Tropical decomposition of symmetric tensors

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1 Introduction

In [2], Comon et al. give an algorithm for decomposing a symmetric tensor into a sum of symmetric rank 1 tensors, provided that some rank conditions hold. This type of problem is very relevant to the field of algebraic statistics: in this context, the space of symmetric rank 1 tensors corresponds to the model of independent, identically distributed random variables, and the space of symmetric tensors of non-negative rank at most k corresponds to the k-fold mixture of this model. In this paper, we consider a tropical version of this problem.

Specifically, we investigate the notion of *tropical symmetric rank* of a symmetric tensor X, defined as the smallest number of symmetric tensors of tropical rank 1 whose sum is X. (A symmetric tensor has *tropical rank* 1 if can be written as a tropical tensor product of a single vector). We show that the tensors of tropical symmetric rank at most k are precisely the k^{th} tropical secant variety of the linear space of symmetric tensors of rank 1. We then to consider the more general setting of tropical secant varieties of linear spaces. In Section 3, we describe a theorem due to Develin characterizing these secant varieties in terms of polytopes [3]. In Section 4, we use this characterization to give an algorithm which, given a point $s \in \mathbb{R}^m$ and a linear subspace $V \subseteq \mathbb{R}^m$, returns a minimum length tropical decomposition of s into vectors lying in V or indicates that no finite decomposition exists. Finally, equipped with our polytopal characterization of tropical secant varieties, we specialize to the case of symmetric tropical rank 1 tensors in Section 5. We characterize those tensors of finite rank, and prove some bounds on the maximum rank of a tensor of order d and size n. The `matlab` code implementing this algorithm is included in Section 6.

2 Symmetric tensors and tropical rank

Our setting is the *max-plus algebra* $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where

$$x \oplus y = \max\{x, y\} \text{ and } x \otimes y = x + y, \text{ for all } x, y \in \mathbb{R}.$$

The element $-\infty$ is the additive identity, and 0 is the multiplicative identity. The operations $\oplus$ and $\otimes$ give a semiring structure on $\mathbb{R} \cup \{-\infty\}$ which we call the *tropical semiring.*

A *tensor* of order d and size n is an array $X = (x)_{i_1 \cdots i_d}$ of real numbers, where $1 \leq i_1, \ldots, i_d \leq n$. We say that X is *symmetric* if it is invariant under permuting indices, that is, if

$$x_{i_1 \cdots i_d} = x_{i_{\sigma 1} \cdots i_{\sigma d}}$$

for all $\sigma \in S_d$. Thus, a symmetric tensor of order 2 is a symmetric matrix, with $x_{ij} = x_{ji}$ for all indices i, j. We note that the space of symmetric tensors of fixed order d and size n is a linear space; one may check that its dimension is $\binom{n+d-1}{d}$.

A tensor has *tropical rank 1* if it can be decomposed as a tropical outer product of d vectors in $\mathbb{R}^n$, that is, if there exist vectors $z^1, \ldots, z^d \in \mathbb{R}^n$ such that for all choices of indices $i_1, \ldots, i_d$, we have

$$x_{i_1 \cdots i_d} = z^1_{i_1} \otimes \cdots \otimes z^d_{i_d}.$$

We emphasize that $\otimes$ denotes tropical multiplication of scalars, that is, addition in the usual sense.

For example, the matrix

$$\begin{pmatrix} 2 & 6 \\ 3 & 7 \end{pmatrix}$$

has tropical rank 1 since it is the tropical outer product of $(\lambda, 1 + \lambda)$ and $(2 - \lambda, 6 - \lambda)$ for any scalar λ.

One can check that the following averaging condition characterizes symmetric, tropical rank 1 tensors.

