Score 8

Math 170 HW #5

Elizabeth Holland

21 February 2017

1 Exercise 4.1

Consider the linear programming problem:

$$\begin{aligned} \text{minimize } & x_1 - x_2 \\ \text{subject to } & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\ & 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\ & -x_1 - x_2 + 2x_3 + x_4 = 6 \\ & x_1 \leq 0 \\ & x_2, x_3 \geq 0 \end{aligned}$$

1

Write down the corresponding dual problem.

$$\begin{aligned} \text{maximize } & 3p_2 + 6p_3 \\ \text{subject to } & 3p_1 + p_2 - p_3 \leq -1 \\ & -p_1 + 4p_2 + 2p_3 \leq 0 \\ & 2p_1 + 3p_2 - p_3 \geq 1 \\ & p_1 - 2p_2 + 3p_3 = 0 \\ & p_1 \leq 0 \\ & p_2 \geq 0 \\ & p_3 \text{ free} \end{aligned}$$

2 Exercise 4.2

1

Consider the primal problem

$$\begin{aligned} \text{minimize } & c'x \\ \text{subject to } & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Form the dual problem and convert it into an equivalent minimization problem. Derive a set of conditions on the matrix A and the vectors b, c, under which the dual is identical to the primal, and construct an example in which these conditions are satisfied.

Dual problem:

$$\begin{aligned} \text{maximize } & p'b \\ \text{subject to } & p'A \leq c \\ & p \geq 0 \end{aligned}$$

Then converting to equivalent minimization, manipulating the constraint to mirror the format of the primal:

$$\begin{aligned} &\text{minimize } -p'b \\ &\text{subject to } -p'A \geq -c' \\ &p \geq 0 \end{aligned}$$

First, we must have A be an $n \times n$ matrix in order to make the dual problem of same dimension. This implies that $-p'A \geq -c' \Leftrightarrow -A'p \geq -c$, so for this to be the same problem we need $b = -c$ (which also guarantees $-p'b = c'x$) and $A = -A'$, which means that A is skew-symmetric. Note that p and x already both have the same nonnegativity constraints. Then these problems are equivalent.

For example, consider the primal below:

$$\begin{aligned} &\text{minimize } x_1 - 2x_2 + 3x_3 \\ &\text{subject to } x_2 - x_3 \geq -1 \\ &-x_1 + 3x_3 \geq 2 \\ &x_1 - 3x_2 \geq 3 \\ &x \geq 0 \end{aligned}$$

Then its dual is:

$$\begin{aligned} &\text{maximize } -p_1 + 2p_2 - 3x_3 \\ &\text{subject to } -p_2 + p_3 \leq 1 \\ &p_1 - 3x_3 \leq -2 \\ &-p_1 + 3p_2 \leq 3 \\ &p \geq 0 \end{aligned}$$

which can be transformed into:

$$\begin{aligned} &\text{minimize } p_1 - 2p_2 + 3x_3 \\ &\text{subject to } p_2 - p_3 \geq -1 \\ &-p_1 + 3x_3 \geq 2 \\ &p_1 - 3p_2 \leq -3 \\ &p \geq 0 \end{aligned}$$

which is equivalent to the primal. ■

3 Exercise 4.4

Let A be a symmetric square matrix. Consider the linear programming problem

(1)

$$\begin{aligned} &\text{minimize } c'x \\ &\text{subject to } Ax \geq c \\ &x \geq 0 \end{aligned}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

Let A be a symmetric square matrix. Suppose x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution. Form the dual of the above linear programming problem, noting that all dimension will be identical to the primal because A is square:

$$\begin{aligned} &\text{maximize } p'c \\ &\text{subject to } p'A \leq c \\ &p \geq 0 \end{aligned}$$

Let $p* = x*$; then clearly $p* = x* \geq 0$. Then consider $p*' A$: $p*' A = A'p* = Ap* = Ax* = c$ because A is symmetric (thus $A = A'$) and square. Thus $p*$ is a feasible solution. Then we can see that $c'x* = x*' c = p*' c$, so the objective functions of the primal and the dual take on equivalent values for solutions $x*, p*$ respectively. Thus by Corollary 4.2, $x*$ is an optimal solution. ■

1

4 Exercise 4.5

Consider a linear programming problem in standard form and assume that the rows of A are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

(a) Let x^* be a basic feasible solution. Suppose that for every basis corresponding to x^*, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less than $c'x^*$

True; by Weak Duality, since x^* is feasible we know that $p'b \leq c'x^*$. If there is no feasible associated basic solution to the dual, we know that x^* is not the optimal solution and thus there is no p such that $p'b = c'x^*$. However, there is some optimal cost which will be equal to the optimal cost in the dual. The associated dual optimal solution must be feasible, so we have that for some p, optimal cost $= p'b < c'x^*$.

