_A \cdot \mathbf{u_i}^T$ (notice that in the definition of Plücker matrix we divide by d). Then we have

Proposition 3.7. *If σ corresponds to $\mathbf{u_i}^T$, then $mult(\sigma) = cont(P_A \cdot \mathbf{u_i}^T)$, where the content of a vector is the greatest common divisor of its components.*

Finally note that the lattice length of $P_A \cdot \mathbf{u_i}^T$ is also $cont(P_A \cdot \mathbf{u_i}^T)$, which means that $P_A \cdot \mathbf{u_i}^T$ is indeed one directed edge of $Newt(Z)$. The proof is finished.

4 An Example

We illustrate our result by one example.

Example 4.1. *Let $\mathcal{H}$ admit the following parameterization over $\mathbb{C}$:*

$$\begin{aligned}\mathcal{H} = cl(\{(&t_1^2(x^2+1) : t_1t_2x^3(x-1) : t_1t_3x(x+1) : \\ &t_2^2(x-2)(x^2+1) : t_3^2(x-1)^2(x+1)) \in \mathbb{P}^4 | \\ &t_1, t_2, t_3, x \in \mathbb{C}\}).\end{aligned}$$

In this example

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}, d = 2.$$

Then

$$P_A = \begin{bmatrix} 0 & -2 & 2 & 1 & -1 \\ 2 & 0 & -4 & 0 & 2 \\ -2 & 4 & 0 & -2 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -2 & 0 & 1 & 0 \end{bmatrix}.$$

The linear factors of f_i's are

$$x, x-1, x+i, x-i, x+1, x-2.$$

But we can combine $x \pm i$ into x^2+1. So the vectors are

$$\begin{aligned}&(0,3,1,0,0), (0,1,0,0,2), (2,0,0,2,0), (0,0,1,0,1), (0,0,0,1,0) \\ \&&(-2,-4,-2,-3,-3).\end{aligned}$$

Then the valuation matrix of $\mathcal{H}$ is

$$W = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & -2 \\ 3 & 1 & 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 0 & 1 & -3 \\ 0 & 2 & 0 & 1 & 0 & -3 \end{bmatrix}.$$

Using ideal elimination in Macaulay2 we can compute the defining polynomial of $\mathcal{H}$ in variables u_0, u_1, u_2, u_3, u_4:

$$\begin{gathered} 16u_1^4u_2^{16}u_3^2 - 40u_0u_1^4u_2^{14}u_3^2u_4 + 8u_0^2u_1^2u_2^{14}u_3^3u_4 - 16u_0u_1^6u_2^{12}u_3u_4^2 \\ +20u_0^2u_1^4u_2^{12}u_3^2u_4^2 + 159u_0^3u_1^2u_2^{12}u_3^3u_4^2 + u_0^4u_2^{12}u_3^4u_4^2 \\ +54u_0^2u_1^6u_2^{10}u_3u_4^3 - 77u_0^3u_1^4u_2^{10}u_3^2u_4^3 + 379u_0^4u_1^2u_2^{10}u_3^3u_4^3 \\ +5u_0^2u_1^8u_2^8u_4^4 - 27u_0^3u_1^6u_2^8u_3u_4^4 - 29u_0^4u_1^4u_2^8u_3^2u_4^4 \\ +163u_0^5u_1^2u_2^8u_3^3u_4^4 - 12u_0^3u_1^8u_2^6u_4^5 - 35u_0^4u_1^6u_2^6u_3u_4^5 \\ -425u_0^5u_1^4u_2^6u_3^2u_4^5 + 4u_0^6u_1^2u_2^6u_3^3u_4^5 + 87u_0^5u_1^6u_2^4u_3u_4^6 \\ +717u_0^6u_1^4u_2^4u_3^2u_4^6 + 103u_0^6u_1^6u_2^2u_3u_4^7 - 115u_0^7u_1^4u_2^2u_3^2u_4^7 \\ +12u_0^7u_1^6u_3u_4^8 + 4u_0^8u_1^4u_3^2u_4^8. \end{gathered}$$

The vertices of Newton polygon of this defining polynomial are

$$(0,4,16,2,0),(2,8,8,0,4),(3,8,0,6,5),$$
$$(7,6,0,1,8),(8,4,0,2,8),(4,0,12,4,2).$$

Then the directed edges are

$$(2,4,-8,-2,4),(-4,4,4,-2,-2),(-4,-4,12,2,-6)$$
$$(1,-2,0,1,0),(4,-2,-6,1,3),(1,0,-2,0,1).$$

While the product $P_A \cdot W$ is

$$\begin{bmatrix} -4 & -4 & 2 & 1 & 1 & 4 \\ -4 & 4 & 4 & -2 & 0 & -2 \\ 12 & 4 & -8 & 0 & -2 & -6 \\ 2 & -2 & -2 & 1 & 0 & 1 \\ -6 & -2 & 4 & 0 & 1 & 3 \end{bmatrix}.$$

5 Application

As for almost toric hypersurfaces, we would like to determine its defining polynomial in $n+2$ variables $u_0, u_1, \cdots, u_{n+1}$ given A and $f_0, \cdots, f_{n+1}$. Classical approach is ideal elimination using Gröbner bases, which is inefficient when n is large. Based on the main theorem we could have the following approach:

1. Write down A directly from parameterization, factor $f_0, f_1, \cdots, f_{n+1}$ into linear factors.
2. Compute P_A,W and their product.
3. Find $Newt(Z)$ from the product. (Actually it's unique!)
4. Determine all monomials in variables $u_0, u_1, \cdots, u_{n+1}$ in the defining polynomial.
5. Use linear algebra to determine all coefficients of these monomials.

Remark 5.1.
- *$P_A \cdot W$ gives us all directed edges of $Newt(Z)$. Then $Newt(Z)$ is unique determined because all vertices have* nonnegative *coordinates in $\mathbb{Z}^{n+2}$ and each coordinate attains 0 at least once.*
- *$P_A \cdot W$ is a rank 2 matrix with integer entries, and all rows and columns sum to 0.*
- *Actually any such matrix could be $P_A \cdot W$ for some Z. We could implement a combinatorial algorithm to find the Newton polygon from $P_A \cdot W$.*

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