Math 274: Tropical Geometry
Assignment IV

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Exercise 1. [Solution due to Alex Fink] Invoking our fan structure for $\mathcal{T}(M)$, we know that a general point $x = (x_1, \ldots, x_n) \in \mathcal{T}(M)$ has the property that

$$x_i = a(\min\{j : i \in F_j\})$$

for some reals $a(1) > \cdots > a(s)$, where $\emptyset = F_0 \subsetneq \cdots \subsetneq F_s = [n]$ is a flag of flats of M of some length. This point x lies in the relative interior of the cone corresponding to the flag $\{F_i\}$.

In particular, the points $x \in \mathcal{T}(M)$ with exactly two distinct components arise from a flag of length $s = 2$, where F_1 is a flat other than $\emptyset$ or $[n]$; and F_1 is the set of indices i such that x_i is the lesser of the two occurring values. This lets us recover all the nonempty flats of M from $\mathcal{T}(M)$ (which in particular is enough data to recover M).

Now, we have that the bases B of M are exactly those subsets of $[n]$ which can be obtained by choosing a maximal flag of flats

$$\emptyset = F_0 \subsetneq \cdots \subsetneq F_r = [n];$$

choosing an element $b_i \in F_i \setminus F_{i-1}$ for each I; and setting $B = \{b_1, \ldots, b_r\}$. If B is in fact a basis then labelling the elements $b_1, \ldots, b_r$ in any fashion and choosing F_i to be the closure of $\{b_1, \ldots, b_i\}$ suffices: we can't have $b_j \in F_i$ for $j < i$ because then the set F_i of rank i contains the set $\{b_1, \ldots, b_{i-1}, b_i, b_j\}$ of rank $i + 1$. Conversely if B is not a basis, then for any ordering $b_1, \ldots, b_r$ of its elements there's some i such that the rank of $b_1, \ldots, b_i$ is $i-1$, and then having chosen a flat F_{i-1} (necessarily of rank $i - 1$) containing $\{b_1, \ldots, b_{i-1}\}$ it must also contain b_i by the closure. As a consequence, let r be the maximum number of distinct reals occurring as coordinates of any point $x \in \mathcal{T}(M)$. Then the bases B are just those sets of r indices for which there exist points $x \in \mathcal{T}(M)$ such that the coordinates of x indexed by B are all distinct. Put differently, B is a basis of M just if B is maximal such that $\mathcal{T}(M)$ does not lie entirely inside the hyperplane arrangement

$$\{\{x \in \mathbb{R}^n : x_i = x_j\} : i \neq j \in B\}.$$

The circuits of M are just the minimal subsets C of $[n]$ contained in no basis: so they are the minimal sets C such that for every $x \in \mathcal{T}(M)$, two of the coordinates of x indexed by C are identical. Indeed this is as we might have hoped, given our initial definition of $\mathcal{T}(M)$ in terms of circuits — we don't get any extra sets whose coordinates have the properties of circuits but aren't themselves coordinates. Of course there is an analogous restatement with hyperplane arrangements: C is a circuit of M just if C is minimal such that $\mathcal{T}(M)$ is contained in the hyperplane arrangement

$$\{\{x \in \mathbb{R}^n : x_i = x_j\} : i \neq j \in C\}.$$

Exercise 2. [Solution due to Daniel Erman] Our solution to this problem is a minor variation of the proof in [2]. We begin with two bivariate Laurent polynomials f_1 and f_2 where the support of f_i is the set A_i of vectors $\mathbf{a} \in \mathbb{Z}^2$. We write: $f_i = \sum_{\mathbf{a} \in A_i} c_{i,\mathbf{a}} x^{\mathbf{a}}$ and we assume that that the $c_{i,\mathbf{a}}$ are chosen generically from $\mathbb{C}^*$. This "genericity assumption" will be used repeatedly. We define $F := (f_1, f_2), A := (A_1, A_2)$.

We wish to prove the following Theorem:

Theorem 1. *The number of isolated solutions of $V(f_1, f_2)$ in $(\mathbb{C}^*)^2$ equals the mixed volume of the pair (A_1, A_2).*

Roughly speaking, our approach is as follows. First, we will deform the system F to a system $\widehat{F}$ over $\mathbb{C}\{\{t\}\}[x_1^{\pm}, x_2^{\pm}]$. Second, we will associate each solution of $V_{\text{trop}}(\widehat{F})$ with a particular type of cell in some subdivision of the Minkowski sum of the pair (A_1, A_2). Finally, we will compute the number of points in the fiber over any solution of $V_{\text{trop}}(\widehat{F})$.