Proposition 2.1. *Let X be a tensor of order d and size n. Then the following are equivalent:*

1. *X is symmetric and has tropical rank 1,*
2. *For all choices of indices $i_1, \ldots, i_d$, we have*

$$x_{i_1 \cdots i_d} = \frac{x_{i_1 \cdots i_1} + \cdots + x_{i_d \cdots i_d}}{d},$$

and

3. *X can be written as a tropical outer product of a vector $v \in \mathbb{R}^n$ with itself d times.*

Thus the space of symmetric tropical rank 1 tensors is an n- dimensional linear subspace of the $\binom{n+d-1}{d}$-dimensional space of symmetric tensors.

Given two tensors X and X' of the same order and size, we may add them tropically by taking the classical maximum in each coordinate. For example,

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \oplus \begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix}. \tag{1}$$

Definition. A symmetric tensor X has *tropical symmetric rank* k if X is the tropical sum of k symmetric tropical rank 1 tensors but is not the tropical sum of $k-1$ such tensors. If X does not have tropical symmetric rank k for any integer k then we say that its rank is infinite.

For example, the two tensors on the left hand side of Equation 1 have tropical symmetric rank 1. Since the tensor on the right hand side does not have rank 1 by Proposition 2.1, Equation 1 shows that it has rank 2.

Note that, unlike the classical case, not all symmetric tensors have finite rank. We will characterize symmetric tensors of finite rank in Propositions 5.2 and 5.3. For a second example, illustrating yet another difference between the classical setting and the tropical setting, consider the tropical equation

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$

Since the matrices on the left have tropical rank 1, the matrix on the right has tropical rank at most 2. However, Proposition 5.2 shows that its tropical symmetric rank is infinite. This is in marked contrast to the classical situation, where rank and symmetric rank coincide for all symmetric matrices and are conjectured to coincide for all symmetric tensors, as discussed in [1].

In the rest of this paper, *rank* will always mean tropical symmetric rank.

3 Tropical secant varieties of linear spaces

We may generalize the above setting as follows.

Definition. Let L be a linear subspace of $\mathbb{R}^m$. Then we define the k^{th} *tropical secant variety* of L, denoted $Sec^k(L)$, to be the set of points

$$\{x \in \mathbb{R}^m \mid x = y^1 \oplus \cdots \oplus y^k \text{ for some } y^1, \ldots, y^k \in L\}$$

where $\oplus$ denotes coordinate-wise tropical addition.

Thus $Sec^1(L) = L$ and $Sec^1(L) \subseteq Sec^2(L) \subseteq \cdots$. Linear subspaces are of particular interest since they correspond to tropicalizations of toric varieties. Note that if L is the n- dimensional linear subspace of $\mathbb{R}^{\binom{n+d-1}{d}}$ corresponding to symmetric tropical rank 1 matrices, then $Sec^k(L)$ corresponds precisely to those symmetric matrices of rank at most k.

The following theorem, due to Develin, gives a polytopal characterization of membership in the k^{th} tropical secant variety of a linear space. We let $\mathbf{1}$ denote the vector $(1, \ldots, 1)$.

Theorem 3.1. *[3] Let V be a linear subspace of $\mathbb{R}^m$ containing $\mathbf{1}$, and let $s = (s_1, \ldots, s_m)$ be a vector in $\mathbb{R}^m$. Let V be spanned by the vectors $v_1, \ldots, v_n, \mathbf{1}$, and let W_V be the point configuration $w_1, \ldots, w_m$ whose m points are the m columns of the matrix*

$$\begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix}$$

whose rows are $v_1, \ldots, v_n$. Let $w'_i \in \mathbb{R}^{n+1}$ be the vector (w_i, s_i), and let C denote the convex polytope in $\mathbb{R}^{n+1}$ formed by taking the convex hull of $w'_1, \ldots, w'_m$.

Then s lies in the k^{th} tropical secant variety of V if and only if there exists a set of at most k facets of the lower envelope of C which meet each w'_i.

We note that choosing a different generating set for V would only change the point configuration W_V by an affine isomorphism, so that the conclusion of the theorem does indeed hold for any choice of $v_1, \ldots, v_n$.

