Math 255: Algebraic Curves

Bernd Sturmfels, UC Berkeley, Fall 2011
Homework # 1, due Thursday, September 1

1. *[Kirwan 2.2]* Find the singular points, and the tangent lines at the singular points, of each of the following curves in $\mathbb{C}^2$:

   (a) $y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy = 0$.

   (b) $x^4 + y^4 - x^2y^2 = 0$.

   (c) $y^2 = x^3 - x$.

2. *[Kirwan 2.4]* Let (a, b) be a singular point of an affine curve C defined by a polynomial $P(x, y)$. Show that (a, b) is an ordinary double point if and only if, at the point (a, b),

$$\left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 \neq \left(\frac{\partial^2 P}{\partial x^2}\right)\left(\frac{\partial^2 P}{\partial y^2}\right).$$

3. *[Kirwan 2.5]* Let C be an affine curve defined by a polynomial $P(x, y)$ of degree d. Show that if (a, b) is a point of multiplicity d in C then $P(x, y)$ is a product of d linear factors, so C is the union of d lines through (a, b).

4. *[Kirwan 2.7]* Show that a complex algebraic curve in $\mathbb{C}^2$ is never compact.

5. *[Kirwan 2.9]* For which values of $\lambda \in \mathbb{C}$ are the following projective curves in $\mathbb{P}^2$ nonsingular? Describe the singularities when they exist:

   (a) $x^3 + y^3 + z^3 + \lambda xyz = 0$.

   (b) $x^3 + y^3 + z^3 + \lambda(x + y + z)^3 = 0$.

6. *[Kirwan 2.8]* The multiplicity of a point $(a : b : c)$ of a projective curve $P(x, y, z) = 0$ is the smallest integer m such that

$$\frac{\partial^m P}{\partial x^i \partial y^j \partial z^k}(a, b, c) \neq 0$$

for some i, j, k such that $i + j + k = m$. Find the singular points and their multiplicities for the following projective curves:

(a) $xy^4 + yz^4 + xz^4 = 0$.

(b) $x^2y^3 + x^2z^3 + y^2z^3 = 0$.

(c) $y^2z = x(x - z)(x - \lambda z) = 0$ for $\lambda \in \mathbb{C}$.

(d) $x^n + y^n + z^n = 0$ for any integer $n > 0$.