Score: 8

Math 170 – Problem Set 3 (Due Tuesday February 7)
Solutions by Frederick Law

1. (B&T 2.18) Consider a polyhedron $P = \{\mathbf{x} : \mathbf{Ax} \geq \mathbf{b}\}$. Given any $\varepsilon > 0$, show that there exists some $\overline{\mathbf{b}}$ with the following two properties: (a) The absolute value of every component $\mathbf{b} - \overline{\mathbf{b}}$ is bounded by ε. (b) Every basic feasible solution in the polyhedron $P = \{\mathbf{x} : \mathbf{Ax} \geq \mathbf{b}\}$ is nondegenerate.

   Solution: Intuitively, we shall find such a $\overline{b}$ by first identifying excessive constraints and perturbing the necessary constraints by some ϵ. That is, if $P = H_1 \cap \cdots \cap H_m$, where H_j are the half-spaces whose intersection gives us P, then we identify those H_j which are unnecessary in defining P. If H_j is excessive, then P is well defined apart from H_j, which means that $\bigcap_{k \neq j} H_k = P$. Let M be the set of all indices of excessive half-spaces. Then $P = \bigcap_{k \notin M} H_k$, that is, we can remove all the half-spaces of M and still get the same polyhedron. This is true just because of our construction, since all the half-spaces removed are excessive. If $H_j = \{\mathbf{x} \in \mathbb{R}^n : a_j'\mathbf{x} \geq b_j\}$, then let $H_k' = \{\mathbf{x} \in \mathbb{R}^n : a_k'\mathbf{x} \geq b_j - \varepsilon\}$. Let $\overline{\mathbf{b}}$ be defined by:

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$$\overline{\mathbf{b}} = \begin{cases} b_j & j \notin M \\ b_j - \varepsilon & j \in M \end{cases}$$

   Then our new polyhedron is $P' = \bigcap_{k \in M} H_k \cap \bigcap_{j \in M} H_j'$. Note that in P, the degenerate basic feasible solutions can be interpreted as having too many hyperplanes, boundary of half-spaces, touching at that point. For example, on a cube in three dimensions, all the points are non-degenerate as they all have 3 hyperplanes touching them, the three facets that connect at a corner. Moreover, every facet of a polyhedron is not excessive, since it serves as a boundary of the polyhedron. Thus, we can interpret the excessive constraints as half-spaces whose hyperplane boundary either touches the polyhedron only at some degenerate solution or at not degenerate solution.. Therefore, by perturbing these outward by ϵ, we remove all the degeneracy from our basic feasible solutions. Thus all the basic feasible solutions in P' are nondegenerate. Also, since we have only perturbed the excessive half-spaces by ϵ, it follows that by construction $|\overline{b}_i - b_i| \leq \varepsilon$ for all i.

2. (B&T 2.21) Suppose that Fourier-Motzkin elimination is used in the manner described at the end of Section 2.8 to find the optimal cost in a linear programming problem. Show how this approach can be augmented to obtain an optimal solution as well.

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   Solution: To get an optimal solution using the Fourier-Motzkin elimination, we first find the optimal solution. This is done by extending our LP by one variable x_0, so we get a new polyhedron in $\mathbb{R}^{n+1}$ defined by $\{(x_0, \mathbf{x}) : \mathbf{x} \in P, \mathbf{c}'\mathbf{x} = x_0\}$. Then we use the Fourier Motzkin elimination to project onto the first variable, which gives us $\{x_0 \in \mathbb{R} : \exists \mathbf{x} \in P\, s.t.\, \mathbf{c}'\mathbf{x} = x_0\}$. Then we minimize over this subset of $\mathbb{R}$ to find our optimal cost, call this c^*. To get an optimal solution, we really just project back upwards on n dimensions, by inverting the Fourier Motzkin algorithm. Moreover, to save time in our algorithm, we really only need to project upwards starting at c^*. That is, if we imagine our entire process as a mapping $\Phi : P \to \mathbb{R}$ where $\mathbf{x} \in P$ gets sent to $\mathbf{c}'\mathbf{x}$, then we take the preimage over c^* which is just the level set $\Phi^{-1}(c^*)$ and see what points lie on the level set. This is also the same as taking the plane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}'\mathbf{x} = c^*$ and finding where this hyperplane intersects P.