4 A an algorithm for computing tropical secant varieties

Theorem 3.1 provides a criterion for membership in the k^{th} tropical secant variety of a linear subspace V. We can turn this criterion into an algorithm as follows.

- Input: A list of n vectors $v_1, \ldots, v_n \in \mathbb{R}^m$, such that $V = span\{v_1, \ldots, v_n, \mathbf{1}\}$, and a vector $s \in \mathbb{R}^m$.
- Output: A list of vectors $y_1, \ldots, y_k \in V$ such that $s = y_1 \oplus \cdots \oplus y_k$ and s has no such decomposition with fewer than k terms; or -1 if s has no finite decomposition as a tropical sum of vectors in V.

1. Compute the lower facets of the convex hull C of the points $w'_1, \ldots, w'_m$ as in Theorem 3.1, represented as a list of point-facet incidences.
2. If some point w'_i is incident to no facets, return -1.
3. Else, find a smallest set of facets meeting all w'_i's. Represent these facets as a list $f_1, \ldots, f_k$ of affine functionals on $\mathbb{R}^n$; that is $f_i : (x_1, \ldots, x_n) \mapsto a_{i1}x_1 + \cdots + a_{in}x_n + b_i$ for real numbers a_{ij}, b_i.
4. For $i = 1, \ldots, k$, let $y_i = (f_i(w_1), \ldots, f_i(w_m)) \in \mathbb{R}^m$. Return the list $y_1, \ldots, y_k$.

We claim, and [3] proves, that $y_1 \oplus \cdots \oplus y_k = s$, and furthermore, that this decomposition is of minimum length.

Some remarks on this algorithm are in order. In Step 1, we can use the matlab command `convhulln`, which returns a list of point- facet incidences of the convex hull C of n given points. Since we are only interested in the facets in the lower envelope of C, we add a "fake" point $(0, \ldots, 0, \infty)$ to our list and then ignore all facets in the output that are incident to this point.

In Step 3, we compute a smallest set of facets meeting all w'_i's. This task is no harder than computing a minimum dominating set in a bipartite graph. We make this precise as follows.

Definition. Given a graph $G = (V, E)$, we say that a vertex v is *covered* by a vertex w if $v = w$ or $\{v, w\} \in E$. A *dominating set* in a graph $G = (V, E)$ is a subset D of V such that every vertex $v \in V$ is covered by some vertex in D.

Definition. A *bipartite graph* is a graph $G = (V, E)$ whose vertex set can be partitioned into subsets A and B such that every edge $\{v, w\} \in E$ has precisely one of v or w in A. We write $G = (A, B, E)$ in this case.

Suppose we have a list of incidences between a set of facets F and a set of points X of a polytope. We wish to compute a smallest set of facets meeting all points in X. Let us construct a bipartite graph $G = (A, B, E)$ as follows. We let $A = F \cup \{a_1, a_2\}$ and $B = X \cup \{b\}$. For each $f \in F$

and $x \in X$, we add an edge $\{f, x\}$ if and only if x is incident to f in the polytope. For each $f \in F$, we add edge $\{f, b\}$. Finally, we add edges $\{a_1, b\}$ and $\{a_2, b\}$.

Proposition 4.1. *Let F be a set of facets and X be a set of points of a polytope, and let $G = (A, B, E)$ be the bipartite graph constructed above. Let D be any minimum size dominating set of G. For each $x \in D \cap X$, let f_x be any neighbor of x, so $f_x \in F$. Then*

$$F' = (D \cap F) \cup \{f_x \mid x \in D \cap X\}$$

is a smallest set of facets which meet all vertices in X.