We need some additional notation. We choose maps $w^{(i)} : A_i \to \mathbb{Z}$ which assign a sufficiently random integer to each element of A_i. Then we define the lifted polytopes $\widehat{A_i}$ where:

$$\widehat{A_i} = \{(\mathbf{a}, w^{(i)}(\mathbf{a})) \in \mathbb{Z}^3 | \mathbf{a} \in A_i\}$$

By considering the lower hull of these lifted polytopes we obtain a subdivision S_w of the Minkowski sum of A_1 and A_2 (c.f. [2, Lemma and Defintion 2.6]). In fact, by [2, p. 1546-7], the genericity of our choice of w ensures that the subdivision S_w will consist only of three types of cells:

- Triangles which correspond to triangles from A_1
- Triangles which correspond to triangles from A_2.
- Parallelograms P_γ spanned by an edge from A_1 and an edge from A_2.

We now deform the system F into a system over $\mathbb{C}\{\{t\}\}[x_1^{\pm}, x_2^{\pm}]$. We define:

$$\widehat{f_i}(\mathbf{x}, t) := \sum_{\mathbf{a} \in A_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{w^{(i)}(\mathbf{a})}$$

We define $\widehat{F}$ to be the system $\widehat{f_1}(\mathbf{x}, t), \widehat{f_2}(\mathbf{x}, t)$, which we think of as a system of equations over $\mathbb{C}\{\{t\}\}[x_1^{\pm}, x_2^{\pm}]$.

Definition 1. *We say that* $(x_1(t), x_2(t)) \in (\mathbb{C}\{\{t\}\}^*)^2$ *is a* root of F of type (γ_1, γ_2) *if:*

$$(x_1(t), x_2(t)) \in V(F) \text{ and } val(x_1(t), x_2(t)) = (\gamma_1, \gamma_2).$$

Our genericity assumptions (on the coefficients of f_i and on the functions $w^{(i)}$) imply that the number of solutions of $\widehat{F}$ in $(\mathbb{C}\{\{t\}\}^*)^2$ equals the number of solutions of F in $(\mathbb{C}^*)^2$. In other words:

$$\#V(f_1, f_2) = \#V(\widehat{f}_1, \widehat{f}_2) \tag{*}$$

By the fundamental theorem of tropical geometry, we know that the valuation map:

$$\nu : V(\widehat{f}_1, \widehat{f}_2) \to V_{trop}(\widehat{f}_1, \widehat{f}_2)$$

has dense image. Since each set above is a finite set, it follows that ν is surjective. Combining our observations thus far, we obtain the formula:

$$\#V(\widehat{f}_1, \widehat{f}_2) = \sum_{\gamma \in V_{trop}(\widehat{f}_1, \widehat{f}_2)} \#\nu^{-1}(\gamma) \tag{**}$$

The proof of Theorem 1 will follow from the following lemma.

Lemma 1. *The following statements hold.*

(i) There is a bijection between $\gamma \in V_{trop}(\widehat{F})$ *and parallelograms* P_γ *appearing in the subdivision* S_w *of* (A_1, A_2)*.*

(ii) The cardinality of $\nu^{-1}(\gamma)$ *equals the area of* P_γ *for every such* γ*.*

Proof. (1) The union of tropical curves $V_{trop}(\widehat{f}_1)$ and $V_{trop}(\widehat{f}_2)$ induces the mixed subdivision S_w of the Minkowski sum $A_1 + A_2$ described above. By our genericity assumption, we may assume that each intersection point $\gamma \in V_{trop}(\widehat{F_1}) \cap V_{trop}(\widehat{f}_2)$ is a 4-valent vertex in the graph. Hence γ it corresponds to a parallelogram P_γ in the subdivision. Conversely, every such parallelogram arises from an intersection point γ.

(2) The edges of P_γ correspond to translates of an edge from A_1 and an edge from A_2. This is equivalent to saying that, at γ, the initial term $f_{i,\gamma} := \text{in}_{(\gamma_1,\gamma_2)}(\widehat{f}_i)$ is a binomial. More precisely,

$$f_{i,\gamma} = x^{\mathbf{a}} - c_i x^{\mathbf{b}}$$

where the vector $\mathbf{a} - \mathbf{b}$ corresponds to the edge of P_γ from A_i. Noting that $\#V(F_\gamma) = \#\nu^{-1}(\gamma)$, we have reduced the problem to showing that:

$$\#V(F_\gamma) = \text{area}(P_\gamma)$$

Multiplying $f_{i,\gamma}$ by a monomial– which doesn't change the number of solutions of F_γ in the torus– we may write:

$$f_{i,\gamma} = x_1^{\lambda_i} x_2^{\zeta_i} - c_i$$

A direction computation as in [2, Lemma 3.2] then shows that the above system F_γ has precisely $|\zeta_1\lambda_2 - \lambda_1\zeta_2| = \text{area}(P_\gamma)$ solutions. $\square$

Proof of Theorem 1. By line (*), it is equivalent to show that $\#V(\widehat{f_1}, \widehat{f_2})$ equals the mixed volume of the pair (A_1, A_2). We may compute this mixed volume by adding up the area of the parallelograms in the fine mixed subdivision induced by w. Lemma 1 part (2) shows that each such parallelogram P_γ corresponds to area(P_γ) solutions of $\widehat{F}$ in $(\mathbb{C}\{\{t\}\}^*)^2$, and Lemma 1 part (1) shows that this accounts for all solutions of $\widehat{F}$. □

