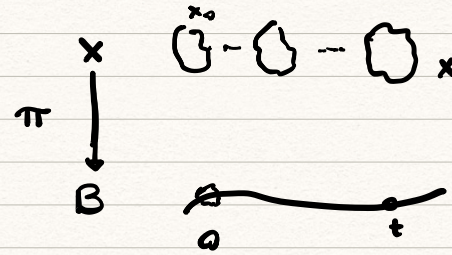
Toric Connections - Degenerations

** "Degeneration" appears 3 time (2x on page 428, 1x in the index) of Cox, Little, Oshea" **

- **Q**: What is a (flat) degeneration?



$X_t = \pi^{-1}(\{t\})$

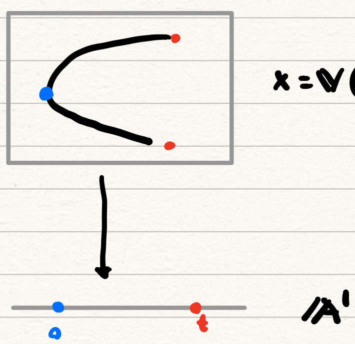
Motto:

- Degeneration $\equiv$ taking limit as $t \longrightarrow 0$

  $\equiv$ general fibre $\rightsquigarrow$ special fibre

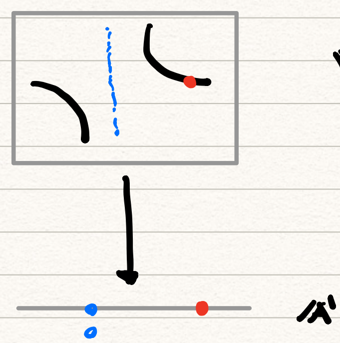
- Flatness $\equiv$ the limit preserves properties of X_t $(t \neq 0)$.

- **Examples**



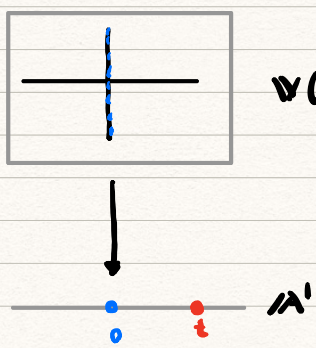
$X = \mathbb{V}(x^2 - t) \subseteq \mathbb{A}^2$

Flat



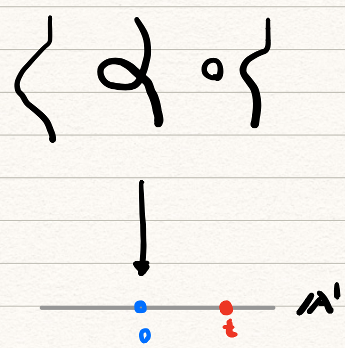
$\mathbb{V}(xt - 1) \subseteq \mathbb{A}^2$

Not Flat



$\mathbb{V}(tx - t) \subseteq \mathbb{A}^2$

Not Flat



$\mathbb{V}(-y^2 + x^3 + x^2 + tx) \subseteq \mathbb{A}^3$

Flat

- **Ex: (Gröbner Bases + Initial Ideals):**

$X \subseteq \mathbb{P}^r$

$$\lambda(t) = \begin{bmatrix} t^{w_0} & & & 0 \\ & t^{w_1} & & \\ & & \ddots & \\ 0 & & & t^{w_r} \end{bmatrix} \subseteq GL(r+1) \quad \text{(} t \neq 0 \text{)}$$

$$f = a_1 \bar{x}^{\bar{v}_1} + a_2 \bar{x}^{\bar{v}_2} + \cdots + a_n \bar{x}^{\bar{v}_n}$$

$$\lambda(t) \cdot f = a_1 t^{\bar{w} \cdot \bar{v}_1} \bar{x}^{v_1} + a_2 t^{\bar{w} \cdot \bar{v}_2} \bar{x}^{v_2} + \cdots\cdots +$$

the limit $t \to 0$ is then

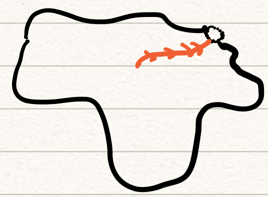
$$\sum_{\substack{\bar{w} \cdot \bar{v}_i \\ \text{minimized}}} a_i \bar{x}^{\bar{v}_i} = in(f) \; !$$