Proof. First, we claim $b \in D$. If not, then we must have $\{a_1, a_2\} \subseteq D$, but then $(D \cup \{b\}) \setminus \{a_1, a_2\}$ would be a dominating set that is smaller than D, contradiction. So $b \in D$, and therefore $a_1, a_2 \notin D$. Now let

$$D' = (D \cap A) \cup \{f_x \mid x \in D \cap X\} \cup \{b\}.$$

We claim D' is a dominating set of G. Indeed, every vertex in G is covered by this set: the vertices in $X \setminus D$ are covered by $D \cap A$; the vertices in $X \cap D$ are covered by $\{f_x \mid x \in D \cap X\}$; and the vertices in $A \cup \{b\}$ are covered by b.

Next, writing

$$D = (D \cap A) \cup (D \cap X) \cup \{b\}$$

and noting that $|\{f_x \mid x \in D \cap X\}| \leq |D \cap X|$, we have $|D'| \leq |D|$, so D' is a dominating set of G of minimum size.

Finally, noting that $F' = D' \setminus \{b\}$, we have that F' covers X; that is, F' is a set of facets meeting all points in X in the polytope. Finally, if some other subset F'' of F meets all points in X, then one may check that $F'' \cup \{b\}$ is a dominating set of G. So $|F'' \cup \{b\}| \geq |F' \cup \{b\}|$ and $|F''| \geq |F'|$ as desired. $\square$

Thus, computing a smallest subset of facets F meeting all points in X is no harder than computing a minimum dominating set in a bipartite graph on $|F| + |X| + 3$ vertices.

Unfortunately, computing a minimum dominating set, even for a bipartite graph, is an NP-hard problem in general. A future direction for research would be to determine whether the bipartite graph constructed above has any nice structure arising from the geometry of the polytope which one could exploit to obtain a faster algorithm.

The code implementing this algorithm, written in `matlab`, is included in the appendix to this paper. It is not yet optimized to run quickly. A future project would be to improve the running time of this code using the ideas described above.

5 An application to symmetric tensors of tropical rank 1

We now discuss the implications of Theorem 3.1 in the context of the linear space of symmetric tropical rank 1 tensors.

Proposition 5.1. *Let S be the n-dimensional linear space of symmetric tropical rank 1 tensors of order d and size n. Then the point configuration W_V associated to V is a d-fold subdivision of the simplex Δ^{n-1}, given by*

$$W_V = \{(w_1, \dots, w_{n-1}) \in \mathbb{R}^{n-1} \mid w_i \in \{0, \dots, d\}, \sum_{i=1}^{n-1} w_i \leq d\}.$$

Proof. We regard a symmetric tensor X of order d and size n as a row vector in $\mathbb{R}^{\binom{n+d-1}{d}}$ whose coordinates are indexed by nondecreasing sequences $(i_1, \dots, i_d)$ of indices where $1 \leq i_1 \leq \cdots \leq i_d \leq n$. For each j with $1 \leq j \leq n$, let v^j denote the row vector in $\mathbb{R}^{\binom{n+d-1}{d}}$ whose value in coordinate $(i_1, \dots, i_d)$ equals the number of times that j appears amongst this sequence. Note that the v^i's are linearly independent, since the projection to the n coordinates corresponding to constant sequences $(i, \dots, i)$ are, and since S has dimension n, the vectors $v^1, \dots, v^n$ must be a basis for S. Moreover, since

$$\sum_{j=1}^{n} v^j = d \cdot \mathbf{1},$$

we have that $\{v^1, \dots, v^{n-1}, \mathbf{1}\}$ spans S. Finally, we note that the columns of the matrix whose rows are $v^1, \dots, v^{n-1}$ are precisely the set of points described in the proposition. Indeed, the vector in $\mathbb{R}^{n-1}$ associated to the column $(i_1, \dots, i_d)$ of this matrix has as its j^{th} coordinate the number of times j appears in the sequence $(i_1, \dots, i_d)$. □

This geometric interpretation of symmetric rank 1 tensors allows us to apply Theorem 3.1 to describe their k^{th} secant varieties, that is, symmetric tensors of rank at most k. We start by characterizing symmetric matrices of finite rank.

