Volumes

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In this lecture we discuss the problem of computing the volume of a subset X of $\mathbb{R}^n$ that is full-dimensional and semialgebraic. Being *semialgebraic* means that X is described by a finite Boolean combination of polynomial inequalities. We say that X is *basic semialgebraic* if that description is a conjunction of polynomial inequalities. In symbols, this means

$$X \;=\; \bigl\{\, x \in \mathbb{R}^n \,:\, f_1(x) \,\geq\, 0 \;\text{ and }\; f_2(x) \,\geq\, 0 \;\text{ and }\; \cdots \;\text{ and }\; f_k(x) \,\geq\, 0 \,\bigr\},$$

where $f_1, f_2, \ldots, f_k$ are polynomials in n unknowns with real coefficients.

The simplest scenario arises when $k = 1$, so X is the domain of nonnegativity of one polynomial $f(x) = f(x_1, \ldots, x_n)$ with real coefficients. Our task is to evaluate the integral

$$\mathrm{Vol}(X) \;=\; \int_X 1 \cdot dx, \tag{1}$$

where dx denotes Lebesgue measure. Of course, it makes perfect sense to also consider integrals $\int_X g(x)dx$, where $g(x)$ is some polynomial function. The value of such an integral is a real number which is called a *period* [3]. Our integrals are known as *period integrals*.

We begin with an instance where the volume can be computed explicitly using calculus.

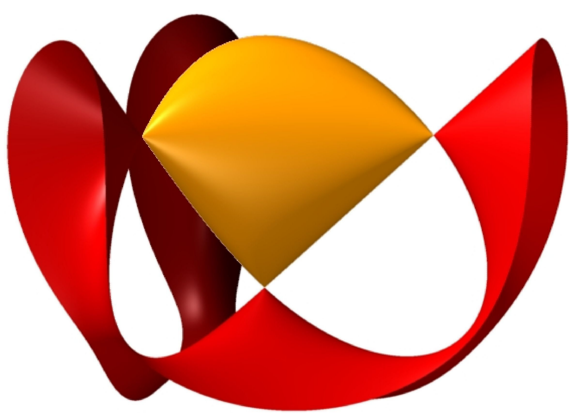


Figure 1: The yellow convex body is the elliptope. It is bounded by Cayley's cubic surface.

Example 1 (Elliptope)**.** Consider the set X of all points (x, y, z) in $\mathbb{R}^3$ such that the matrix

$$M \;=\; \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}$$

is positive semidefinite. The set X is convex and semialgebraic: it consists of all points (x, y, z) in the cube $[-1,1]^3$ such that $\det(M) = 2xyz - x^2 - y^2 - z^2 + 1$ is nonnegative. Figure 1 appears in [7, Figure 1.1] and it serves as the logo of the Nonlinear Algebra group at the Max-Planck Institute for Mathematics in the Sciences in Leipzig. It illustrates several applications of algebraic geometry. In statistics, it is the set of all correlation matrices. In optimization, it is the feasible region of a semidefinite programming problem [7, Chapter 12].

We now compute the volume of the elliptope. We begin by rewriting of its boundary surface. Solving the equation $\det(M) = 0$ for z with the quadratic formula, we obtain

$$z \;=\; xy \pm \sqrt{x^2y^2 - x^2 - y^2 + 1} \;=\; xy \pm \sqrt{(1-x^2)(1-y^2)} \qquad \text{for } (x,y) \in [-1,1]^2,$$

The plus sign gives the upper yellow surface and the minus sign gives the lower yellow surface. The volume of the elliptope X is obtained by integrating the difference between the upper function and the lower function over the square. Hence the desired volume equals

$$\mathrm{vol}(X) \;=\; \int_{-1}^{1}\int_{-1}^{1} 2\sqrt{(1-x^2)(1-y^2)}\,dxdy \;=\; 2\left[\int_{-1}^{1}\sqrt{1-t^2}\,dt\right]^2.$$

The univariate integral on the right gives the area of a semicircle with radius 1. We know from trigonometry that this area equals $\pi/2$, where $\pi = 3.14159265...$. We conclude that

$$\mathrm{vol}(X) \;=\; \pi^2/2 \;=\; 4.934802202...$$

Thus our elliptope covers about 61.7 % of the volume of the cube $[-1,1]^3$ that surrounds it.

The number $\pi^2/2$ we found is an example of a period. It is generally much more difficult to accurately evaluate such integrals. In fact, this challenge has played an important role in the history of mathematics. Consider the problem of computing the arc length of an ellipse. This requires us to integrate the reciprocal square root of cubic polynomial $f(t)$. Such integrals are called *elliptic integrals*, and they represent periods of elliptic curves. Furthermore, in an 1841 paper, Abel introduced *abelian integrals*, where $g(t)$ is an algebraic function in one variable t. How to evaluate such an integral? This question leads us to Riemann surface and then to their Jacobians. And, violà, we arrived at the theory of *abelian varieties*.

This lecture presents two current paradigms for accurately computing integrals like (1). The first method rests on the theory of D-modules, that is, on the algebraic study of linear differential equations with polynomial coefficients. Our volume is found as a special value of a parametric volume function that is encoded by means of its *Picard-Fuchs differential equation*. This method, which tends to appeal to algebraic geometers, was introduced by Lairez, Mezzarobba and Safey El Din in [4]. We shall closely follow the exposition in [9].

The second approach is due to Lasserre and his collaborators [2, 10, 11]. On first glance it might appeal more to analysts and optimizers, but there is also plenty of deep algebraic structure under the hood. The idea is to consider all moments $m_a = \int_X x^a dx$ of our semialgebraic set X and to use relations among these moments to infer an accurate approximation of $m_0 = \mathrm{vol}(X)$. That numerical inference rests on semidefinite programming [7, Chapter 12].

In calculus, we learn about definite integrals in order to determine the area under a graph. Likewise, in multivariable calculus, we examine the volume enclosed by a surface. We are here interested in areas and volumes of semi-algebraic sets. When these sets depend on one or more parameters, their volumes are holonomic functions of the parameters. We explain what this means and how it can be used for highly accurate evaluation of volume functions.

Suppose that M is a D-module. The letter D denotes the *Weyl algebra* (cf. [8, 9]):

$$D \;=\; \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle.$$

In applications, M is usually a space of infinitely differentiable functions on a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. Such D-modules are torsion-free. For a function $f \in M$, its *annihilator* is the D-ideal

$$\mathrm{Ann}_D\,(f) \;\coloneqq\; \{\, P \in D \mid P \bullet f = 0 \,\}\,.$$

In general, it is a non-trivial task to compute this annihilating ideal. But, in some cases, computer algebra systems can help us to compute holonomic annihilating ideals. For rational functions $r \in \mathbb{Q}(x)$ this can be done in `Macaulay2` with a built-in command as follows:

```
needsPackage "Dmodules"; D = QQ[x1,x2,d1,d2, WeylAlgebra => {x1=>d1,x2=>d2}];
rnum = x1;  rden = x2;   I = RatAnn(rnum,rden)
```

This code fragment shows that $r = x_1/x_2$ has $\mathrm{Ann}_D(r) = D\{\,\partial_1^2,\, x_1\partial_1 - 1,\, x_2\partial_1\partial_2 + \partial_1\}$.

Suppose now that $f(x_1, \ldots, x_n)$ is an algebraic function. This means that f satisfies some polynomial equation $F(f, x_1, \ldots, x_n) = 0$. Using the polynomial F as its input, the `Mathematica` package `HolonomicFunctions` can compute a holonomic representation of f. The output is a linear differential operator of lowest degree annihilating f. See Example 3.