For $t \neq 0$ we have

$$X_t = \lambda(t) \cdot X \cong X$$

$$X_0 \text{ " = " } \lim_{t \to 0} X_t = \mathbb{V}(in_{\bar{w}}(X))$$

Why? "Compact" Moduli



X

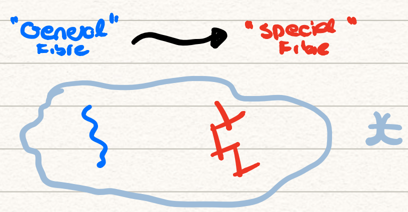
- **Goals of 6.6:**
  - Move from toric varieties over K to toric schemes over a valuation ring.
  - Understand the special fibre of a toric scheme over a valuation ring.
  - Study subvarieties of toric varieties.

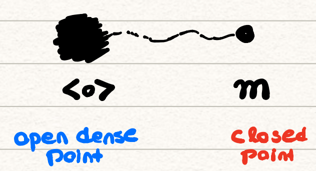
- **Set-up:**

(K, val, Γ_{val}) = non-trivial valued field

$\updownarrow$

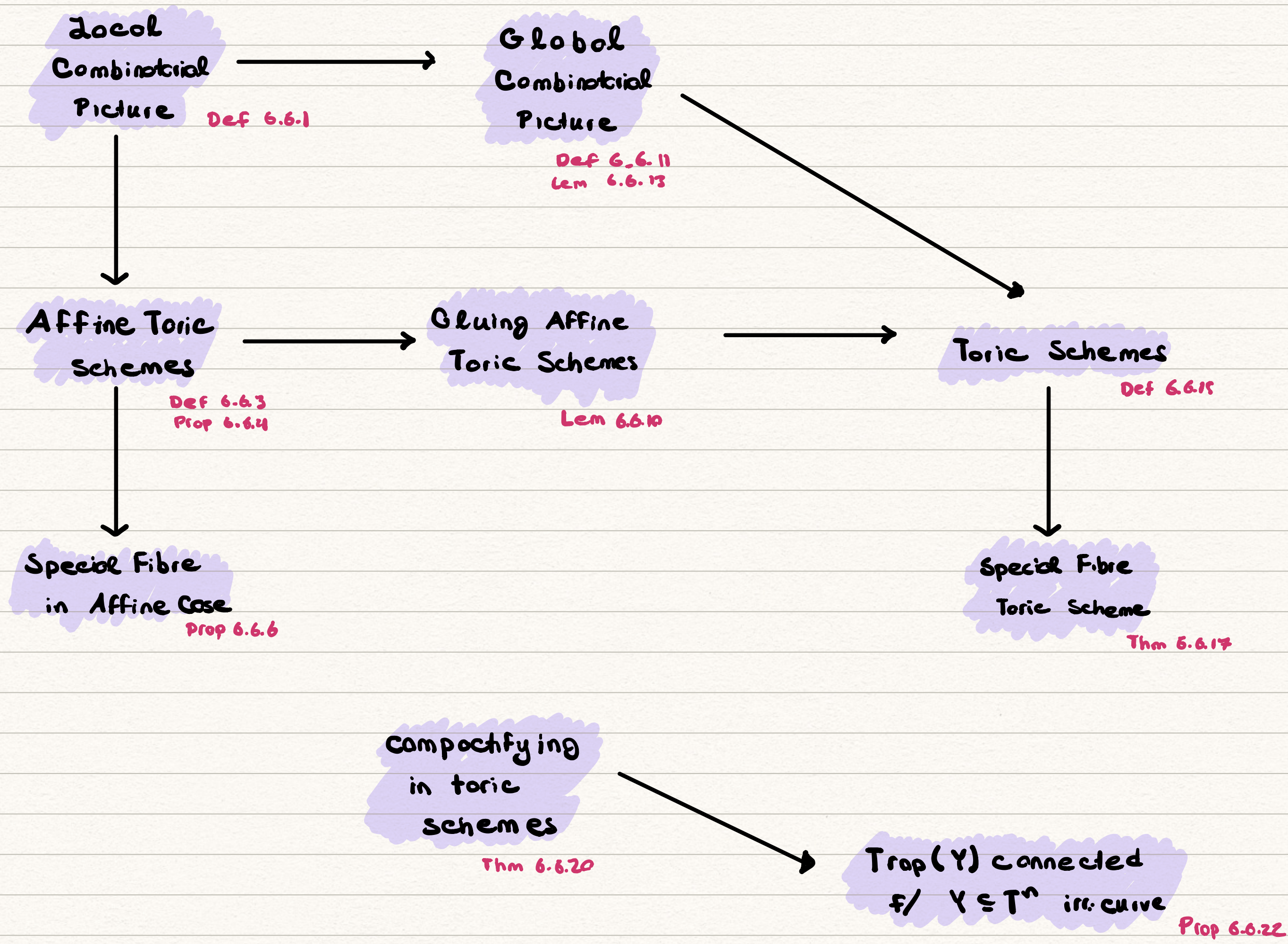
(R, m) = valuation ring





Spec(R)

- **Approach:**



- **Definition 6.6.1:** Given $\bar{u}_1, \bar{u}_2, \dots \bar{u}_r \in N_\mathbb{R}$ and $c_1, c_2, \dots, c_r \in \Gamma_{val}$ the polyhedral cone

$$\sigma = \{ (\bar{w}, s) \in N_\mathbb{R} \times \mathbb{R}_{\geq 0} \mid \bar{w} \cdot u_i \geq -s c_i \quad \text{for } i = 1, 2, \dots r \}$$

is Γ_{val}-admisible if σ does not contain a line.

- Let $\sigma_s = \{ \bar{w} \in N_\mathbb{R} \mid (\bar{w}, s) \in \sigma \}$.

$$K[M]^\sigma = \left\{ \sum_{\bar u \in \sigma_0^\vee \cap M} c_{\bar u}\, \bar x^{\bar u} \;\middle|\; s\,\mathrm{val}(c_{\bar u}) + \bar w \cdot \bar u \geq 0 \quad \text{for all } (\bar w, s) \in \sigma \right\}$$

$\cup\!\!|$

R why??

"twisted Laurent ring"

• Ex 6.6.9:

$$\sigma = \left\{ (w_1, w_2, s) \;\middle|\; \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ s \end{pmatrix} \geq \bar 0 \right\} \subseteq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$$

surface

$$= \mathbb{R}_{\geq 0} \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -8 \\ 3 \\ 2 \end{pmatrix} \right\rangle$$

① F.G. R-algebra

$$K[M]^\sigma = R[t^4 x,\ t^{5/2} xy,\ t\, xy^2,\ t^{1/2} xy^3,\ xy^4] \subseteq K[M]$$

② Flat R-module

$t^a xy \in K[M]^\sigma \iff as + w_1 + w_2 \geq 0$ for all $(\bar w, s) \in \sigma$

$$\Longleftarrow \begin{aligned} 0 + 0 + 1 &\geq 0 \\ 0 + 4 - 1 &\geq 0 \\ 2a + -4 + 1 &\geq 0 \\ 2a + -8 + 3 &\geq 0 \end{aligned} \quad \rightarrow \quad a \geq 5/2$$

$$= R[a, b, c, d, e] \Big/ \left\langle \begin{matrix} ac - b^2,\ ad - t \cdot be,\ ae - t^2 c^2 \\ bd - tc^2,\ be - tcd,\ ce - d^2 \end{matrix} \right\rangle$$

③ Toric Variety
"General Fibre"

$$K[M]^\sigma \otimes_R K = K[a, b, c, d, e] \Big/ \left\langle \begin{matrix} ac - b^2,\ ad - t \cdot be,\ ae - t^2 c^2 \\ bd - tc^2,\ be - tcd,\ ce - d^2 \end{matrix} \right\rangle$$

$C \subseteq \mathbb{P}^4$ rational normal curve
deg 4

④ Union of toric varieties.
"Special Fibre" $(t \to 0)$

$$K[M]^\sigma \otimes_R R/\mathfrak{m} = \Bbbk[a, b, c, d, e] / \mathrm{in}_0(I)$$

$$= \langle a, b, ce - d^2 \rangle \cap \langle ac - b^2, d, e \rangle$$