3. (B&T 2.22) Let P and Q be polyhedra in $\mathbb{R}^n$. Let $P + Q = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in P, \mathbf{y} \in Q\}$.

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   (a) Show that $P + Q$ is a polyhedron.

   Solution: Let us define M as $M = \{(\mathbf{z}, \mathbf{x}, \mathbf{y}) : \mathbf{x} \in P, \mathbf{y} \in Q, z = \mathbf{x} + \mathbf{y}\}$. If P is constructed with n_1 linear constraints and Q is constructed with n_2 linear constraints, then M is constructed with $n_1 + n_2 + n$ linear constraints, where n of them come from

$\mathbf{z} = \mathbf{x} + \mathbf{y}$ component wise. Therefore since M is constructed using linear constraints, M is a polyhedron in $\mathbb{R}^{3n}$. Then we use Fourier-Motzkin elimination to reduce to the first n coordinates: $\Pi_n(M) = \{\mathbf{z} \in \mathbb{R}^n : \exists \mathbf{x} \in P, \mathbf{y} \in Q\, s.t.\, \mathbf{x} + \mathbf{y} = \mathbf{z}\}$. This can be rewritten as $\Pi_n(M) = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^n : \mathbf{x} \in P, \mathbf{y} \in Q\} = P + Q$. By the Fourier-Motzkin elimination algorithm, we know that $\Pi_n(M)$ is a polyhedron, and thus $P + Q$ is a polyhedron. $\square$

(b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

Solution: Suppose not. Then there exists $\mathbf{x} + \mathbf{y}$ which is extreme in $P + Q$ but either $\mathbf{x}$ is not extreme in P or $\mathbf{y}$ is not extreme in Q or both. WLOG, suppose that $\mathbf{x}$ is not an extreme in P, $\mathbf{y}$ may or may not be extreme in Q. Since $\mathbf{x}$ is not extreme in P then that means there exists $\mathbf{z}, \mathbf{z}' \in P$, $\lambda \in [0, 1]$ such that $\mathbf{x} \neq \mathbf{z}$ and $\mathbf{x} \neq \mathbf{z}'$ and $\mathbf{x} = \lambda \mathbf{z} + (1 - \lambda)\mathbf{z}'$. Then we have

$$\mathbf{x} + \mathbf{y} = \lambda \mathbf{z} + (1 - \lambda)\mathbf{z}' + \mathbf{y} = \lambda(\mathbf{z} + \mathbf{y}) + (1 - \lambda)(\mathbf{z}' + \mathbf{y})$$

But now we have written $x + y$ as a convex combination of $\mathbf{z} + \mathbf{y}$ and $\mathbf{z}' + \mathbf{y}$, where $\mathbf{x} + \mathbf{y} \neq \mathbf{z} + \mathbf{y}$ and $\mathbf{z}' + \mathbf{y}$, since $\mathbf{x} \neq \mathbf{z}$ and $\mathbf{x} \neq \mathbf{z}'$. Therefore $\mathbf{x} + \mathbf{y}$ is not an extreme point. This is a contradiction, so we are done. $\square$

4. (B&T 3.2) **(Optimality conditions)** Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ over a polyhedron P. Prove the following:

(a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

Solution: First we prove the forward direction. Suppose that $\mathbf{x}$, a feasible solution, is optimal. Then it follows that $\forall \mathbf{y} \in P$, $\mathbf{c}'\mathbf{x} \leq \mathbf{c}'\mathbf{y}$. Suppose $\mathbf{d}$ is an arbitrary feasible direction at $\mathbf{x}$. Then there exists $\theta > 0$ such that $\mathbf{x} + \theta\mathbf{d} \in P$. Then $\mathbf{c}'\mathbf{x} \leq \mathbf{c}'(\mathbf{x} + \theta\mathbf{d}) = \mathbf{c}'\mathbf{x} + \theta\mathbf{c}'\mathbf{d}$. Thus $\theta\mathbf{c}'\mathbf{d} \geq 0$, and since $\theta > 0$, we divide by θ and get $\mathbf{c}'\mathbf{d} \geq 0$.