Exercise 3. The references for the Tropical Riemann-Roch Theorem for tropical curves are:

(i) [BN] Matthew Baker, Serguei Norine: "Riemann-Roch and Abel-Jacobi theory on a finite graph". http://arxiv.org/abs/math/0608360

(ii) [GK] Andreas Gathmann, Michael Kerber: "A Riemann-Roch theorem in tropical geometry". http://aps.arxiv.org/abs/math/0612129

(iii) [MZ] Grigory Mikhalkin, Ilia Zharkov: "Tropical curves, their Jacobians and Theta functions". http://arxiv.org/abs/math.AG/0612267

The first paper discusses a version of the Riemann-Roch Thm for finite graphs, whereas the second one generalizes this to metric graphs, hence providing a version of the Thm for abstract tropical curves. The third paper contains a completely independent proof of the Riemann-Roch theorem for tropical curves, using Jacobians of tropical curves (for more details, see Section 7, Thm. 7.3.)

We describe the theorem, the definitions and the extension from [BN] to [GK].

Definition 2. *An* abstract tropical curve *is a connected metric graph* Γ*, which admits bounded and unbounded edges. Each edge of length* ∞ *is identified with the real interval* $[0, \infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$ *in such a way that the* ∞ *end of the edge has valence 1. The infinity points are called the (unbounded) ends of* Γ*. In particular, we allow vertices of valence 1 and 2.*

The genus of a graph Γ *is the first Betti number of* Γ*, i.e.* $g = E - V + C$ *where* E *is the number of edges of* Γ*,* V *is the number of vertices and* C *is the number of connected components (in this case 1.)*

If all edges of a metric graph Γ *are integers (resp. rational numbers),* Γ *is called a* $\mathbb{Z}$*-graph (resp.* $\mathbb{Q}$*-graph). The set of points of* Γ *with integer (resp. rational) distance to the set* $V(\Gamma)$ *is denoted by* $\Gamma_{\mathbb{Z}}$ *(resp* $\Gamma_{\mathbb{Q}}$*).*

A rational function *on* Γ *is simply a continuous piecewise linear real-valued function* $f : \Gamma \to \mathbb{R} \cup \{\pm\infty\}$ *with integer slopes. For each point* $P \in \Gamma$ *we define the* order *of* f *in* P *as the sum of the slopes of* f *for all edges emanating from* P*. In particular,* f *can achieve the values* $\pm\infty$ *only at the unbounded edges of* Γ*.*

It is easy to show that the order of f in P equals zero for all but finitely many points.

Definition 3. *A* divisor *on* Γ *will be a formal* $\mathbb{Z}$*-linear combination of points of* Γ*. A divisor is* effective *is all of its coefficients are non-negative. The* degree *of a divisor* $D = \sum_{finite} a_P \cdot P$ *is* $\sum_P a_P$*.*

In case of a $\mathbb{Z}$*-graph (resp* $\mathbb{Q}$*-graph), D is called a* $\mathbb{Z}$*-divisor (resp* $\mathbb{Q}$*-divisor) if* $SuppD \subset \Gamma_{\mathbb{Z}}$ *(resp.* $SuppS \subset \Gamma_{\mathbb{Q}}$*).*

Just as in the classical case, for any rational function f we have a natural associated divisor $(f) = \sum_{P \in \Gamma} ord_P(f)P$.

Definition 4. *Given* Γ *as above we define the* canonical divisor K *of* Γ *ias the sum of all vertices of* Γ *counted with multiplicity equal to their respective valence minus* 2*:*

$$K = K_\Gamma := \sum_{P \in V(\Gamma)} (val(P) - 2) \cdot P.$$

Note that K *is a* $\mathbb{Z}$*-divisor (resp* $\mathbb{Q}$*-divisor) if* Γ *is a* $\mathbb{Z}$*-graph (resp* $\mathbb{Q}$*-graph.)*

For a given divisor D we define the space $R(D)$ of all rational functions f on Γ s.t. $(f)+D$ is an effective divisor. Following the classical Riemann-Roch we would like to make a statement about the dimension of theses spaces. Unfortunately, in the tropical cases we cannot do this because $R(D)$ is a polyhedral complex which is **not** of pure dimension. Therefore, we replace the notion of dimension of $R(D)$ by a new magnitud $r(D)$:

$$r(D) = \max\{n \in \mathbb{N} \mid R(D - P_1 - \ldots - P_n) \neq \emptyset \, \forall P_1, \ldots, P_n \in \Gamma\}$$

This number is closely related to the dimensions of the cells of $R(D)$. We should also remark that the points P_i need not be distinct.