Let M be a D-module and $f \in M$. We say that f is *holonomic* if, for each $i \in \{1, \ldots, n\}$, there is an operator $P_i \in \mathbb{C}[x_1, \ldots, x_n]\langle \partial_i \rangle \backslash \{0\}$ that annihilates f. If this holds then $\mathrm{Ann}_D(f)$ is a *holonomic D-ideal*. Suppose this holds, and fix a general point $x_0 \in \mathbb{C}^n$. Let $m_1, \ldots, m_n$ denote the orders of the differential operators $P_1, \ldots, P_n$ in the definition of holonomic. Thus, P_k is an operator in ∂_k of order m_k whose coefficients are polynomials in $x_1, \ldots, x_n$.

We fix initial conditions for f by specifying the following $m_1 m_2 \cdots m_n$ numerical values:

$$(\partial_1^{i_1} \cdots \partial_n^{i_n} \bullet f)|_{x=x_0} \quad \text{where } \; 0 \le i_k < m_k \; \text{ for } \; k = 1, \ldots, n. \tag{2}$$

Then the operators $P_1, \ldots, P_n$ together with the initial conditions (2) specify the function f.

Many interesting functions are holonomic. To begin with, every rational function r in $x_1, \ldots, x_n$ is holonomic, because r is annihilated by $r(x)\partial_i - \partial r/\partial x_i$ for $i = 1, 2, \ldots, n$. By clearing denominators in this operator, we obtain a non-zero $P_i \in \mathbb{C}[x]\langle \partial_i \rangle$ with $m_i = 1$ that annihilates r. See the `Macaulay2` example above. These operators, together with fixing the value $r(x_0)$ at a general point $x_0 \in \mathbb{C}^n$, constitute a canonical holonomic representation of r.

Holonomic functions in one variable are solutions to ordinary linear differential equations with rational function coefficients. Examples include algebraic functions, some elementary trigonometric functions, hypergeometric functions, Bessel functions, period integrals, and many more. But, not every nice function is holonomic. A necessary condition for a meromorphic function $f(x)$ to be holonomic is that it has only finitely many poles in $\mathbb{C}$.

For a concrete example, we start with the holonomic function $\sin(x)$. This is annihilated by the operator $\partial^2 + 1$. Its reciprocal $f(x) = \frac{1}{\sin(x)}$ has infinitely many poles, so is not holonomic. Hence the class of holonomic functions is not closed under division. It is also not closed under composition of functions, since both $\frac{1}{x}$ and $\sin(x)$ are holonomic. We record:

Proposition 2. *Let $f(x)$ be holonomic and $g(x)$ algebraic. Then $f(g(x))$ is holonomic.*

For the proof see [9, Proposition 2.3]. The term "holonomic function" is due to Zeilberger [12]. Koutschan [1] developed practical algorithms for manipulating holonomic functions. These are implemented in his `Mathematica` package `HolonomicFunctions`, as seen below.

Example 3. Every algebraic function $f(x)$ is holonomic. Consider the function $y = f(x)$ that is defined by $y^4 + x^4 + \frac{xy}{100} - 1 = 0$. Its annihilator in D can be computed as follows:

```
<< RISC‘HolonomicFunctions‘
q = y^4 + x^4 + x*y/100 - 1
ann = Annihilator[Root[q, y, 1], Der[x]]
```

This `Mathematica` code determines an operator P of lowest order in $\mathrm{ann}_D(f)$. We find

$$\begin{aligned} P \;=\;& (2x^4+1)^2(25600000000x^{12} - 76800000000x^8 + 76799999973x^4 - 25600000000)\,\partial^3 \\ & +6x^3(2x^4+1)(51200000000x^{12} + 76800000000x^8 - 307199999946x^4 + 179199999973)\,\partial^2 \\ & +3x^2(102400000000x^{16} + 204800000000x^{12} + 2892799999572x^8 - 3507199999444x^4 + 307199999953)\,\partial \\ & -3x(102400000000x^{16} + 204800000000x^{12} + 1459199999796x^8 - 1049599999828x^4 + 51199999993). \end{aligned}$$

This operator is an encoding of the algebraic function $y = f(x)$ as a holonomic function.

In computer algebra, one represents a real algebraic number as a root of a polynomial with coefficients in $\mathbb{Q}$. However, this *minimal polynomial* does not specify the number uniquely. For that, one also needs an isolating interval or sign conditions on derivatives. The situation is analogous for encoding a holonomic function f in n variables. We specify f by a holonomic system of linear PDEs together with a list of initial conditions. The canonical holonomic representation is one possibility. Initial conditions such as (2) are designed to determine the function uniquely inside the linear space $\mathrm{Sol}(I)$, where $I \subseteq \mathrm{Ann}_D(f)$. For instance, in Example 3, we would need three initial conditions to specify the function $f(x)$ uniquely inside the 3-dimensional solution space to our operator P. We could fix the values at three distinct points, or we could fix the value and the first two derivatives at one special point.

To be more precise, we generalize the canonical representation (2) as follows. A *holonomic representation* of a function f is a holonomic D-ideal $I \subseteq \mathrm{ann}_D(f)$ together with a list of linear conditions that specify p uniquely inside the finite-dimensional solution space of holomorphic solutions. The existence of this representation makes f a *holonomic function.* The next example shows the relevance of holonomic functions for metric algebraic geometry.

Example 4 (The area of a TV screen)**.** Let

$$q(x,y) \;=\; x^4 + y^4 + \frac{1}{100}xy - 1. \tag{3}$$

We are interested in the semi-algebraic set $S = \{(x,y) \in \mathbb{R}^2 \mid q(x,y) \leq 0\}$. This convex set is a slight modification of a set known in the optimization literature as "the TV screen". Our aim is to compute the area of the semi-algebraic convex set S as accurately as is possible.

One can get a rough idea of the area of S by sampling. This is illustrated in Figure 2. From the equation we find that S is contained in the square defined by $-1.2 \leq x, y \leq 1.2$.

We sampled 10000 points uniformly from that square, and for each sample we checked the sign of q. Points inside S are drawn in blue and points outside S are drawn in pink. By multiplying the area $(2.4)^2 = 5.76$ of the square with the fraction of the number of blue points among the samples, we learn that the area of the TV screen is approximately 3.7077.

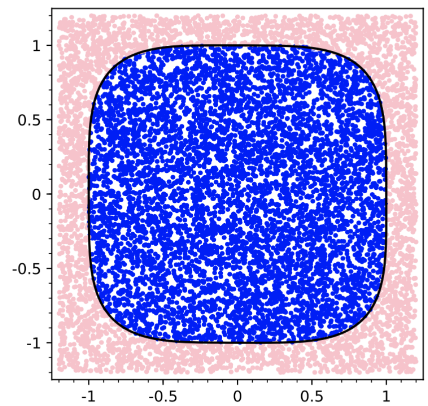


Figure 2: The TV screen is the convex region consisting of the blue points.

We now compute the area more accurately using D-modules. Let $\mathrm{pr}\colon S \to \mathbb{R}$ be the projection on the x-coordinate, and write $v(x) = \ell\,(\mathrm{pr}^{-1}(x) \cap S)$ for the length of a fiber. This function is holonomic and it satisfies the third-order differential operator in Example 3.