Now we prove the backward direction. Suppose that $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$. Let $\mathbf{y} \in P$ be arbitrary. Then let $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Then if we let $\theta = 1$, then $\mathbf{x} + \theta\mathbf{d} = \mathbf{y} \in P$, so $\mathbf{d}$ is a feasible direction at $\mathbf{x}$ and so $\mathbf{c}'\mathbf{d} \geq 0$. Thus it follows that $\mathbf{c}'(\mathbf{y} - \mathbf{x}) \geq 0$, and so $\mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{x} \geq 0$ and $\mathbf{c}'\mathbf{x} \leq \mathbf{c}'\mathbf{y}$. Since y was an arbitrary point in P, it follows that x is an optimal solution. This proves the equality. $\square$

(b) A feasible solution $\mathbf{x}$ is the unique optimal solution if and only if $\mathbf{c}'\mathbf{d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

Solution: Our argument will be similar to that in part (a). First we prove the forward direction. Suppose that $\mathbf{x}$, a feasible solution, is unique. Then this means that for any $\mathbf{y} \in P$ such that $\mathbf{y} \neq \mathbf{x}$, then $\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$. Let $\mathbf{d}$ be any nonzero feasible direct at $\mathbf{x}$. Then there exists $\theta > 0$ such that $\mathbf{x} + \theta\mathbf{d} \in P$. Since $\theta > 0$ and $\mathbf{d} \neq 0$, then it follows that $\mathbf{x} \neq \mathbf{x} + \theta\mathbf{d}$. Therefore $\mathbf{c}'\mathbf{x} < \mathbf{c}'(\mathbf{x} + \theta\mathbf{d}) = \mathbf{c}'\mathbf{x} + \theta\mathbf{c}'\mathbf{d}$. Subtracting $\mathbf{c}'\mathbf{x}$ from both sides and dividing by $\theta > 0$, it follows that $\mathbf{c}'\mathbf{d} > 0$.

Now we prove the backward direction. Suppose that $\mathbf{c}'\mathbf{d} > 0$ for any nonzero feasible direction at $\mathbf{x}$. Let $\mathbf{y} \in P$ be an arbitrary point such that $\mathbf{y} \neq \mathbf{x}$. Since $\mathbf{y} \neq \mathbf{x}$, then $\mathbf{d} = \mathbf{y} - \mathbf{x} \neq \mathbf{0}$. Using $\theta = 1$, then $\mathbf{x} + \theta\mathbf{d} = \mathbf{y} \in P$, so $\mathbf{d}$ is a feasible, nonzero, direction at $\mathbf{x}$. Therefore it follows that $\mathbf{c}'\mathbf{d} > 0$. Thus $\mathbf{c}'(\mathbf{y} - \mathbf{x}) = \mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{x} > 0$. And thus it follows that $\mathbf{c}'\mathbf{y} > \mathbf{c}'\mathbf{x}$. Since $\mathbf{y}$ was any point in P not equal to $\mathbf{x}$, it follows that $\mathbf{x}$ is the unique optimal solution. This proves the equality. $\square$

5. (B&T 3.3) Let $\mathbf{x}$ be an element of the standard form polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Prove that a vector $\mathbf{d} \in \mathbb{R}^n$ is a feasible direction at $\mathbf{x}$ if and only if $\mathbf{A}\mathbf{d} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$.

Solution: We first prove the forward direction. Suppose that $\mathbf{d}$ is feasible at $\mathbf{x}$. Then there exists $\theta > 0$ such that $\mathbf{x} + \theta\mathbf{d} \in P$. This means $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{A}\mathbf{x} + \theta\mathbf{A}\mathbf{d} = \mathbf{b}$. Since $\mathbf{x} \in P$, then

$\mathbf{Ax} = \mathbf{b}$, thus subtracting $\mathbf{b}$ from both sides, and dividing by $\theta > 0$, it follows that $\mathbf{Ad} = \mathbf{0}$. If i is an index such that $x_i = 0$, and since $\mathbf{x} + \theta\mathbf{d} \geq 0$, then $x_i + \theta d_i = \theta d_i \geq 0$. Dividing by $\theta > 0$, it follows that $d_i \geq 0$ for any index i such that $x_i \geq 0$.