Proposition 5.2. *Let X be a symmetric $n \times n$ matrix. Then X has finite tropical symmetric rank if and only if*

$$x_{ij} \leq \frac{x_{ii} + x_{jj}}{2} \tag{2}$$

for all i, j with $1 \leq i, j \leq n$.

Proof. Suppose X has finite rank, so that X is the coordinate-wise maximum of $Y^1, \ldots, Y^m$, where each Y^i is a symmetric tropical rank 1 matrix. Suppose for a contradiction that there exist indices i, j such that (2) does not hold. Since $x_{ij} = {y^k}_{ij}$ for some k, we have

$$\frac{x_{ii} + x_{jj}}{2} < x_{ij} = {y^k}_{ij} = \frac{{y^k}_{ii} + {y^k}_{jj}}{2},$$

and hence

$$x_{ii} < {y^k}_{ii} \text{ or } x_{jj} < {y^k}_{jj},$$

contradiction.

Conversely, if (2) is satisfied for all i and j, then we can give an explicit decomposition of X into symmetric rank 1 matrices. Note that the size of this particular decomposition will in general be far from optimal. For each k with $1 \leq k \leq n$, let Y^{kk} be the matrix whose (k, k) entry is x_{kk} and whose remaining entries are all $-\infty$. For each i, j with $1 \leq i < j \leq n$, pick any s and t subject to the conditions

$$s \leq x_{ii}, t \leq x_{jj}, \text{ and } \frac{s+t}{2} = x_{ij}.$$

That such s and t exist is guaranteed by (2). Now let Y^{ij} be the matrix whose restriction to the 2×2 submatrix of rows and columns $\{i.j\}$ is

$$\begin{pmatrix} s & x_{ij} \\ x_{ij} & t \end{pmatrix}$$

and whose remaining entries are all $-\infty$.

Then it is straightforward to check, using Proposition 2.1, that for each i, j with $1 \leq i \leq j \leq n$, the matrix Y^{ij} has tropical rank 1, and furthermore that

$$X = \bigoplus_{1 \leq i \leq j \leq n} Y^{ij}$$

as desired. ∎

One could alternatively view Proposition 5.2 through the lens of Theorem 3.1. In this way, one may check that in the point configuration corresponding to symmetric $n \times n$ matrices, the point associated to x_{ij} is the midpoint of the points associated to x_{ii} and x_{jj}, and furthermore that these are the only instances in which one point lies in the convex hull of others. Then for a given symmetric matrix, which we regard as a height vector on the point configuration, we need only require that the height at x_{ij} is at most the average of the heights at x_{ii} and x_{jj}, and so Proposition 5.2 follows.

Unfortunately, for a general tensor of order d, the condition of having finite rank does not have quite such a nice description. In this case, there are points in the corresponding configuration which lie in the convex hull of full-dimensional regions of our subdivision. The following characterization of finite rank symmetric tensors follows immediately from Theorem 3.1 and Proposition 5.1, however.

Proposition 5.3. *Let X be a symmetric tensor of order d and size n. Let W_V be the set*

$$\{(w_1, \ldots, w_{n-1}) \in \mathbb{R}^{n-1} \mid w_i \in \{0, \ldots, d\}, \sum_{i=1}^{n-1} w_i \leq d\} \subset \mathbb{R}^{n-1},$$

as in Proposition 5.1.

Then X has finite tropical symmetric rank if and only if, for every $w, z^1, \ldots, z^n \in W_V$ such that

$$w = \lambda_1 z^1 + \cdots \lambda_n z^n$$

for some scalars λ_j satisfying $\lambda_j \geq 0$ and $\sum \lambda_j = 1$, the inequality

$$x_{\underbrace{1\cdots1}_{w_1}\underbrace{2\cdots2}_{w_2}\cdots\underbrace{n\cdots n}_{1-\sum w_i}} \leq \sum_{j=1}^{n} \lambda_j x_{\underbrace{1\cdots1}_{z^j{}_1}\underbrace{2\cdots2}_{z^j{}_2}\cdots\underbrace{n\cdots n}_{1-\sum z^j{}_i}}$$

holds.