(b) The dual of the auxiliary primal problem considered in Phase I of the simplex method is always feasible.

True; the whole purpose of the primal auxiliary problem is to create a problem that is always feasible. Since it is feasible and its optimal cost is bounded by 0, then we must have that the dual is also feasible.

(c) Let p_i be the dual variable associated with the ith equality constraint in the primal. Eliminating the ith primal equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.

True; Let $a_i'x = b_i$ be the ith inequality constraint. Removing it frees p_i, but also removes p_i from the cost function. The same result occurs if you set $p_i = 0$.

(d) If the unboundedness criterion in the primal simplex algorithm is satisfied, then the dual problem is infeasible.

True; if the unboundedness criterion in the primal simplex algorithm is satisfied, then the optimal cost is $-\infty$. From the Weak Duality Theorem, this implies that for any feasible solution to the dual p, $p'b \leq c'x = -\infty$. This is impossible, and thus the dual problem has no feasible solutions. ■

5 Exercise 4.6 (Duality in Chebychev approximation)

Let A be an $m \times n$ matrix and let b be a vector in $\mathbb{R}^m$. We consider the problem of minimizing $||Ax - b||_\infty$ over all $x \in \mathbb{R}^n$. Here $||\cdot||_\infty$ is the vector norm defined by $||y||_\infty = max_i|y_i|$. Let v be the value of the optimal cost.

(a) Let p be any vector in $\mathbb{R}^m$ that satisfies $\sum_{i=1}^m |p_i| \leq 1$ and $p'A = 0'$. Show that $p'b \leq v$.

1

Let $y = Ax - b$. This implies that $Ax = y - b \Leftrightarrow p'Ax = p'(y-b) \Leftrightarrow p'y = p'b$. Then consider $|p'y|$; since $\sum_{i=1}^m |p_i| \leq 1$, the upper bound of this is the maximum value of y, which we were given from the beginning to be v, because no weighted sum of smaller numbers will ever be larger than a larger number. Thus $v \geq |p'y| = |p'b|$, so $p'b \leq v$. ■

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{gathered}\text{maximize } p'b \\ \text{subject to } p'A = 0' \\ \sum_{i=1}^m |p_i| \leq 1\end{gathered}$$

Show that the optimal cost in this problem is equal to v.

This is in fact the dual to our original problem:

$$\begin{gathered}\text{minimize } ||Ax - b||_\infty = v \\ \text{x free}\end{gathered}$$

Thus the optimal cost of the dual is the same as the primal, from the Strong Duality Theorem. Thus v is the optimal cost in this problem.

6 Exercise 4.12 (Degeneracy and Uniqueness - I)

Consider a general linear programming problem and suppose that we have a nondegenerate basic feasible solution to the primal. Show that the complementary slackness conditions lead to a system of equations for the dual vector that has a unique solution.

Consider the general linear programming problem below, with both primal and dual:

Primal

$$\begin{aligned} &\text{minimize } c'x \\ &\text{subject to } a_i'x \geq b_i, && \forall i \in M_1 \\ &a_i'x \leq b_i, && \forall i \in M_2 \\ &a_i'x = b_i, && \forall i \in M_3 \\ &x_j \geq 0, && \forall j \in N_1 \\ &x_j \leq 0, && \forall j \in N_2 \\ &x_j \text{ free}, && \forall j \in N_3 \end{aligned}$$

1

Dual

$$\begin{aligned} &\text{maximize } p'b \\ &\text{subject to } p_i \geq 0, && \forall i \in M_1 \\ &p_i \leq 0, && \forall i \in M_2 \\ &p_i \text{ free}, && \forall i \in M_3 \\ &p'A_j \leq c_j, && \forall j \in N_1 \\ &p'A_j \geq c_j, && \forall j \in N_2 \\ &p'A_j = c_j, && \forall j \in N_3 \end{aligned}$$

Let x be a nondegenerate basic feasible solution to the primal. Thus by definition of basic feasible solution, it satisfies exactly n linearly independent active constraints, all equality constraints are active, and all inequality constraints are satisfied. Note that this implies that the other inequality constraints are not active and therefore are strict inequalities. Therefore, $|M_3| = n$.

