Math 113 (Bernd Sturmfels), **Midterm Exam # 1**
Thursday, July 13, 9:00 a.m.–10:00 a.m.

Please start by writing your name and your student ID on the cover of your blue book. **This exam is closed book**. Do not use any notes, calculators, cell phones etc. Show all your work, and write full sentences if time permits. Each problem is worth 20 points, for a total of 100 points.

(1) Let a, b be two integers and let p be a prime number. Prove that

$$(a+b)^p \equiv a^p + b^p \text{ (mod p)}.$$

(2) Determine the set of all integers x which satisfy the three congruences

$$x \equiv 1 \text{ (mod } 6), \quad x \equiv 3 \text{ (mod } 10) \quad \text{and} \quad x \equiv 7 \text{ (mod } 15).$$

(3) Prove that every subgroup of an abelian group is normal.

(4) Consider the product of cyclic groups $G = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z}$.
- (a) Show that the group G is cyclic.
- (b) How many elements in G are generators of G?
- (c) How many elements in G have order 10?

(5) The cycles $\sigma = (123)$ and $\tau = (124)$ are in the symmetric group S_4.
- (a) Compute the two products $\sigma\tau$ and $\tau\sigma$ in S_4.
- (b) Write both $\sigma\tau$ and $\tau\sigma$ as products of disjoint cycles.