The map pr has two branch points $x_0 < x_1$. They are the real roots of the resultant

$$\mathrm{Res}_y(q, \partial q/\partial y) \;=\; 25600000000x^{12} - 76800000000x^8 + 76799999973x^4 - 25600000000. \quad (4)$$

These values can be written in radicals, but we take an accurate floating point representation:

$$x_1 \;=\; -x_0 \;=\; 1.000254465850258845478545766643566750080196276158976351763236\ldots$$

The desired area equals $\mathrm{vol}(S) = w(x_1)$, where w is the holonomic function

$$w(x) \;=\; \int_{x_0}^{x} v(t)dt.$$

One operator that annihilates w is $P\partial$, where $P \in \mathrm{ann}_D(v)$ is the third-order operator above. To get a holonomic representation of w, we also need some initial conditions. Clearly, $w(x_0) = 0$. Further initial conditions on w' are derived by evaluating v at other points. By plugging values for x into (3) and solving for y, we find $w'(0) = 2$ and $w'(\pm 1) = 1/\sqrt[3]{100}$. Thus, we now have four linear constraints on our function w, albeit at different points.

Our goal is to determine a unique function $w \in \mathrm{Sol}(P\partial)$ by incorporating these four initial conditions, and then to evaluate w at x_1. To this end, we proceed as follows. Let $x_{\mathrm{ord}} \in \mathbb{R}$ be any point at which $P\partial$ is not singular. Using the command `local_basis_expansion` that is built into the `SAGE` package `ore_algebra`, we compute a basis of local series solutions to

$P\partial$ at the point x_{ord}. Since that point is non-singular, that basis has the following form:

$$\begin{array}{lclcl} s_{x_{\text{ord}},0}(x) & = & 1 & + & O((x-x_{\text{ord}})^4), \\ s_{x_{\text{ord}},1}(x) & = & (x-x_{\text{ord}}) & + & O((x-x_{\text{ord}})^4), \\ s_{x_{\text{ord}},2}(x) & = & (x-x_{\text{ord}})^2 & + & O((x-x_{\text{ord}})^4), \\ s_{x_{\text{ord}},3}(x) & = & (x-x_{\text{ord}})^3 & + & O((x-x_{\text{ord}})^4). \end{array} \tag{5}$$

Locally at x_{ord}, our solution is given by a unique choice of four coefficients $c_{x_{\text{ord}},i}$, namely

$$w(x) \;=\; c_{x_{\text{ord}},0} \cdot s_{x_{\text{ord}},0}(x) \,+\, c_{x_{\text{ord}},1} \cdot s_{x_{\text{ord}},1}(x) \,+\, c_{x_{\text{ord}},2} \cdot s_{x_{\text{ord}},2}(x) \,+\, c_{x_{\text{ord}},3} \cdot s_{x_{\text{ord}},3}(x).$$

At a regular singular point x_{rs}, complex powers of x and $\log(x)$ can appear in the local basis extension at x_{rs}. Any initial condition at that point determines a linear constraint on these coefficients. For instance, $w'(0) = 2$ implies $c_{0,1} = 2$, and similarly for our initial conditions at $-1, 1$ and x_0. One challenge is that the initial conditions pertain to different points. To address this, we calculate transition matrices that relate the basis (5) of series solutions at one point to the basis of series solutions at another point. These are invertible 4×4 matrices.

With the method described above, we find the basis of series solutions at x_1, along with a system of four linear constraints on the four coefficients $c_{x_1,i}$. These constraints are derived from the initial conditions at 0, ± 1 and x_0, using the 4×4 transition matrices. By solving these linear equations, we compute the desired function value up to any desired precision:

$$w(x_1) \;=\; 3.708159944742162288348225561145865371243065819913934709438572....$$

In conclusion, this number is the area of the TV screen S defined by the polynomial $q(x, y)$.

Let us now come back to properties of holonomic functions. Holonomic functions are very well-behaved with respect to many operations. They turn out to have remarkable closure properties. In the following, let f and g be functions in n variables $x_1, \ldots, x_n$.

Proposition 5. *If f, g are holonomic functions, then both $f + g$ and $f \cdot g$ are holonomic.*

Proof. For each $i \in \{1, 2, \ldots, n\}$, there exist non-zero operators $P_i, Q_i \in \mathbb{C}[x]\langle\partial_i\rangle$, such that $P_i \bullet f = Q_i \bullet g = 0$. Set $n_i = \text{order}(P_i)$ and $m_i = \text{order}(Q_i)$. The $\mathbb{C}(x)$-span of $\left\{\partial_i^k \bullet f\right\}_{k=0,\ldots,n_i}$ has dimension $\leq n_i$. Similarly, the $\mathbb{C}(x)$-span of the set $\left\{\partial_i^k \bullet g\right\}_{k=0,\ldots,m_i}$ has dimension $\leq m_i$.

Now consider $\partial_i^k \bullet (f+g) = \partial_i^k \bullet f + \partial_i^k \bullet g$. The $\mathbb{C}(x)$-span of $\left\{\partial_i^k \bullet (f+g)\right\}_{k=0,\ldots,n_i+m_i}$ has dimension $\leq n_i + m_i$. Hence, there exists a non-zero operator $S_i \in \mathbb{C}[x]\langle\partial_i\rangle$, such that $S_i \bullet (f+g) = 0$. Since this holds for all indices i, we conclude that the sum $f+g$ is holonomic.

A similar proof works for the product $f \cdot g$. For each $i \in \{1, 2, \ldots, n\}$, we now consider the set $\left\{\partial_i^k \bullet (f \cdot g)\right\}_{k=0,1,\ldots,n_i m_i}$. By applying Leibniz' rule for taking derivatives of a product, we find that the $m_i n_i + 1$ generators are linear dependent over $\mathbb{C}(x)$. Hence, there is a non-zero operator $T_i \in \mathbb{C}[x]\langle\partial_i\rangle$ such that $T_i \bullet (f \cdot g) = 0$. We conclude that $f \cdot g$ is holonomic. $\square$

The proof above gives a linear algebra method for computing an annihilating D-ideal I of finite holonomic rank for $f + g$ (resp. of $f \cdot g$), starting from such D-ideals for f and g. The following example, similar to one in [12, Section 4.1], illustrates Proposition 5.

Example 6 ($n = 1$)**.** Consider the functions $f(x) = \exp(x)$ and $g(x) = \exp(-x^2)$. Their canonical holonomic representations are $I_f = \langle \partial - 1 \rangle$ with $f(0) = 1$ and $I_g = \langle \partial + 2x \rangle$ with $g(0) = 1$. We are interested in the function $h = f + g$. Its first partial derivatives are

$$\begin{pmatrix} h \\ \partial \bullet h \\ \partial^2 \bullet h \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 \\ 1 & -2x \\ 1 & 4x^2 - 2 \end{pmatrix} \cdot \begin{pmatrix} f \\ g \end{pmatrix}.$$

By computing the left kernel of this 3×2-matrix, we find that $h = f + g$ is annihilated by

$$I_h \;=\; \langle (2x+1)\partial^2 + (4x^2 - 3)\partial - 4x^2 - 2x + 2 \rangle, \quad \text{with } \; h(0) = 2,\; h'(0) = 1.$$

For the product $j = f \cdot g$ we have $j' \;=\; f'g + fg' = f \cdot g + f \cdot (-2xg) = (1 - 2x)j$, so the canonical holonomic representation of j is the D-ideal $I_j = \langle \partial + 2x - 1 \rangle$ with $j(0) = 1$.