Now we prove the backward direction. Suppose $\mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{Ad} = \mathbf{0}$ and $d_i \geq 0$ for any index i such that $x_i = 0$. Let us choose θ^* satisfying

$$0 < \theta^* < \inf\left\{-\frac{x_j}{d_j} \; : \; x_j > 0, d_j < 0\right\}$$

Since $\mathbf{x} \in P$, then $x_i \geq 0$ for all i. If $x_i = 0$, then $x_i + \theta d_i = \theta d_i \geq 0$, since $d_i \geq 0$ and $\theta > 0$. If $x_i > 0$, there are two cases. If $d_i \geq 0$, then $x_i + \theta d_i \geq 0$. If $d_i < 0$, then $\theta^* < -\frac{x_i}{d_i}$, and multiplying through by $d_i < 0$, we get $\theta^* d_i > -x_i$ and thus $x_i + \theta^* d_i \geq 0$. Thus $\mathbf{x} + \theta^*\mathbf{d} \geq \mathbf{0}$. Also, $\mathbf{A}(\mathbf{x} + \theta^*\mathbf{d}) = \mathbf{Ax} + \theta^*\mathbf{Ad} = \mathbf{Ax} = \mathbf{b}$ since $\mathbf{x} \in P$. Thus it follows that $\mathbf{x} + \theta^*\mathbf{d} \in P$. This proves the equality, and we are done. $\square$

6. (B&T 3.5) Let $P = \{\mathbf{x} \in \mathbb{R}^3 \; : \; x_1 + x_2 + x_3 = 1, \mathbf{x} \geq \mathbf{0}\}$ and consider the vector $\mathbf{x} = (0, 0, 1)$. Find the set of feasible directions at $\mathbf{x}$.

   Solution: We find the set of feasible directions by working with the definition of feasible directions. We say that $\mathbf{d}$ is feasible at $\mathbf{x}$ if there ~~exists~~ some $\theta > 0$ such that $\mathbf{x} + \theta\mathbf{d} \in P$. Let $\mathbf{d} = (d_1, d_2, d_3)$. Then $\mathbf{x} + \theta\mathbf{d} = (\theta d_1, \theta d_2, 1 + \theta d_3)$. To require this to be in P, we need that $\theta(d_1 + d_2 + d_3) + 1 = 1$, which means $d_1 + d_2 + d_3 = 0$, and thus $d_3 = -d_1 - d_2$. We also require d_1, d_2 to be non-negative, since then for any $\theta > 0$, $\theta d_1, \theta d_2 \geq 0$. Lastly, we shall require $1 + \theta d_3 \geq 0$. To do this, we shall choose θ so that $1 + \theta d_3 = 0$. Then $1 = \theta(d_1 + d_2)$ and $\theta = \frac{1}{d_1+d_2}$. Note this θ will always be positive, unless $d_1 = d_2 = 0$. But in that case, $d_3 = 0$, and $\mathbf{d} = \mathbf{0}$, which is always a feasible direction. Thus our set of feasible directions is

$$F = \{(d_1, d_2, -d_1 - d_2) \; : \; d_1 \geq 0, d_2 \geq 0\}$$

   As a safe mental check, if $d_1 = d_2 = 0$, then $\mathbf{d} = \mathbf{0}$, and for $\theta = 1$, $\mathbf{x} + \mathbf{d} = \mathbf{x} \in P$. If either d_1 or $d_2 > 0$, then let $\theta = \frac{1}{d_1+d_2} > 0$, and then $\mathbf{x} + \theta\mathbf{d} = \left(\frac{d_1}{d_1+d_2}, \frac{d_2}{d_1+d_2}, 0\right) \in P$, since $\frac{d_1}{d_1+d_2} + \frac{d_2}{d_1+d_2} = \frac{d_1+d_2}{d_1+d_2} = 1$, and $\frac{d_i}{d_1+d_2} \geq 0$ for $i = 1, 2$. $\square$

7. (B&T 3.6) **(Conditions for a unique optimum)** Let $\mathbf{x}$ be a basic feasible solution associated with some basis matrix $\mathbf{B}$. Prove the following:

   (a) If the reduced cost of every nonbasic variable is positive, then $\mathbf{x}$ is the unique optimal solution.