In this regard, then, the tropical setting is quite different from the classical setting: it is by no means the case that every symmetric tensor has finite rank.

We can ask, however, for the maximum finite rank attained by a symmetric tensor of order d and size n. We first derive a naive upper bound from Theorem 3.1.

Proposition 5.4. *A symmetric tensor of order d and size n either has infinite rank or has rank at most $\binom{n+d-1}{d} - n + 1$.*

Proof. A regular subdivision of $\binom{n+d-1}{d}$ points in $\mathbb{R}^{n-1}$ can have at most $\binom{n+d-1}{d} - n + 1$ regions. The result then follows from Theorem 3.1. $\square$

In some small cases, the rank of a symmetric matrix is bounded by its size. We prove these cases below.

Proposition 5.5. *A 2×2 symmetric matrix of finite rank has rank at most 2.*

Proof. Let

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where since X has finite rank, we have $b \leq \frac{a+c}{2}$ by Proposition 5.2. Then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & b \\ b & 2b-a \end{pmatrix} \oplus \begin{pmatrix} -\infty & -\infty \\ -\infty & c \end{pmatrix}$$

is a tropical decomposition of X into two symmetric rank 1 tensors. $\square$

Proposition 5.6. *A 3×3 symmetric matrix of finite rank has rank at most 3.*

Proof. Let

$$X = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

where since X has finite rank, we have

$$b \leq \frac{a+d}{2}, c \leq \frac{a+f}{2}, e \leq \frac{d+f}{2}$$

by Proposition 5.2. Then one can check that

$$\begin{pmatrix} a & b & -\infty \\ b & 2b-a & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix} \oplus \begin{pmatrix} -\infty & -\infty & -\infty \\ -\infty & d & e \\ -\infty & e & 2e-d \end{pmatrix} \oplus \begin{pmatrix} 2c-f & -\infty & c \\ -\infty & -\infty & -\infty \\ c & -\infty & f \end{pmatrix}$$

is a decomposition of X into symmetric tensors of rank 1. $\square$

Proposition 5.7. *A 4×4 symmetric matrix of finite rank has rank at most 4.*

Proof. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{pmatrix}$$

be a symmetric matrix of finite rank. By Proposition 5.2, we have

$$x_{ij} \leq \frac{x_{ii} + x_{jj}}{2} \tag{3}$$

for each i, j. We use this fact repeatedly in the following argument.

We first claim that

$$x_{23} + x_{11} \geq x_{12} + x_{13} \quad \text{or} \quad x_{13} + x_{22} \geq x_{12} + x_{23},$$

for if not, then adding the inequalities yields

$$x_{11} + x_{22} < 2x_{12},$$

contradicting (3). Then, by permuting indices 1 and 2 if necessary, we may assume that

$$x_{23} + x_{11} \geq x_{12} + x_{13}. \tag{4}$$

Now, suppose that

$$x_{24} + x_{11} \geq x_{12} + x_{14} \quad \text{and} \quad x_{34} + x_{11} \geq x_{13} + x_{14}. \tag{5}$$

Then let

$$\begin{aligned}
Y_1 &= \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{12} & 2x_{12} - x_{11} & x_{12} + x_{13} - x_{11} & x_{12} + x_{14} - x_{11} \\ x_{13} & x_{12} + x_{13} - x_{11} & 2x_{13} - x_{11} & x_{13} + x_{14} - x_{11} \\ x_{14} & x_{12} + x_{14} - x_{11} & x_{13} + x_{14} - x_{11} & 2x_{14} - x_{11} \end{pmatrix} \\
Y_2 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & x_{22} & x_{23} & -\infty \\ -\infty & x_{23} & 2x_{23} - x_{22} & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} \\
Y_3 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & x_{33} & x_{34} \\ -\infty & -\infty & x_{34} & 2x_{34} - x_{33} \end{pmatrix} \\
Y_4 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & 2x_{24} - x_{44} & -\infty & x_{24} \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & x_{24} & -\infty & x_{44} \end{pmatrix}
\end{aligned}$$

Then we can check that $X = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$ and the Y_i's have rank 1.