Proposition 7. *Let f be holonomic in n variables and $m < n$. Then the restriction of f to the coordinate subspace $\{x_{m+1} = \ldots = x_n = 0\}$ is a holonomic function in $x_1, \ldots, x_m$.*

Proof. For $i \in \{m+1, \ldots, n\}$, we consider the right ideal $x_i D$ in the Weyl algebra D. This ideal is a left module over $D_m = \mathbb{C}\langle x_1, \ldots, x_m, \partial_1, \ldots, \partial_m \rangle$. The sum of these ideals with $\mathrm{Ann}_D(f)$ is hence a left D_m-module. Its intersection with D_m is called the *restriction ideal*:

$$(\mathrm{Ann}_D(f) + x_{m+1}D + \cdots + x_n D) \;\cap\; D_m. \tag{6}$$

By [8, Prop. 5.2.4], this D_m-ideal is holonomic and it annihilates $f(x_1, \ldots, x_m, 0, \ldots, 0)$. □

Proposition 8. *The partial derivatives of a holonomic function are holonomic functions.*

Proof. Let f be holonomic and $P_i \in \mathbb{C}[x]\langle \partial_i \rangle \backslash \{0\}$ with $P_i \bullet f = 0$ for all i. We can write P_i as $P_i = \widetilde{P}_i \partial_i + a_i(x)$, where $a_i \in \mathbb{C}[x]$. If $a_i = 0$, then $\widetilde{P}_i \bullet \frac{\partial f}{\partial x_i} = 0$ and we are done. Assume $a_i \neq 0$. Since both a_i and f are holonomic, by Proposition 5, there is a non-zero linear operator $Q_i \in \mathbb{C}[x]\langle \partial_i \rangle$ such that $Q_i \bullet (a_i \cdot f) = 0$. Then $Q_i \widetilde{P}_i$ annihilates $\partial f / \partial x_i$. □

A key insight from the theory of D-modules (see [8, Section 5.5]) is that integration is dual, in the sense of the Fourier transform, to restriction. Here is the dual to Proposition 7.

Proposition 9. *Let $f \colon \mathbb{R}^n \to \mathbb{C}$ be a holonomic function. Then the definite integral*

$$F(x_1, \ldots, x_{n-1}) \;=\; \int_a^b f(x_1, \ldots, x_{n-1}, x_n) dx_n$$

is a holonomic function in $n - 1$ variables, assuming the integral converges.

By dualizing (6), we obtain the following D_m-ideal, known as the *integration ideal*:

$$\big(\mathrm{Ann}_D(f) \;+\; \partial_{m+1}D + \cdots + \partial_n D\big) \cap D_m \qquad \text{for } m < n.$$

The expression is dual to the restriction ideal (6) under the Fourier transform. This exchanges x_i and ∂_i. If $m = n - 1$ then the integration ideal annihilates the holonomic function F above.

Equipped with our tools for holonomic functions, we now return to the computation of volumes of compact semi-algebraic sets. We follow the work of P. Lairez, M. Mezzarobba and M. Safey El Din in [4]. They compute this volume by deriving a differential operator that encodes the period of a certain rational integral [3]. Here is the key definition.

Let $R(t, x_1, \ldots, x_n)$ be a rational function and consider the formal period integral

$$\oint R(t, x_1, \ldots, x_n) dx_1 \cdots dx_n. \tag{7}$$

Fix an open subset Ω of either $\mathbb{R}$ or $\mathbb{C}$. An analytic function $\phi \colon \Omega \to \mathbb{C}$ is a *period* of the integral (7) if, for any $s \in \Omega$, there exists a neighborhood $\Omega' \subseteq \Omega$ of s and an n-cycle $\gamma \subset \mathbb{C}^n$ with the following property. For all $t \in \Omega'$, γ is disjoint from the poles of $R_t \coloneqq R(t, \bullet)$ and

$$\phi(t) \;=\; \int_\gamma R(t, x_1, \ldots, x_n) dx_1 \cdots dx_n. \tag{8}$$

If this holds, then there exists an operator $P \in D\backslash\{0\}$ of the Fuchsian class annihilating $\phi(t)$.

Let $S = \{f \leq 0\} \subset \mathbb{R}^n$ be a compact basic semi-algebraic set, defined by a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$. Let $\mathrm{pr} \colon \mathbb{R}^n \to \mathbb{R}$ denote the projection on the first coordinate. The set of *branch points* of pr is the following subset of the real line, which is assumed to be finite:

$$\Sigma_f = \left\{ p \in \mathbb{R} \mid \exists\, x = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} : f(p, x) = 0 \;\text{ and }\; \tfrac{\partial f}{\partial x_i}(p, x) = 0 \;\text{ for } i = 2, \ldots, n \right\}.$$

The polynomial in the unknown p that defines Σ_f is obtained by eliminating $x_2, \ldots, x_n$. It can be represented as a multivariate resultant, generalizing the Sylvester resultant in (4).

Fix an open interval I in $\mathbb{R}$ with $I \cap \Sigma_f = \emptyset$. For any $x_1 \in I$, the set $S_{x_1} \coloneqq \mathrm{pr}^{-1}(x_1) \cap S$ is compact and semi-algebraic in $(n-1)$-space. We are interested in its volume. By [4, Theorem 9], the function $v \colon I \to \mathbb{R},\; x_1 \mapsto \mathrm{vol}_{n-1}(S_{x_1})$ is a period of the rational integral

$$\frac{1}{2\pi i} \oint \frac{x_2}{f(x_1, x_2, \ldots, x_n)} \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_2} dx_2 \cdots dx_n. \tag{9}$$

Let $e_1 < e_2 < \cdots < e_K$ be the branch points in Σ_f and set $e_0 = -\infty$ and $e_{K+1} = \infty$. This specifies the pairwise disjoint open intervals $I_k = (e_k, e_{k+1})$. They satisfy $\mathbb{R}\backslash\Sigma_f = \bigcup_{k=0}^{K} I_k$. Fix the holonomic functions $w_k(t) = \int_{e_k}^{t} v(x_1) dx_1$. The volume of S then is obtained as

$$\mathrm{vol}_n(S) \;=\; \int_{e_1}^{e_K} v(x_1) dx_1 \;=\; \sum_{k=1}^{K-1} w_k(e_{k+1}).$$

How does one compute such an expression? As a period of the rational integral (9), the volume v is a holonomic function on each interval I_k. A key step is to compute an operator $P \in D_1$ that annihilates $v|_{I_k}$ for all k. Then the product $P\partial$ annihilates the functions $w_k(x_1)$ for all k. By imposing sufficiently many initial conditions, we can reconstruct the functions w_k from the operator $P\partial$. One initial condition that comes for free for each k is $w_k(e_k) = 0$.

The differential operator P is known as the *Picard–Fuchs equation* of the period in question. The following software packages can be used to compute such Picard–Fuchs equations:

- `HolonomicFunctions` by C. Koutschan in `Mathematica`,
- `ore_algebra` by M. Kauers in `SAGE`,
- `periods` by P. Lairez in `MAGMA`,
- `Ore_Algebra` by F. Chyzak in `Maple`.

We next show how one can compute the volume in practice. Starting from the polynomial f, we compute the Picard–Fuchs operator $P \in D_1$ and we find sufficiently many compatible initial conditions. Therefore, for each interval I_k, where $k = 1, \dots, K-1$, we perform the following steps. We describe this for the `ore_algebra` package in `SAGE`:

(i) Using the command `local_basis_expansion`, compute a local basis of series solutions for the linear differential operator $P\partial$ at various points in $[e_k, e_{k+1})$.