Theorem 2 (Riemann-Roch for tropical curves)**.** *For* Γ*,* D *and* K *as above:*

$$r(D) - r(K - D) = \deg D + 1 - g$$

where g *is the genus of* D*.*

We remark that the definition of $\mathbb{Z}$-graphs and divisors correspond to the Riemann-Roch for finite graphs discussed in [BN].

The goal of the second paper is achieved via analyzing two cases: the case of finite $\mathbb{R}$-metric graphs and the case of metric graphs admiting infinit lengths for its edges (i.e. abstract tropical curves.)

The extension from the discrete [BN] setting to the finite $\mathbb{R}$-metric setting is done in two steps. First, they go from the $\mathbb{Z}$-case to the $\mathbb{Q}$-case via rescaling of Γ by $\lambda \in \mathbb{R}_{>0}$: the new metric graph equals Γ as a graph, but we replace the weights by λ times the corresponding weight. Divisors and rational functions on Γ also rescale to divisors and rational functions on $\lambda \cdot \Gamma$. This will prove the result for $\mathbb{Q}$-metric graphs with a finite number of edges and no unbounded edges, where they require the divisor D in the theorem to be a $\mathbb{Q}$-divisor.

The final step to go from $\mathbb{Q}$ to $\mathbb{R}$ uses a continuity argument: for a given finite metric graph Γ and a divisor D on Γ of degree n they find a nearby $\mathbb{Q}$-graph Γ' and a $\mathbb{Q}$-divisor D' on it such that $r_{\Gamma'}(D') = r_\Gamma(D)$ and $r_{\Gamma'}(K_{\Gamma'} - D) = r(K_\Gamma - D)$.

To extend the result to abstract tropical curves the authors introduce the notion of equivalence of divisors just as in the classical case:

Definition 5. *Two divisors D and D' on a tropical curve Γ are called equivalent ($D \sim D'$) if there exists a rational function f on Γ s.t. $D' = D + (f)$.*

The final step in the proof consists of relating divisors on tropical curves Γ with divisors on the corresponding metric graph $\overline{\Gamma}$ obtained by removing all unbounded edges. Namely,

Lemma 2 (3.4)**.** *With Γ and $\overline{\Gamma}$ as above, every divisor $D \in Div(\overline{\Gamma})$ is equivalent on $\overline{\Gamma}$ to a divisor D' with $SuppD' \subset \Gamma$. Moreover, if D is effective, then D' can be chosen to be effective as well.*

Exercise 4. Given a planar smooth (i.e. with nine nodes) tropical cubic curve, the tropical j-invariant is the lattice lenght of the boundary of its unique bounded cell ([6]). Our goal is to find a smooth cubic curve in $\mathbb{TP}^2$ with tropical j-invariant equal to 17. For this we'll pick a cubic curve with a d-cycle and assign lenghts to each edge so that its perimeter has lattice lenght equal to 17.

For this we obtain our candidate by interpolation techniques and drawing the tropical curve with the tropical.lib package. We'll start with an 8 cycle, dilate it so that it lenght exceeds 17 and then reduce some edges to arrive to the appropiate length. Our proposed vertices for the cycle (in clock-wise order) are: $(0,0), (0,1), (1,3), (2,4), (6,4), (7,3), (7,1)$ and $(6,0)$. We add the point $(4,0)$ to get nine points and interpolate.

After doing this we get the (homogeneous) cubic curve:

$$F := 23 \odot z^3 \oplus 17 \odot xz^2 + 20 \odot yz^2 \oplus 15 \odot x^2z \oplus 19 \odot y^2z \oplus 13 \odot xyz \oplus 14 \odot x^3 \oplus 19 \odot y^3 \oplus 13 \odot x^2y \oplus 13 \odot xy^2.$$

We confirm this by drawing the corresponding tropical plane curve (setting $z = 1$) and taking coefficients in the ring of Puiseux series s.t. the valuation give the corresponding coefficients in F. See Figure 1.

To compute the discriminant we need to use the notion of tropical A-discriminants (see [7]). In this case, the discriminant of a cubic hypersurface in $\mathbb{P}^2$ is given by Δ_A, where A is the matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

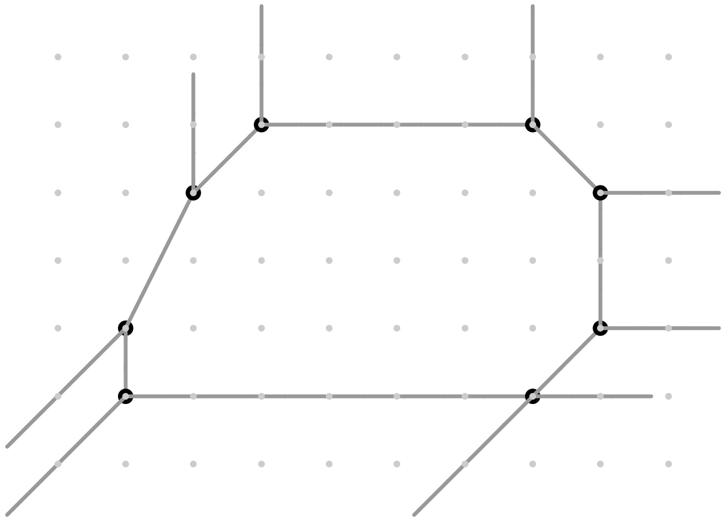
The degree of the polynomial is 12 and it has 2040 terms.