So we may assume that not both inequalities in (5) hold. By permuting indices 2 and 3, we may assume that

$$x_{24} + x_{11} < x_{12} + x_{14}. \tag{6}$$

Then we claim that

$$x_{12} + x_{44} \geq x_{14} + x_{24}, \tag{7}$$

for otherwise (6) yields $x_{11} + x_{44} < 2x_{14}$, contradicting (3). Now let

$$\begin{aligned}
Y_1 &= \begin{pmatrix} x_{11} & x_{12} & x_{13} & -\infty \\ x_{12} & 2x_{12} - x_{11} & x_{12} + x_{13} - x_{11} & -\infty \\ x_{13} & x_{12} + x_{13} - x_{11} & 2x_{13} - x_{11} & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} \\
Y_2 &= \begin{pmatrix} 2x_{14} - x_{44} & x_{14} + x_{24} - x_{44} & -\infty & x_{14} \\ x_{14} + x_{24} - x_{44} & 2x_{24} - x_{44} & -\infty & x_{24} \\ -\infty & -\infty & -\infty & -\infty \\ x_{14} & x_{24} & -\infty & x_{44} \end{pmatrix} \\
Y_3 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & x_{33} & x_{34} \\ -\infty & -\infty & x_{34} & 2x_{34} - x_{33} \end{pmatrix} \\
Y_4 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & x_{22} & x_{23} & -\infty \\ -\infty & x_{23} & 2x_{23} - x_{22} & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}
\end{aligned}$$

Then one may check, using (3), (4), and (7), that $X = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$ and the Y_i's have rank 1.

□

Observation. It is not true, however, that every 5×5 symmetric matrix of finite rank has rank at most 5. For example, we claim that the matrix

$$X = \begin{pmatrix} 0 & -4 & -16 & -16 & -2 \\ -4 & 0 & -4 & -1 & -8 \\ -16 & -4 & 0 & -8 & -1 \\ -16 & -1 & -8 & 0 & -4 \\ -2 & -8 & -1 & -4 & 0 \end{pmatrix}$$

has tropical symmetric rank 6. We let

```
A = [2,0,0,0,0,1,1,1,1,0,0,0,0,0,0;
      0,2,0,0,0,1,0,0,0,1,1,1,0,0,0;
      0,0,2,0,0,0,1,0,0,1,0,0,1,1,0;
      0,0,0,2,0,0,0,1,0,0,1,0,1,0,1;
      1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]
```

and

```
x = [0,0,0,0,0,-4,-16,-16,-2,-4,-1,-8,-8,-1,-4]
```

and run `tsv(A,x)` in `matlab`, the code for which is given in the appendix to this paper, to obtain the output

```
facets =

      4     0   -12   -12    -8
      2    -4   -14   -14    -4
      4     8    -4    -4   -16
     -4     8     4     4   -16
     -7     8     1     7   -16
    -12     0     4    -4    -8
    -15    -6     1    -7    -2
     -9     6    -1     7   -14
    -12     0    -4     4    -8
     -2    -8   -14   -14     0
    -12    -8    -4    -4     0
    -15    -8    -1    -7     0


incidences =

      1     6     7     8    12     0     0     0     0     0
0     0     0     0     0
      1     7     8     9    12     0     0     0     0     0
0     0     0     0     0
      2     6     7     8    12     0     0     0     0     0
0     0     0     0     0
      2     7     8    10    12    13     0     0     0     0
0     0     0     0     0
```

```
       2     8    11    12    13     0     0     0     0     0
0      0     0     0     0
       3     7    10    12    13     0     0     0     0     0
0      0     0     0     0
       3     7    12    13    14     0     0     0     0     0
0      0     0     0     0
       4     8    11    12    13     0     0     0     0     0
0      0     0     0     0
       4     8    12    13    15     0     0     0     0     0
0      0     0     0     0
       5     7     8     9    12     0     0     0     0     0
0      0     0     0     0
       5     7     8    12    13    15     0     0     0     0
0      0     0     0     0
       5     7    12    13    14     0     0     0     0     0
0      0     0     0     0


ans =

       0    -8   -32   -32    -8    -4   -16   -16    -4   -20
-20     -8   -32   -20   -20
       0   -12   -32   -32    -4    -6   -16   -16    -2   -22
-22     -8   -32   -18   -18
     -24     0    -8    -8   -16   -12   -16   -16   -20    -4
-4     -8    -8   -12   -12
     -32   -14     0   -16    -2   -23   -16   -24   -17    -7
-15     -8    -8    -1    -9
     -32    -2   -16     0   -14   -17   -24   -16   -23    -9
-1     -8    -8   -15    -7
     -24   -16    -8    -8     0   -20   -16   -16   -12   -12
-12     -8    -8    -4    -4
```