(ii) Using the command `op.numerical_transition_matrix`, numerically compute a transition matrix for the series solution basis from one point to another one.

(iii) From the initial conditions construct linear relations between the coefficients in the local basis extensions. Using step (ii), transfer them to the branch point e_{k+1}.

(iv) Plug in to the local basis extension at e_{k+1} and thus evaluate the volume of $S \cap \mathrm{pr}^{-1}(I_k)$.

We now illustrate this recipe by computing the volume of a convex body in 3-space.

Example 10 (Quartic surface)**.** Fix the quartic polynomial

$$f(x,y,z) \;=\; x^4 + y^4 + z^4 + \frac{x^3 y}{20} - \frac{xyz}{20} - \frac{yz}{100} + \frac{z^2}{50} - 1, \tag{10}$$

and consider the set $S = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) \leq 0\}$. Our aim is to compute $\mathrm{vol}_3(S)$.

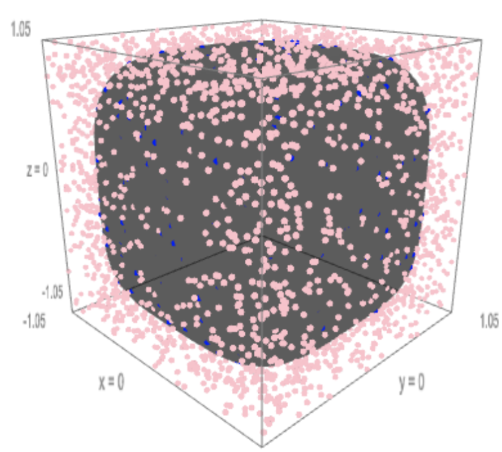


Figure 3: The quartic bounds the convex region consisting of the gray points.

As in Example 4 with the TV screen, we can get a rough idea of the volume of S by sampling. This is illustrated in Figure 3. Our set S is compact, convex, and contained in the cube defined by $-1.05 \leq x, y, z \leq 1.05$. We sampled 10000 points uniformly from that cube. For each sample we checked the sign of $f(x, y, z)$. By multiplying the volume $(2.1)^3 = 9.261$ of the cube by the fraction of the number of gray points and the number of sampled points, `SAGE` found within few seconds that the volume of the quartic body S is ≈ 6.4771. In order to obtain a higher precision, we now compute the volume of our set S with help of D-modules.

Let $\mathrm{pr}\colon \mathbb{R}^3 \to \mathbb{R}$ be the projection onto the x-coordinate. Let $v(x) = \mathrm{vol}_2\left(\mathrm{pr}^{-1}(x) \cap S\right)$ denote the area of the fiber over any point x in $\mathbb{R}$. We write $e_1 < e_2$ for the two branch points of the map pr restricted to the quartic surface $\{f = 0\}$. They can be computed with resultants. The projection has 36 complex branch points. The first two of them are real and therefore are the branch points of pr. We obtain $e_1 \approx -1.0023512$ and $e_2 \approx 1.0024985$.

By [4, Theorem 9], the area function $v(x)$ is a period of the rational integral

$$\frac{1}{2\pi i} \oint \frac{y}{f(x,y,z)} \frac{\partial f(x,y,z)}{\partial y} dydz.$$

We set $w(t) = \int_{e_1}^{t} v(x)dx$. The desired 3-dimensional volume equals $\mathrm{vol}_3(S) = w(e_2)$.

Using Lairez' implementation `periods` in `MAGMA`, we compute a differential operator P of order eight that annihilates $v(x)$. Again, $P\partial$ then annihilates $w(x)$. One initial condition is $w(e_1) = 0$. We obtain eight further initial conditions $w'(x) = \mathrm{vol}_2(S_x)$ for points $x \in (e_1, e_2)$ by running the same algorithm for the 2-dimensional semi-algebraic slices $S_x = \mathrm{pr}^{-1}(x) \cap S$. In other words, we make eight subroutine calls to an area measurement as in Example 4.

From these nine initial conditions we derive linear relations of the coefficients in the local basis expansion at e_2. These computations are run in `SAGE` as described in steps (i), (ii), (iii) and (iv) above. We find the approximate volume of our convex body S to be

$$\begin{gathered}\approx 6.43883248057289354474073389596995618895842088923511697626632892312882\\ 6915527388764216209149558398903829431137608893452690352556009760102417\\ 1190804769405534826558114212766135380613959757935305271022089419155701\\ 521586470170874002194384529140686856227759541715097113399134734059\\ 61763289220607208551633239796916338376007073876010731824775206150471\\ 4367250460900923409066377732273390396822296235214963623286613117557\\ 9306875441483607212256810534811787600582647388671058103268189\\ 11578448323758536767168707442532146029753762594261578920477859.\end{gathered}$$

This numerical value is guaranteed to be accurate up to 550 digits.

We now present the second method for computing volumes, based on semidefinite programming. This was developed by Lasserre and his collaborators. See [2, 10, 11] and references therein. We consider an inclusion of semialgebraic sets $K \subset B \subset \mathbb{R}^n$, where K and B are compact. Here B is a set that serves as a bounding box, like $B = [-1, 1]^n$. We assume that the moments of Lebesgue measure on B are known or easy-to-compute, i.e. we are given

$$\beta_u \;=\; \int_B \mathbf{x}^u dx \;=\; \int_B x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}\, dx_1 dx_2 \cdots dx_n \qquad \text{for } u \in \mathbb{N}^n.$$

The moments m_u of Lebesgue measure on X are unknown. These are our decision variables:

$$m_u \;=\; \int_K \mathbf{x}^u dx \;=\; \int_X x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}\, dx_1 dx_2 \cdots dx_n \qquad \text{for } u \in \mathbb{N}^n. \tag{11}$$

Our aim is to compute $m_0 = \mathrm{vol}(K)$. The idea is to use the following infinite-dimensional linear program: *Maximize the integral* $\int d\mu$*, where* μ *and* $\hat{\mu}$ *range over measures on* $\mathbb{R}^n$*, where* μ *is supported on* K*,* $\hat{\mu}$ *is supported on* B*, and the sum* $\mu + \hat{\mu}$ *is Lebesgue measure on* B*.* The unique optimal solution $(\mu, \hat{\mu})$ to this linear program is given as follows: μ^* is Lebesgue measure on K, $\hat{\mu}^*$ is Lebesgue measure on $B \backslash K$, and the optimal value is $\mathrm{vol}(K) = \int d\mu^*$. This is described in [10, equation (1)]. The linear programming dual is given in [10, equation (2)].

We can express our linear program in terms of the moment sequences $\mathbf{m} = (m_u)$ and $\hat{\mathbf{m}} = (\hat{m}_u)$ of the two unknown measures μ and $\hat{\mu}$. Namely, we paraphrase: *Maximize* m_0 *subject to* $m_u + \hat{m}_u = \beta_u$ *for all* $u \in \mathbb{N}^d$*, where* $\mathbf{m}$ *and* $\hat{\mathbf{m}}$ *are valid moment sequences of measures on* $\mathbb{R}^n$*, with* m *supported on* X*.* This brings us to the moment problem, which is the question how to characterize valid moment sequences. This is problem with a long history in mathematics, and an exact characterization is very difficult. However, in recent years, it has been realized that there are effective necessary conditions. These involve semidefinite programming formulations in finite dimensions, which are built via *localizing matrices*.

