Math 113, Solutions to Midterm Exam # 2

(1) There is only one group of order 65, namely the cyclic group $\mathbf{Z}/65\mathbf{Z}$. The proof uses the same argument as in Example 2.10.20 and Exercise 2.53, which goes as follows: Let G be any group of order $65 = 5 \cdot 13$. The Third Sylow Theorem tells us that $|\mathrm{Syl}_5(G)|$ is in $\{1, 13\}$ and is congruent to 1 modulo 5, and that $|\mathrm{Syl}_{13}(G)|$ is in $\{1, 5\}$ and is congruent to 1 modulo 13. These facts imply that G has a unique Sylow 5-subgroup H_1 and a unique Sylow 13-subgroup H_2. Both H_1 and H_2 are normal by the Second Sylow Theorem. This implies that the map $H_1 \times H_2 \to G$, $(h_1, h_2) \mapsto h_1 h_2$ is a group isomorphism. Since the H_1 and H_2 have prime order, they must be cyclic, and, using the Chinese Remainder Theorem, we conclude

$$G \quad \simeq \quad \mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/13\mathbf{Z} \quad \simeq \quad \mathbf{Z}/65\mathbf{Z}.$$

(2) Let $a = (123)$ and $b = (234)$. Then a, a^2, b and b^2 are four distinct three-cycles. Note that the three disjoint transpositions in A_4 can be written as follows:

$$(12)(34) = ab\,, \quad (13)(24) = ba\,, \quad (14)(23) = ab^2a.$$

Hence the subgroup of A_4 generated by a and b contains at least eight distinct elements, namely, $a, a^2, b, b^2, ab, ba, ab^2a$ and the identity. Using Lagrange's Theorem, we conclude that this subgroup must have order 12 and hence be equal to A_4.

(3) The greatest common divisor of 12 and 20 equals 4, and hence $I = \langle 4 \rangle$. The greatest common divisor of 18 and 30 equals 6, and hence $J = \langle 6 \rangle$. The sum ideal $I + J$ is generated by $\gcd(4, 6) = 2$, the intersection ideal $I \cap J$ is generated by $\mathrm{lcm}(4, 6) = 12$, and the product ideal $I \cdot J$ is generated by $4 \cdot 6 = 24$.

(4) (a) In the integers $R = \mathbf{Z}$, the ideal $I = \langle 3 \rangle$ is principal and maximal.
(b) In the integers $R = \mathbf{Z}$, the ideal $I = \langle 6 \rangle$ is principal and not prime.
(c) This is impossible because every maximal ideal is prime (Remark 3.2.8).
(d) In the polynomial ring over the integers, $R = \mathbf{Z}[x]$, the ideal $I = \langle 3, x \rangle$ is maximal but not principal. Reason for maximal: $R/I \simeq \mathbf{F}_3$ is a field.
(e) In the polynomial ring over the integers, $R = \mathbf{Z}[x]$, the ideal $I = \langle x \rangle$ is principal and prime but not maximal. Reason: $R/I \simeq \mathbf{Z}$ is a domain but not a field.

(5) We compute $x^4 + x^3 + 7 \,=\, (x^2 + x + 2)(x^2 - 2) \,+\, (2x + 11)$
and $x^2 - 2 \,=\, (1/2x - 11/4)(2x + 11) \,+\, 113/4$.
Hence the answer in (a) is $2x+11$, and the answer in (b) is $113/4$. These computations show that the unit $113/4$ is in the ideal generated by f and g, and hence $\langle f, g \rangle \,=\, \mathbf{Q}[x]$.