Math 255 Homework 3

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4.1 Let C, D be nonsingular projective curves of degrees n and m in $\mathbb{P}^2$. Suppose that C is homeomorphic to D. By the degree-genus formula, $\frac{1}{2}(n-1)(n-2) = \frac{1}{2}(m-1)(m-2)$. Hence either $n = m$ or $\{n, m\} = \{1, 2\}$.

4.2 Let C be a nonsingular projective curve in the projective plane not containing $[0,1,0]$, defined by a homogeneous polynomial $P(x,y,z)$. Let $\phi : C \to \mathbb{P}^1$ be defined by $\phi[x,y,z] = [x,z]$. Suppose that C has degree $d > 1$. Since $[0,1,0] \notin C$, the coefficient of y^d in $P(x,y,z)$ is nonzero, so $\frac{\partial P}{\partial y}(x,y,z)$ is not identically zero and has degree $d-1$, thus defining a projective curve D of degree $d-1$. By Theorem 3.8, C and D intersect in at least one point, so by Remark 4.4(ii), ϕ has at least one ramification point.

Suppose $d = 1$. Then the coefficient of y in $P(x,y,z)$ is nonzero and is a constant, so $\frac{\partial P}{\partial y}(x,y,z)$ is never zero, so by Remark 4.4(ii), ϕ has no ramification points and is a homeomorphism.

4.3 Let $P(x,y,z) = y^2z - x^3$ be the polynomial defining D. To compute the singular points, we solve the system of equations $\frac{\partial P}{\partial x} = -3x^2 = 0$, $\frac{\partial P}{\partial y} = 2yz = 0$, $\frac{\partial P}{\partial z} = y^2 = 0$ for the unique solution $[0,0,1]$.

Let $f : \mathbb{P}^1 \to D$ be the map defined by $f[s,t] = [s^2t, s^3, t^3]$. We can define an inverse map by sending $[x,y,z] \mapsto [y,x]$ for $x, y \neq 0$, since $[s^3, s^2t] = [s,t]$, and $[0,0,1] \mapsto [0,1]$. Thus f is a homeomorphism, which means that D has genus 0, contradicting the degree-genus formula which implies that the genus should be 1. Hence the degree-genus formula cannot be applied to singular curves in $\mathbb{P}^2$.

4.4 Let C be the quartic curve in $\mathbb{P}^2$ defined by $yz^3 = (x+z)^4$, and let D be the line in $\mathbb{P}^2$ defined by $z = 0$. Then the map $f : D \to C$ defined by $[s,t,0] \mapsto [st^3, (s+t)^4, t^4]$ is a homeomorphism, where we can define an inverse map by $[x,y,z] \mapsto [x,z,0]$ for $x, z \neq 0$ and $[0,1,0] \mapsto [1,0,0]$.

This does not contradict exercise 4.1 since the quartic curve C is singular. Indeed, if we let $P(x,y,z) = yz^3 - (x+z)^4$ be the polynomial defining C, then we can solve the system of equations $\frac{\partial P}{\partial x} = -4(x+z)^3 = 0$, $\frac{\partial P}{\partial y} = z^3 = 0$, $\frac{\partial P}{\partial z} = 3yz^2 - 4(x+z)^3 = 0$, to get the singular point $[0,1,0]$.

4.5 Let C be an irreducible projective cubic curve in $\mathbb{P}^2$ with a singular point p. By making a projective transformation, we may assume that p is the point $[0,0,1]$. Then by Exercise 3.9, C is equivalent under a projective transformation to $y^2z = x^3$ or $y^2z = x^2(x+z)$. We identify $\mathbb{P}^1$ with the set of lines in $\mathbb{P}^2$ which pass through p, and define the map $f : \mathbb{P}^1 \to C$ by mapping a line L through p to p if it is tangent to C at p, otherwise to the unique other point of intersection of L with C.

If C is a cuspidal cubic, then p is in the image of f since the line $y = 0$ is tangent to C at p. For any other point $[a,b,c]$ on C, the line L through $[a,b,c]$ and p is of the form $bx - ay = 0$. So we can write the map f as $f[s,t] = [-t, s, -t^3/s^2] = [-s^2t, s^3, -t^3]$, which is a homeomorphism by the same argument as in Exercise 4.3.

If f is a nodal cubic, then the lines $y = x, y = -x$ are tangent to C at p. So f maps the two points $[1,1]$ and $[1,-1]$ to p. For the other points of $\mathbb{P}^1$, we can write f as $f[s,t] = [-t, s, t^3/(t^2 - s^2)]$, and we can define an inverse map by sending $[x,y,z] \mapsto [y,-x]$. So f is a continuous bijection and hence a homeomorphism away from p.