Suppose for simplicity that $K = \{x \in \mathbb{R}^n : f(x) \geq 0\}$ is defined by a single polynomial $f = \sum_w c_w x^w$ in n variables, and fix an integer d that is larger than the degree of f. We shall construct three symmetric matrices of format $\binom{n+d}{d} \times \binom{n+d}{d}$ whose entries are linear in the decision variables. The rows and columns of our matrices are indexed by elements $u \in \mathbb{N}^n$ with $|u| = u_1 + \cdots + u_n$ at most d. These correspond to monomials x^u of degree $\leq d$.

Our first matrix $M_d(\mathbf{m})$ has the entry m_{u+v} in row u and column v. Our second matrix $M_d(\hat{\mathbf{m}})$ as the entry $\hat{m}_{u+v}$ in row u and column v. And, finally, our third matrix $M_d(f\mathbf{m})$ has the entry $\sum_w c_w m_{u+v+w}$ in row u and column v. We consider the semidefinite program

$$\begin{gathered}\text{Maximize } m_0 \text{ subject to } m_u + \hat{m}_u = \beta_u \text{ for all } u \in \mathbb{N}^d \text{ with } |u| \leq d, \text{ where} \\ \text{the symmetric matrices } M_d(m), M_d(\hat{m}) \text{ and } M_d(f\mathbf{m}) \text{ are positive semidefinite.}\end{gathered} \tag{12}$$

Here, the third matrix is usually replaced by $M_{d'}(f\mathbf{m})$ where $d' = d - \lceil \deg(f)/2 \rceil$. The objective function value depends on d, it decreases as d increases, and the limit for $d \to \infty$ is equal to the volume of X. Indeed, this sequence of SDP problems is an approximation to the infinite-dimensional linear programming problem above, and convergence is shown in [2].

The remainder of this lecture shows how to solve (12) in practise. It is based on [2, 10, 11], and we discuss an implementation `Mathematica`. This material was developed by Chiara Meroni, and we are very grateful to her for allowing us to include it in these lecture notes.

Our point of departure is the following question: given a sequence of real numbers $\mathbf{m} = (m_\alpha)_\alpha$, does there exist a set S and a measure μ_S supported on S such that (11) holds? Given $d \in \mathbb{N}$, denote by $\mathbb{N}^n_d$ the set of multiindices $\alpha \in \mathbb{N}^n$ such that $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq d$. Fix a set K as above, let $r = \lceil \frac{\deg f}{2} \rceil$, and consider a sequence of real numbers $\mathbf{m} = (m_\alpha)_\alpha$.

The associated *moment matrix* and the *localizing matrix* are respectively

$$M_d(\mathbf{m}) = \Big(m_{\alpha+\beta} \Big)_{\alpha,\beta \in \mathbb{N}^n_d}, \qquad M_{d-r}(f\mathbf{m}) = \Big(\sum_{w \in W} c_w m_{w+\alpha+\beta} \Big)_{\alpha,\beta \in \mathbb{N}^n_d}. \tag{13}$$

The moment matrix has size $\binom{n+d}{d} \times \binom{n+d}{d}$ whereas the localizing matrix has size $\binom{n+d-r}{d-r} \times \binom{n+d-r}{d-r}$. A necessary condition for a sequence $\mathbf{m} = (m_\alpha)_\alpha$ to have a representing measure supported on K is that for every $d \in \mathbb{N}$ the matrix inequalities $M_d(\mathbf{m}) \succcurlyeq 0$ and $M_{d-r}(f\mathbf{m}) \succcurlyeq$

0 hold. This result is a formulation of Putinar's Positivstellensatz [2, Theorem 2.2]. In particular, the positive definiteness of the moment matrix is a necessary condition for $\mathbf{m}$ to have a representing measure; the inequality with the localizing matrix forces the support of the representing measure to be contained in the superlevel set $\{f(x) \geq 0\}$, namely K.

Example 11. As a sanity check, consider the disc $K = \{(x, y) \in \mathbb{R}^2 \mid f = 1 - x^2 - y^2 \geq 0\}$. One can compute its moments via the formula

$$m_{(\alpha_1,\alpha_2)} = ((-1)^{\alpha_1} + 1)\,((-1)^{\alpha_2} + 1)\,\frac{\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)}{4\Gamma\left(\frac{1}{2}(\alpha_1 + \alpha_2 + 4)\right)}.$$

For $d = 3$, the moment and localizing matrices in (13) are

$$M_3(\mathbf{m}) = \begin{pmatrix} \pi & 0 & \frac{\pi}{4} & 0 & 0 & 0 & 0 & \frac{\pi}{4} & 0 & 0 \\ 0 & \frac{\pi}{4} & 0 & \frac{\pi}{8} & 0 & 0 & 0 & 0 & \frac{\pi}{24} & 0 \\ \frac{\pi}{4} & 0 & \frac{\pi}{8} & 0 & 0 & 0 & 0 & \frac{\pi}{24} & 0 & 0 \\ 0 & \frac{\pi}{8} & 0 & \frac{5\pi}{64} & 0 & 0 & 0 & 0 & \frac{\pi}{64} & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{4} & 0 & \frac{\pi}{24} & 0 & 0 & \frac{\pi}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{\pi}{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{24} & 0 & \frac{\pi}{64} & 0 & 0 & \frac{\pi}{64} \\ \frac{\pi}{4} & 0 & \frac{\pi}{24} & 0 & 0 & 0 & 0 & \frac{\pi}{8} & 0 & 0 \\ 0 & \frac{\pi}{24} & 0 & \frac{\pi}{64} & 0 & 0 & 0 & 0 & \frac{\pi}{64} & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{8} & 0 & \frac{\pi}{64} & 0 & 0 & \frac{5\pi}{64} \end{pmatrix}, \quad M_2(f\mathbf{m}) = \begin{pmatrix} \frac{\pi}{2} & 0 & \frac{\pi}{12} & 0 & 0 & \frac{\pi}{12} \\ 0 & \frac{\pi}{12} & 0 & 0 & 0 & 0 \\ \frac{\pi}{12} & 0 & \frac{\pi}{32} & 0 & 0 & \frac{\pi}{96} \\ 0 & 0 & 0 & \frac{\pi}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{96} & 0 \\ \frac{\pi}{12} & 0 & \frac{\pi}{96} & 0 & 0 & \frac{\pi}{32} \end{pmatrix},$$

which are indeed positive definite.

We consider the infinite-dimensional *linear program* on measures whose optimal value is the volume of $K \subset B$. The program is stated in [2, Equation 3.1] and [10, Equation 1]:

$$\begin{aligned} P: \qquad & \max_{\mu_K, \mu_{B\setminus K}} \int \mathrm{d}\mu_K \\ & \text{s.t.} \quad \mu_K + \mu_{B\setminus K} = \mu_B^*, \end{aligned} \tag{14}$$

where μ_S is a positive finite Borel measure supported on S, and μ_B^* is the Lebesgue measure on B. The adjective "infinite-dimensional" refers to the fact that we are optimizing over a set of measures, which is uncountable. Based on the theory of dual Banach spaces, one can talk about dual convex bodies or convex cones, and construct the theory of dual programming. In our case, the dual to the space of positive finite Borel measures is the set of positive continuous functions. This observation leads to the definition of an LP dual to P:

$$\begin{aligned} P^*: \qquad & \inf_{\gamma} \int \gamma \, \mathrm{d}\mu_B^* \\ & \text{s.t.} \quad \gamma \geq \mathbf{1}_K, \end{aligned} \tag{15}$$

where γ is a positive continuous function on B and $\mathbf{1}_K$ is the indicator function of K. It is known that there is no duality gap between P and P^*, i.e. the optimal values of (14) and (15) coincide. Notice that the optimal value of P^* is an infimum and not a minimum, since we are trying to approximate the *discontinuous* indicator function $\mathbf{1}_K$ using continuous functions. This detail is relevant for the slow rate of approximation of the basic method.