The Tropicalisation of

$$f = t^{14} \cdot x^3 + t^{13} \cdot x^2 y + t^{13} \cdot xy^2 + t^{19} \cdot y^3 + t^{15} \cdot x^2 + t^{13} \cdot xy + t^{19} \cdot y^2 + t^{17} \cdot x + t^{20} \cdot y + t^{23}$$

The vertices of the tropical curve are:

$$(0,1), \ (1,3), \ (0,0), \ (6,0), \ (2,4), \ (7,1), \ (6,4), \ (7,3)$$



The Newton subdivision of the tropical curve is:

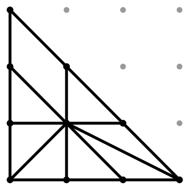


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Figure 1: A tropical smooth plane curve with an 8 cycle and j-invariante equal to 17.

Another alternative to this is to compute the classical discriminant and tropicalize this polynomial using the *trivial* valuation. The polynomial is obtained via elimination: if we consider the plane curve $f(x,y) = F(x,y,1)$ with the coefficients being variables $(a,\ldots,j) \in \mathbb{P}^9$ then $\mathrm{disc}(f)$ equals the unique primitive irreducible integer polinomial in the variables $(a,\ldots,j)$ living in the ideal $I = (F(x,y,z), \frac{\partial F}{\partial x}(x,y,z), \frac{\partial F}{\partial y}(x,y,1), \frac{\partial F}{\partial z}(x,y,z), z-1)$.

```
ring R = 0, (x,y,z,a,b,c,d,e,f,g,h,i,j), (dp(3),dp);
poly P = a*x^3+b*x^2*y+c*x*y^2+d*y^3+e*x^2*z+f*x*y*z+g*y^2*z+h*x*z^2+y*i*z^2+
j*z^3;
ideal I = (P, diff(P,x), diff(P,y), diff(P,z),z-1);
eliminate(I,x*y*z);
```

To finish, we write a code to evaluate the polynomial (tropically) at the point: $(a,\ldots,j) = (14,13,13,19,15,13,19,17,20,23)$. The answer is 173.

[**Alternative Solution by Alex Fink**] The tropical j-invariant of a smooth cubic curve in $\mathbb{TP}^2$, when this is positive, is the length of its cycle. We'll construct one by choosing a cycle of length 17 and then providing more polyhedra until we have a cubic curve.

We'll go for a hexagonal cycle with all its edges in directions e_i, since I know the edge lengths of edges of these types (e_i has length 1). Keeping the hexagon regular, we want each of its edge-lengths to be 17/6. So we can get this by classically scaling my favourite example of a tropical cubic, whose equation is

$$\begin{aligned} f(x,y) = &\frac{17}{2} \odot x^{\odot 3} \oplus \frac{17}{6} \odot x^{\odot 2} \odot y \oplus \frac{17}{6} \odot x \odot y^{\odot 2} \oplus \frac{17}{2} \odot y^{\odot 3} \\ &\oplus \frac{17}{6} \odot x^{\odot 2} \oplus 0 \odot x \odot y \oplus \frac{17}{6} \odot y^{\odot 2} \\ &\oplus \frac{17}{6} \odot x \oplus \frac{17}{6} \odot y \\ &\oplus \frac{17}{2} \end{aligned}$$

and which is drawn in Figure 2.

Then there's this matter of the tropical discriminant. The main task here seems to be getting one's hands on the actual classical discriminant, as a polynomial. Morgan points out we can do this with some elimination theory. Let g be the polynomial describing the universal curve in $\mathbb{P}^9 \times \mathbb{P}^2$ over the space $\mathbb{P}^9$ of cubics. The singular cubics are those with a point in $\mathbb{P}^2$ at which g and all its partials all vanish. But as formulated this has the problem that we'll pick up the point $(x,y,z) = (0,0,0)$ in each fiber which is not actually a point of projective space. To circumvent this we look at the cubics in affine space; this will also lose cubics with a singularity at infinity, but we hope to get those back with a closure. So we tell `Singular` the following.