Then suppose the complementary slackness conditions hold. That is, for some dual feasible p, we have:

$$p_i(a_i'x - b_i) = 0 \forall j$$

$$(c_j - p'A_j)x_j = 0 \forall j$$

Then consider $i \in M_1$. We know that $a_i'x > b_i$, so we must have that $p_i = 0$. Similarly for $i \in M_2$, we know that $a_i'x < b_i$, so we must have that $p_i = 0$. Thus $p_i = 0 \ \forall i \in M_1 \cup M_2$.

Note also that $x_j \neq 0$ because x is a nondegenerate solution. Thus we must have that $c_j = p'A_j$.

Then form a matrix B from a_i for $i \in M_3$. B is thus an $n \times n$ matrix with linearly independent columns. WLOG, let M_3 be the first n rows of matrix A. Remembering that $p_i = 0 \ \forall i \in M_1 \cup M_2$, this implies that $p_j = 0 \ \forall j \in \{n, n+1, ..., m\}$. Thus only the first n rows of A are actually used in $c_j = p'A_j$, so we can write that $c = r'B$, where r is the first n nonzero components of p. B must be invertible because it is a square linearly independent matrix, so we can show that $r' = cB^{-1}$. This will uniquely define p. ■

7 Exercise 4.13 (Degeneracy and Uniqueness - II)

Consider the following pair of problems that are duals of each other:

$$\begin{aligned} \text{minimize } & c'x \\ \text{subject to } & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize } & p'b \\ \text{subject to } & p'A \leq c \end{aligned}$$

(a) Prove that if one problem has a nondegenerate and unique optimal solution, so does the other.

Suppose the primal has a nondegenerate and unique optimal solution. Since the complementary slackness conditions hold for optimal solutions, we can conclude from 4.12 that the dual has a unique optimal solution.

Further recognizing that the primal is the dual of the dual, we can say WLOG that the same result applies. ■

(b) Suppose that we have a nondegenerate optimal basis for the primal and that the reduced cost for one of the nonbasic variables is zero. What does the result of part (a) imply? Is it true that there must exist another optimal basis?

The result in part (a) does not hold because the optimal solution for the primal must be unique. When one of the nonbasic variables is zero, this implies we can find multiple optimal solutions and possibly another optimal basis.

This does not imply that there must exist another optimal basis. Consider the problem of minimizing x_1 subject to $x_2 = 3$ and $x \geq 0$. Thus the set of solutions is $(0, 3, x_3)$ for $x_3 \geq 0$, so we have multiple optimal solutions but only one optimal basis.

8 Exercise 4.16

Give an example of a pair (primal and dual) of linear programming problems, both of which have multiple optimal solutions.

Primal:

$$\begin{aligned} \text{minimize } & x_1 + 2x_2 + x_3 \\ \text{subject to } & x_1 + x_2 = 1 \\ & x_2 + x_3 = 1 \\ & x_1, x_2 \geq 0 \\ & x_3 \leq 0 \end{aligned}$$

Here the only possible cost is 2, from the constraints. Thus our optimal solutions are any possible solutions to the constraints; in particular, $(1, 0, 1)$, $(0, 1, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ are optimal solutions, so the primal has multiple optimal solutions.

Dual:

$$\begin{aligned} \text{maximize } & p_1 + p_2 \\ \text{subject to } & p_1 \leq 1 \\ & p_1 + p_2 \leq 2 \\ & p_2 \geq 1 \end{aligned}$$

Clearly the maximum value for $p_1 + p_2$ is 2, again from the constraints. Thus we can see that $(1, 1)$ and $(0, 2)$ and in fact many others are solutions to the dual; thus the dual has multiple optimal solutions.