The infinite-dimensional LP can be approximated as closely as desired by a hierarchy of finite-dimensional *Semidefinite Programs*, see [5]. The sequence of optimal values of the hierarchy converges monotonically to the optimal value of the LP [2, Theorem 3.2]. There is again a primal and dual version of the SDP problems. In our setting, the primal hierarchy is

$$\begin{aligned} P_d: \qquad & \max_{\mathbf{m},\widehat{\mathbf{m}}} \ m_{\mathbf{0}} \\ & \text{s.t.} \ \ \mathbf{m} + \widehat{\mathbf{m}} = \mathbf{b}, \\ & \qquad M_d(\mathbf{m}) \succcurlyeq 0,\ M_d(\widehat{\mathbf{m}}) \succcurlyeq 0,\ M_{d-r}(f\mathbf{m}) \succcurlyeq 0, \end{aligned} \tag{16}$$

where $\mathbf{m} = (m_\alpha)_{\alpha\in\mathbb{N}^n_{2d}}$, $\widehat{\mathbf{m}} = (\widehat{m}_\alpha)_{\alpha\in\mathbb{N}^n_{2d}}$, and $\mathbf{b}$ collects the moments of B indexed by $\mathbb{N}^n_{2d}$. This formulation is [10, Equation 3]. The optimal value of P_d is an upper bound for $\mathrm{vol}(K)$, since we are optimizing over a larger set. The corresponding dual SDP is [2, Equation 3.6], which is formulated using sums of squares of polynomials. The authors of [2, 10, 11] implemented the SDPs using `GloptiPoly MATLAB`. Our computations in the next examples are performed in `Mathematica`. We are going to include the linear condition $\mathbf{m} + \widehat{\mathbf{m}} = \mathbf{b}$ inside the condition on the moment matrix of $\widehat{\mathbf{m}}$, by imposing directly that $M_d(\mathbf{b}-\mathbf{m}) \succcurlyeq 0$.

Example 12 (TV screen)**.** Fix the convex set $K_1 = \{x, y \in [-1.2, 1.2]^2 \mid f_1(x, y) \geq 0\} \subset \mathbb{R}^2$ where $f_1 = -q$ is the quartic in (3). This shown in Figures 2 and 4. We saw that $\mathrm{vol}(K_1) =$ 3.7081599447.... Let us now try the SDP formulation, with $d = 10$. The moment matrices $M_{10}(\mathbf{m})$ and $M_{10}(\mathbf{b}-\mathbf{m})$ have format 66×66. For instance, the second matrix looks like

$$M_{10}(\mathbf{b}-\mathbf{m}) \;=\; \begin{pmatrix} 4-m_{(0,0)} & -m_{(0,1)} & \frac{4}{3}-m_{(0,2)} & -m_{(0,3)} & \cdots \\ -m_{(0,1)} & \frac{4}{3}-m_{(0,2)} & -m_{(0,3)} & \frac{4}{5}-m_{(0,4)} & \cdots \\ \frac{4}{3}-m_{(0,2)} & -m_{(0,3)} & \frac{4}{5}-m_{(0,4)} & -m_{(0,5)} & \cdots \\ -m_{(0,3)} & \frac{4}{5}-m_{(0,4)} & -m_{(0,5)} & \frac{4}{7}-m_{(0,6)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The localizing matrix $M_8(f_1\mathbf{m})$ has format 45×45. Its (α, β) entry equals

$$m_{\alpha+\beta} - m_{(4,0)+\alpha+\beta} - m_{(0,4)+\alpha+\beta} - \frac{1}{100} m_{(1,1)+\alpha+\beta}.$$

The optimal value of the semidefinite program P_{10} is 4.4644647361..., the optimal value of P_{15} is 4.3251948878..., and for P_{20} we get 4.3329467504.... These numbers are upper bounds for the actual volume, as predicted. However, these bounds are still far from the truth.

Example 13 (Elliptope)**.** Set $f_2(x, y) = 1 - x^2 - y^2 - z^2 + 2xyz$. This defines the elliptope $K_2 = \{x, y \in [-1, 1]^3 \mid f_2(x, y) \geq 0\} \subset \mathbb{R}^3$, shown in Figures 1 and 4. We know $\mathrm{vol}\, K_2 = \frac{\pi^2}{2} =$ 4.934802202.... The upper bounds computing from the semidefinite program for $d = 4, 8, 12$ are respectively 7.3254012963..., 6.6182632506..., and 6.303035372.... This is still pretty bad.

We saw in Examples 12 and 13 that the convergence of the approximation via the SDP method is quite slow. One method to improve the convergence is the method of *Stokes constraints*. This was introduced and analyzed in [6, 10, 11] and we shall now explain it.

In the infinite-dimensional linear program P^* (and in its corresponding SDP hierarchy) we aim to approximate a piecewise-differentiable function, $\mathbf{1}_K$ with continuous functions

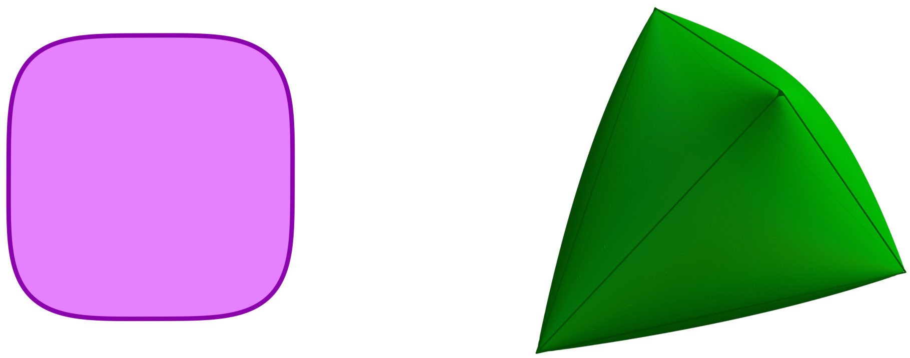


Figure 4: Left: the TV screen from Example 12. Right: the elliptope from Example 13.

(respectively, polynomials). This produces the well-known *Gibbs effect*, creating many oscillations near the boundary of K in the polynomial solutions of the SDP. To remedy this, we add certain linear constraint that do not modify the infinite-dimensional LP problem but add more information to the finite-dimensional SDP. One concrete way to do this uses Stokes' theorem (and its consequences) and the fact that f vanishes on the boundary of K.