to obtain a smallest decomposition of X into 6 rank 1 tensors, represented by row vectors in the output. One may check that their tropical sum is x, as desired.

The above discussion suggests the following question.

Question 1. *Let $n \geq 5$. What is the maximum finite rank of a symmetric*

$n \times n$ matrix?

6 Appendix: a `matlab` program for computing tropical secant varieties

The following is our code for a `matlab` function `tsv.m`, which takes as input a matrix M and a vector s whose length equals the number of columns of M. We require that the rows of M are linearly independent and that the last row of M is the vector **1**. (These requirements are just artifacts of the particular implementation and could be disposed of without too much difficulty).

The output is a list of facets of the lower envelope of the relevant polytope, given as affine functionals; a list of vertex-facet incidences; and a decomposition of s into a tropical sum of vectors in the rowspan of M, or -1 if no such decomposition exists.

```
function ans = tsv (M,s);

[m,n]=size(M);
tuples = nchoosek( [1:n], m );
epsilon = .000000001; % Error tolerance

facets = zeros (1, m);
incidences = zeros (1, n);
nxt = 1;
for i = 1:size(tuples,1)
  if (abs(det(M(:,tuples(i,:)))) > epsilon)
   f =(M(:, tuples(i,:))') \ (s(tuples(i,:)))' ;
   nxtv = 1;
   goodf = 1;
   vertices = zeros(1,n);
   for j=1:n
     temp = M(:,j)'*f - s(j);
     if (temp >= -1*epsilon && temp <= epsilon)
       vertices(nxtv) = j;
       nxtv = nxtv + 1;
     end
     if (temp > epsilon)
       goodf = 0;
```

```
      end
    end
    if ((goodf == 1) )
      flag = 1;
      for i = 1:size(incidences,1)
        if (vertices == incidences(i,:))
          flag = 0;
        end
      end
      if (flag == 1)
        facets(nxt,:) = f;
        incidences(nxt,:) = vertices;
        nxt = nxt+1;
      end
    end
  end
end

facets
incidences

for i=1:n
   if (isempty(find(incidences==i)))
      ans= -1;
      return;
   end
end

for k=1:size(incidences,1)
   tuplesk = nchoosek([1:size(incidences,1)],k);
   for j=1:size(tuplesk,1)
      flag = 1;
      for i=1:n
        if (isempty(find(incidences(tuplesk(j,:),:)==i)))
          flag = 0;
        end
      end
      if (flag == 1)
        ans = facets(tuplesk(j,:),:)*M;
        return;
```

```
    end
  end
end
```

References

[1] P. Comon, G. Golub, L. Lim, B. Mourrain, Symmetric tensors and symmetric tensor rank, SIAM. J. Matrix Anal. & Appl. 30:1254 (2008).

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[3] M. Develin, Tropical secant varities of linear spaces, Discrete and Computational Geometry 35:117-129 (2006).