**Math 113, Solutions to Midterm Exam # 1**

(1) This is part (ii) of Problem 30 on page 45. By the Binomial Theorem,

$$(a+b)^p \;=\; \sum_{i=0}^{p} \binom{p}{i} \cdot a^i \cdot b^{p-i}, \qquad (*)$$

where

$$\binom{p}{i} \;=\; \frac{p\cdot(p-1)\cdot(p-2)\cdots(p-i+1)}{i\cdot(i-1)\cdot(i-2)\cdots 1}.$$

All factors in the denominator are relatively prime from $p$, and hence $p$ divides $\binom{p}{i}$ provided $0 < i < p$. Hence the expression (*) is congruent to $a^p + b^p$ modulo $p$.

(2) The requirement $x \equiv 3 \pmod{10}$ implies $x \equiv 3 \pmod 5$, and the requirement $x \equiv 7 \pmod{15}$ implies $x \equiv 7 \equiv 2 \pmod 5$. These two statements are inconsistent. Hence these congruences have no solutions at all: the set of solutions is the empty set.

(3) Let $G$ be an abelian group and $H$ any subgroup. For $g \in G$ and $h \in H$ we have $g^{-1}h = hg^{-1}$ and hence

$$g^{-1}hg \;=\; (g^{-1}h)g \;=\; (hg^{-1})g \;=\; h(g^{-1}g) \;=\; h \in H.$$

This shows that $H$ is a normal subgroup of $G$.

(4)

(a) The residues 3, 4 and 5 are pairwise relatively prime, and their product is 60. The Chinese Remainder Theorem (Section 2.8.3) implies that $G$ is isomorphic to the group $\mathbf{Z}/60\mathbf{Z}$.

(b) By Proposition 2.7.4 (iii), the number of generators is $\phi(60) = \phi(3)\cdot\phi(4)\cdot\phi(5) = 2\cdot 2\cdot 4 = 16$.

(b) By Proposition 2.7.4 (iii), the number of elements of order 10 is $\phi(10) = \phi(2)\cdot\phi(5) = 1\cdot 4 = 4$.

(5)

(a) We write the two cycles as permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

Therefore

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

(b) In cycle notation, we have

$$\sigma\tau = (13)(24) \quad \text{and} \quad \tau\sigma = (14)(23).$$