Let U be an open set such that the Euclidean closure of U is our set K. Since ∂K is smooth almost everywhere, the classical Stokes Theorem applies. This states that

$$\int_{\partial K} \omega \;=\; \int_K \mathrm{d}\omega$$

for any $(n-1)$-differential form ω on $\mathbb{R}^n$. One consequences of this theorem is *Gauss formula*

$$\int_{\partial K} V(x) \cdot \widehat{n}(x) \,\mathrm{d}\mathcal{H}^{n-1}(x) \;=\; \int_K \operatorname{div} V(x) \,\mathrm{d}x.$$

Here $V(x)$ is a vector field, div denotes divergence, $\widehat{n}(x)$ is the exterior normal vector at $x \in \partial K$, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. If the vector field is a scalar field times a constant vector, say $V(x) = v(x)c$, then we obtain the following equations:

$$c \cdot \left(\int_{\partial K} v(x)\widehat{n}(x) \,\mathrm{d}\mathcal{H}^{n-1}(x) \right) \;=\; \int_K \operatorname{div}\big(v(x)c\big) \,\mathrm{d}x \;=\; c \cdot \left(\int_K \nabla v(x) \,\mathrm{d}x \right)$$

because $\operatorname{div}\big(v(x)c\big) = \nabla v(x) \cdot c + v(x) \operatorname{div} c$ and the divergence of a constant vector is zero. Since this equality must be valid for every $c \in \mathbb{R}^n$, we have

$$\int_{\partial K} v(x)\widehat{n}(x) \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_K \nabla v(x) \,\mathrm{d}x. \tag{17}$$

If $v = 0$ on ∂K, then the left hand side of (17) is zero. This condition can be expressed in terms of measures and distributions, and added to (14) and (15) as in [10, Equation 17 and Remark 3]. In the setting of our SDP, the Stokes constraints are written as follows. Let $v(x) = f(x)x^\alpha$ for any multiindex $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d + 1 - \deg f$. Then we require

$$\nabla\big(f(x)x^\alpha\big)\big|_{x^\beta \to m_\beta} \;=\; 0.$$

We now replacing each monomial with the corresponding moment. This yields n new linear conditions for each α as above.

Example 14. For the SDP in Examples 12 and 13, the Stokes constraints for a given α are:

$$
\begin{aligned}
&K_1:\\
&\alpha_1 m_{\alpha+(-1,0)} - (\alpha_1+4)m_{\alpha+(3,0)} - \alpha_1 m_{\alpha+(-1,4)} - \tfrac{\alpha_1+1}{100} m_{\alpha+(0,1)} = 0,\\
&\alpha_2 m_{\alpha+(0,-1)} - \alpha_2 m_{\alpha+(4,-1)} - (\alpha_2+4)m_{\alpha+(0,3)} - \tfrac{\alpha_2+1}{100} m_{\alpha+(1,0)} = 0,\\
&K_2:\\
&\alpha_1 m_{\alpha+(-1,0,0)} - (\alpha_1+2)m_{\alpha+(1,0,0)} - \alpha_1 m_{\alpha+(-1,2,0)} - \alpha_1 m_{\alpha+(-1,0,2)} + 2(\alpha_1+1)m_{\alpha+(0,1,1)} = 0,\\
&\alpha_2 m_{\alpha+(0,-1,0)} - \alpha_2 m_{\alpha+(2,-1,0)} - (\alpha_2+2)m_{\alpha+(0,1,0)} - \alpha_2 m_{\alpha+(0,-1,2)} + 2(\alpha_2+1)m_{\alpha+(1,0,1)} = 0,\\
&\alpha_3 m_{\alpha+(0,0,-1)} - \alpha_3 m_{\alpha+(2,0,-1)} - \alpha_3 m_{\alpha+(0,2,-1)} - (\alpha_3+2)m_{\alpha+(0,0,1)} + 2(\alpha_3+1)m_{\alpha+(1,1,0)} = 0.
\end{aligned}
$$

Table 1 compares the optimal values of the SDP (13) with and without Stokes constraints.

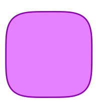
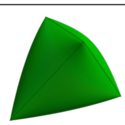
K	Volume	d	without Stokes		with Stokes	
			$\max P_d$	time	$\max P_d$	time
	3.708159...	10	4.464464...	0.621093	3.709994...	0.482376
		15	4.325194...	5.694727	3.708185...	6.413848
		20	4.332946...	29.069448	3.708163...	45.568927
	4.934802...	4	7.325401...	0.124392	5.612716...	0.077315
		8	6.618263...	7.222441	4.976796...	7.178571
		12	6.303035...	696.886298	4.937648...	1105.619231

Table 1: The optimal values of (13) with and without Stokes constraints for Examples 12 and 13. The column "$\max P_d$" displays the optimal value, whereas the column "time" gives the time, in seconds, for running the command `SemidefiniteOptimization` in `Mathematica`.

As Table 1 shows, the convergence with Stokes constraints is much faster than without constraints. The heuristics is that now, with the (dual) Stokes constraints added to P^*, the function we are trying to approximate is not just the indicator function of K. A more precise explanation is given in [11], for a slightly different type of Stokes constraints. The authors, in fact, prove that when adding this new type of constraints, obtained again from Stokes theorem, the optimal solution of the new P^* becomes a minimum. This eliminates any kind of Gibbs effect, and guarantees faster convergence. In [11], the authors mention that, from numerical experiments, it is reasonable to expect that the original Stokes constraints and the new Stokes constraints are equivalent, but there is no formal proof of this statement yet. We close with the remark that general semialgebraic sets fit into this framework; see [2, 10, 11].

References

[1] C. Koutschan: *HolonomicFunctions: A Mathematica package for dealing with multivariate holonomic functions, including closure properties, summation, and integration.* Available at `www3.risc.jku.at/research/combinat/software/ergosum/RISC/HolonomicFunctions.html`.

[2] D. Henrion, J.B. Lasserre and C. Savorgnan: *Approximate volume and integration for basic semialgebraic sets*, SIAM Review **51** (2009) 722–743.

[3] P. Lairez: *Computing periods of rational integrals*, Math. of Computation **85** (2016) 1719–1752.

[4] P. Lairez, M. Mezzarobba and M. Safey El Din: *Computing the volume of compact semi-algebraic sets*, ISSAC 2019, Proceedings of the 2019 ACM International Symposium on Symbolic and Algebraic Computation, 259–266, ACM, New York, 2019.

[5] J.B. Lasserre: *Moments, Positive Polynomials and their Applications*, volume 1 of Imperial College Press Optimization Series. Imperial College Press, London, 2010.

[6] J.B. Lasserre: *Computing Gaussian and exponential measures of semi-algebraic sets*, Adv. Appl. Math. **91** (2017) 137–163.

[7] M. Michałek and B. Sturmfels: *Invitation to Nonlinear Algebra*, Graduate Studies in Mathematics, vol 211, American Mathematical Society, Providence, 2021.

[8] M. Saito, B. Sturmfels, and N. Takayama: *Gröbner Deformations of Hypergeometric Differential Equations*, Algorithms and Computation in Mathematics **6**, Springer, Heidelberg, 1999.

[9] A.-L. Sattelberger and B. Sturmfels: *D-modules and holonomic functions*, `arxiv:1910.01395`, to appear in: Varieties, Polyhedra and Computation, EMS Series of Congress Reports.

[10] M. Tacchi, T. Weisser, J.B. Lasserre and D. Henrion: *Exploiting sparsity for semi-algebraic set volume computation*, Foundations of Computational Mathematics **22** (2022) 161–209.

[11] M. Tacchi, J.B. Lasserre and D. Henrion: *Stokes, Gibbs, and volume computation of semi-algebraic sets*, Discrete and Computational Geometry **69** (2023) 260-283.

[12] D. Zeilberger: *A holonomic systems approach to special functions identities*, Journal of Computational and Applied Mathematics **32** (1990), 321–368.