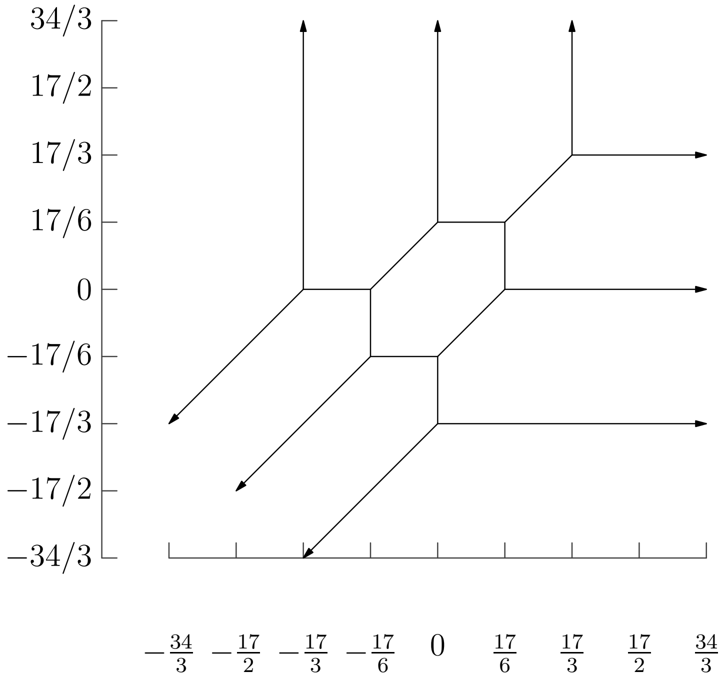

Figure 2: A tropical cubic with j-invariant 17.

```
ring R=0,(x,y,a,b,c,d,e,f,g,h,i,j),(dp(2),dp);
ideal I=x3a+x2yb+xy2c+y3d+x2e+xyf+y2g+xh+yi+j,
3x2a+2xyb+y2c+2xe+yf+h,
x2b+2xyc+3y2d+xf+2yg+i,
x2e+xyf+y2g+2xh+2yi+3j;
```

Then we project to $\mathbb{P}^9$, i.e. eliminate x, y, z, by taking a Gröbner basis. As the first generator of the result we get the determinant, a degree 12 polynomial with 2040 terms, long enough that I haven't included it here.

The next step is to tropicalise and evaluate this. Doing this (by replacing times and plus by plus and min in the Singular output, and dropping constants) we find that the tropical discriminant of this curve is 17.

Exercise 5. As usual, we draw the polytopes in $\mathbb{R}^{n-1}$ using the convention that the first coordinates of the points equal 0.

The tropical hexagon is drawn by taking the segments between any two pairs of points in $\{(k, 2k) : k = 0, \ldots, 5\}$ and filling in the bounded regions of the arrangement of segments. It corresponds to Figure 3.

Regarding the triangle (convex hull of three points in $\mathbb{TP}^5$), we draw the segment between the three pair of points. By definition, this set equals:

$$tconv(\underline{0}, P_1, P_2) = \{\underline{0} \oplus a \odot (0,1,2,3,4,5) \oplus b \odot (0,2,4,6,8,10) = \{P_{a,b} := (\min\{0,a,b\}, \min\{0,a+1,b+2\}, \min\{0,a+2,b+4\}, \min\{0,a+3,b+6\}, \min\{0,a+4,b+8\}, \min\{0,a+5,b+10\}) : a,b, \in \mathbb{R}\}.$$

Call $d = -\min\{0,a,b\}$. If we add $d \cdot \underline{1}$ to each point $P_{a,d}$ we get a representative of $P_{a,d}$ in $\mathbb{TP}^5$ with first coord equal 0, hence we can draw our polytope in $\mathbb{R}^5$.

Thus, if we call $a = a + d$ and $b = b + d$ our points are

$$P_{a,b} = (\min\{d, a+1, b+2\}, \min\{d, a+2, b+4\}, \min\{d, a+3, b+6\}, \min\{d, a+4, b+8\}, \min\{0, a+5, b+10\}) \in \mathbb{R}^5$$

where $d, a, b \in \mathbb{R}_{\geq 0}\}$. if we call $m_i = \min\{a+i, b+2i\}$ we get that for each sequence (increasing) sequence $(m_1, m_2, \ldots, m_5)$ the set of points $P_{a,b,d}$ belongs to the union of the classical polygonal $L_{m_1,\ldots,m_5}$ along the points $\underline{0}$, $(m_1, m_1, \ldots, m_1)$, $(m_1, m_2, m_2, \ldots, m_2)$, $(m_1, m_2, m_3, m_3, m_3)$, $(m_1, m_2, m_3, m_4, m_4)$ and $(m_1, m_2, m_3, m_4, m_5)$.

