# Math 255: Algebraic Curves

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Homework # 5, due Tuesday, November 3

1. *[Fulton 2.26]* Fix a DVR $R$ with quotient field $K$ and maximal ideal $\mathbf{m}$.

   (a) Show that if $z \in K \backslash R$ then $z^{-1} \in \mathbf{m}$.

   (b) Suppose $R \subseteq S \subset K$, where $S$ is a DVR whose maximal ideal contains $\mathbf{m}$. Prove that $R = S$.

2. *[Fulton 2.44]* Let $V$ be a variety in $\mathbb{A}^n$, $I$ its ideal in $k[X_1, \ldots, X_n]$, and $P \in V$. Prove that $\mathcal{O}_P(V)$ is isomorphic to $\mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n)$. How would you perform computations in this ring using `Macaulay2`?

3. *[Fulton 4.27]* Show that the pole set of a rational function on a variety in any multispace $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$ is an algebraic subset.

4. Fix points $P_1, P_2, \ldots, P_n$ in $\mathbb{P}^2_k$ and consider the function

$$\Phi \,:\, \mathbb{N}^{n+1} \to \mathbb{N}\,,\ (d, r_1, \ldots, r_n) \mapsto \dim_k\bigl(V(d, r_1P_1, \ldots, r_nP_n)\bigr)$$

   True or false: The function $\Phi$ is piecewise polynomial.

5. *[Fulton 6.45]* Let $C$ and $C'$ be curves and $f$ a rational map from $C$ to $C'$.

   (a) Prove that $f$ is either dominating or constant.

   (b) Show: if $f$ is dominating then $k(C')$ is a finite extension of $k(C)$.

6. *[Fulton 7.9]* Draw the set of real points of the quartic curve

$$C = \mathcal{V}(X^4 + Y^4 - XYZ^2) \subset \mathbb{P}^2.$$

Write down equations for a nonsingular curve $X$ in some $\mathbb{P}^N$ that is birationally equivalent to $C$? Does there exist a computer program that can perform both of these two tasks for arbitrary plane curves?

7. *[Fulton 8.11]* Let $D$ be a divisor on an irreducible projective curve. Show that $l(D) > 0$ if and only if $D$ is linearly equivalent to an effective divisor.

8. **Bonus Problem:** The *$k$-ellipse* is the Zariski closure in $\mathbb{P}^2_{\mathbb{C}}$ of the curve consisting of all points in $\mathbb{R}^2$ that have a fixed distance from $k$ given generic points. Determine the singularities and the genus of the $k$-ellipse, i.e. answer the Open Question #1 in §5 of `arXiv/0702005`.