Math 255: Algebraic Curves

Bernd Sturmfels, UC Berkeley, Fall 2011
Homework # 3, due Tuesday, September 27

1. *[Kirwan 4.5]* Prove: Every cuspidal cubic curve in the plane $\mathbb{P}^2$ is projectively equivalent to the curve $\{y^2z = x^3\}$, and every nodal cubic curve in $\mathbb{P}^2$ is projectively equivalent to the curve $\{y^2z = x^2(x+z)\}$.

2. *[Kirwan 5.4]* Show that if R is a compact connected Riemann surface then there are no nonconstant holomorphic functions $f : R \to \mathbb{C}$.

3. Prove the Addition Theorem for Weierstrass' $\wp$-function:
   If x, y, z are complex numbers satisfying $x + y + z = 0$ then
$$\det \begin{pmatrix} \wp(x) & \wp'(x) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(z) & \wp'(z) & 1 \end{pmatrix} \quad = \quad 0.$$

4. *[Kirwan 5.12]* Let Λ be a lattice in $\mathbb{C}$. Show that $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \neq 0$.

5. *[Kirwan 5.14]* Let C and $\tilde{C}$ be the non-singular cubic curves in $\mathbb{P}^2$ defined by $y^2z = 4x^3 - g_2xz^2 - g_3z^3$ and $y^2z = 4x^3 - \tilde{g}_2xz^2 - \tilde{g}_3z^3$. Show that there is a projective transformation of $\mathbb{P}^2$ given by a diagonal matrix taking C to $\tilde{C}$ if and only if $J(C) = J(\tilde{C})$, where
$$J(C) = \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

6. Consider the non-singular cubic curve C defined by $y^2z = 4x^3 + xz^2 + z^3$. Using a numerical computation, find a lattice Λ in $\mathbb{C}$ such that $C = C_\Lambda$. To be precise, identify generators $\omega_1, \omega_2 \in \mathbb{C}$ of your lattice Λ and determine their real and imaginary parts up to ten digits of accuracy.

7. Let $\tilde{C}$ denote the *dual curve* of the previous C, as in [Kirwan, 5.18]. Find the defining polynomial of $\tilde{C}$ and identify all singular points of $\tilde{C}$.