Therefore

$$tconv(\underline{0}, P_1, P_2) = \bigcup_{m_1,\ldots,m_5} L_{m_1,\ldots,m_5}.$$

To finish, we need to check which are the possible values of the sequence $m_1 \leq \ldots \leq m_5$. For this, let $k \leq a - b < k + 1$ for some $k \in \mathbb{Z}$. Thus

$$(b + 2i) + (k - i) \leq a + i \leq (b + 2i) + (k - i) + 1.$$

If $k - i \geq 0$ then $m_i = b + 2i$, whereas if $k - i < 0$ $m_i = a + i$.We consider all possible cases and we get that the values of the sequence (m_i) are:

$$\begin{cases} (m_1, m_1+1, m_1+2, m_1+3, m_1+4) & \text{for } m_1 \geq 1, \\ (m_1, m_2, m_2+1, m_2+2, m_2+3) & \text{for } 0 \leq m_2 - m_1 < 1, m_1 \geq 2 \\ (m_1, m_1+2, m_2, m_2+1, m_2+2) & \text{for } 2 \leq m_2 - m_1 < 3, m_1 \geq 2 \\ (m_1, m_1+2, m_1+4, m_2, m_2+1) & \text{for } 4 \leq m_2 - m_1 < 5, m_1 \geq 2 \\ (m_1, m_1+2, m_1+4, m_1+6, m_2) & \text{for } 6 \leq m_2 - m_1 < 7, m_1 \geq 2 \\ (m_1, m_1+2, m_1+4, m_1+6, m_1+8) & \text{for } m_1 \geq 2. \end{cases}$$

To finish, we need to prove that the two objects are isomorphic as tropical polytopes. An isomorphism should be a piecewise linear map between these two polytopes. The first attempt is to define a morphism as a tropicalization of a linear map from $\mathbb{R}^3$ to $\mathbb{R}^6$. In addition, this map would descend to the corresponding tropical projective spaces, thus defining a map $\mathbb{TP}^2 \to \mathbb{TP}^5$. We would also need a map in the opposite direction, i.e. a 6×3 matrix. In addition, if we take a 3×6 matrix, the tropical map would be $A \odot z = \bigoplus z_i \odot A_i$

where A_i denotes the i-th column of A. Since we want $A \cdot z$ to be a point of the hexagon, we should take A_i to be the 6 vertices of the hexagon. Therefore, we pick A as our original matrix.

Similarly, for the map from $\mathbb{TP}^5 \to \mathbb{TP}^2$ we pick the corresponding matrix as A^t. Unfortunately, this guess doesn't work, since the points $\underline{0}, P_1, P_2$ all map to 0. Therefore, we need to take a map $z \mapsto A \odot L(z)$ where L is a linear map to be determined. Likewise $w \mapsto A^t \odot L'(w)$ for a linear map L'. The easiest linear maps are multiplication by constant numbers.

Say $z \mapsto A \odot (\lambda z)$. To avoid the problem with injectivity we had before, we need to take $\lambda \leq -1$. We pick $\lambda = -1$. With this constant, $\underline{0} \mapsto 0$, $P_1 \mapsto (-5, 0, 0) \sim (0, 5, 5)$ and $P_2 \mapsto (-10, -5, 0) \sim (0, 5, 10)$.

To finish, we need to pick a constant corresponding to L'. Call it μ. Since we want these maps to give an isomorphism between the polytope μ must satisfy $A^t \odot (\mu\, \underline{0}) = A^t \odot \underline{0} = \underline{0}$, $A^t \odot (\mu(0, 5, 5)) \sim P_1$ and $A^t \odot (\mu(0, 5, 10)) \sim P_2$.

If $\mu > 0$ then $A^t \odot (\mu(0, 5, 5)) = \underline{0} \neq P_1$. So $\mu \leq 0$. In this case, we get $A^t \odot (\mu(0, 5, 5)) = (0, \min\{-5\mu, 1\}, \min\{-5\mu, 2\}, \min\{-5\mu, 3\}, \min\{-5\mu, 4\}, \min\{-5\mu, 5\}) = P_1$ iff $\mu \leq -1$. The second condition imposed also gives the same restriction $\mu \leq -1$. Therefore, we would try with $\mu = -1$.

To show that these maps are isomorphisms over the tropical polytopes, it suffices to prove that the corresponding compositions give the identity over each polytope, i.e. over points that equal $A \odot z$ or $A^t \odot z$ resp. An exhaustive case by case analysis confirms the later and so we have that the two polytopes are isomorphic via the two maps described above.

Exercise 6. This question was stated by Develin, Santos and Sturmfels in [8] Elena Rubei tried to prove it in [9] giving a negative answer, but her argument has a gap.

At the moment I'm running `gfan` to compute the tropical variety of the 4×4 minors using the symmetry of the ideal by permutation of rows and of columns of the matrix. Since the problem has a lot of symmetries (permuting rows and columns) it is plausible to do this computation without a crash. I'm also computing the prevariety given by the intersection of the 25 minors and try to see if this set is a tropical variety (this will give a positive answer to the question). It has been running for a couple of days up to now with no output.

References

[1] Federico Ardila, Caroline Klivans, "*The Bergman complex of a matroid and phylogenetic trees*," 2003, Preprint arxiv:math.CO/0311370.

[2] Birkett Huber and Bernd Sturmfels, "*A polyhedral method for solving sparse polynomial systems.*" Mathematics of computation **64** (1995), no. 212, 15441-1555.

[3] Matthew Baker and Serguei Norine: "*Riemann-Roch and Abel-Jacobi theory on a finite graph.*" http://arxiv.org/abs/math/0608360

[4] Andreas Gathmann, Michael Kerber: "*A Riemann-Roch theorem in tropical geometry.*" http://aps.arxiv.org/abs/math/0612129

[5] Grigory Mikhalkin, Ilia Zharkov: "*Tropical curves, their Jacobians and Theta functions.*" http://arxiv.org/abs/math.AG/0612267

[6] Eric Katz, Hannah Markwig and Thomas Markwig, "*The tropical j-invariant.*" http://arxiv.org/pdf/0803.4021

[7] Alicia Dickenstein, Eva-Maria Feichtner and Bernd Sturmfels, "*Tropical discriminants.*" J. Amer. Math. Soc. 20 (2007), 1111-1133.

[8] Mike Develin, Francisco Santos and Bernd Sturmfels, "*On the rank of a tropical matrix.*" In "Discrete and Computational Geometry" (E. Goodman, J. Pach and E. Welzl, eds), MSRI Publications, Cambridge Univ. Press, 2005. ISBN-10: 0521848628.

[9] Elena Rubei, "*On tropical and Kapranov ranks of tropical matrices.*" Preprint. http://arxiv.org/abs/0712.3007.

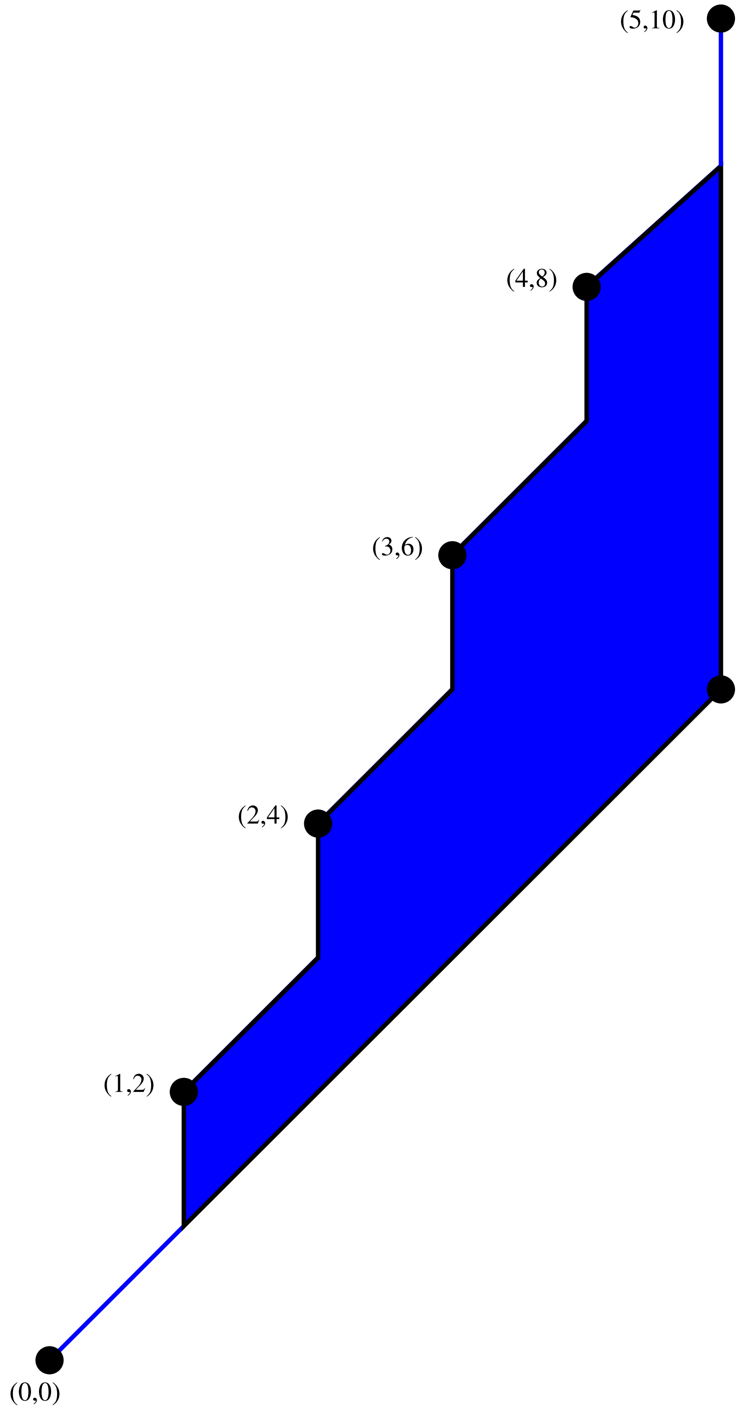

Figure 3: The tropical hexagon in $\mathbb{TP}^2$.