   Solution: Suppose that $\bar{c}_j > 0$ for every nonbasic variable j. Let $\mathbf{y} \in P$ be arbitrary such that $\mathbf{y} \neq \mathbf{x}$. Then $\mathbf{Ax} = \mathbf{Ay} = \mathbf{b}$. Thus it follows that $\mathbf{A}(\mathbf{y} - \mathbf{x}) = \mathbf{0}$. Let $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Then using $\theta = 1$, $\mathbf{x} + \mathbf{d} = \mathbf{y} \in P$, so $\mathbf{d}$ is a feasible direction at $\mathbf{x}$ and $\mathbf{Ad} = \mathbf{0}$. We can rewrite this as $\mathbf{B}\mathbf{d}_B + \sum_{j \in I} A_j d_j = 0$ where I is the set of nonbasic indices. Since $\mathbf{B}$ is invertible, then $d_B = -\sum_{j \in I} \mathbf{B}^{-1} A_j d_j$. Then

$$\mathbf{c}'\mathbf{d} = \mathbf{c}_B'\mathbf{d}_B + \sum_{j \in I} c_j d_j = \sum_{j \in I} \left(c_j - \mathbf{c}_B'\mathbf{B}^{-1}A_j\right) d_j = \sum_{j \in I} \bar{c}_j d_j$$

   Since $\mathbf{y} \neq \mathbf{x}$, there must exist some $j \in I$ such that $y_j \neq x_j = 0$, where $x_j = 0$ for all $j \in I$ since $\mathbf{x}$ is a basic feasible solution. If $\mathbf{y} \in P$ such that $y_j = 0$ for all $j \in I$, then $\mathbf{Ay} = \mathbf{B}\mathbf{y}_B = \mathbf{b}$. Then $\mathbf{y}_B = -\mathbf{B}^{-1}\mathbf{b} = \mathbf{x}_B$, and so $\mathbf{y} = \mathbf{x}$. Therefore there must exist some nonbasic index on which $\mathbf{y}$ and $\mathbf{x}$ disagree. Let $k \in I$ be such an index. Since $\mathbf{y} \in P$, then $y_k \geq 0$, and since $y_k \neq x_k = 0$, then $y_k > 0$. Since $d_j = y_j - x_j$ and since $x_j = 0$ for all $j \in I$:

$$\mathbf{c}'\mathbf{d} = \sum_{j \in I} \bar{c}_j d_j = \sum_{j \; inI} \bar{c}_j y_j \geq \bar{c}_k y_k > 0$$

where we use the fact that $\bar{c}_j \geq 0$ for all $j \in I$ and $y_j \geq 0$ for all $j \in I$ since $\mathbf{y} \in P$. Thus we have $\mathbf{c}'(\mathbf{y} - \mathbf{x}) > 0$ and thus $\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$. Since this holds for all $\mathbf{y} \neq \mathbf{x}$, it follows that $\mathbf{x}$ is the unique optimal solution. □

(b) If $\mathbf{x}$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

Solution: Suppose not. Then there exists an index $m \in I$, borrowing notation from part (a), such that $\bar{c}_m \leq 0$. Let $\mathbf{d}$ be the mth basic direction. We know from our results in class and in Bertsimas and Tsitsiklis that $\mathbf{d}$ is always a feasible direction at $\mathbf{x}$ since $\mathbf{x}$ is nondegenerate. Then there exists $\theta > 0$ such that $\mathbf{x} + \theta\mathbf{d} \in P$. Then

$$\mathbf{c}'\mathbf{d} = \sum_{j \in I} \bar{c}_j d_j = \bar{c}_k \leq 0$$

since $\mathbf{d}$ is the kth basic direction, so $d_k = 1$ and $d_j = 0$ for $j \in I$, $j \neq k$. Therefore $\mathbf{c}'\mathbf{d} \leq 0$. Choosing $\theta > 0$ sufficiently small so that $\mathbf{x} + \theta\mathbf{d} \in P$, we get $\theta\mathbf{c}'\mathbf{d} \leq 0$ which implies that $\mathbf{c}'(\mathbf{x} + \theta\mathbf{d}) = \mathbf{c}'\mathbf{x} + \theta\mathbf{c}'\mathbf{d} \leq \mathbf{c}'\mathbf{x}$. But $\mathbf{x} + \theta\mathbf{d} \in P$, which means that $\mathbf{x}$ is not the unique optimal solution. This is a contradiction, which completes the proof. □

8. (B&T 3.10) Show that if $n - m = 2$, then the simplex method will not cycle, no matter which pivoting rule is used.

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Solution: Note that there is a lot of possible choices in the simplex method, thus if this fact is indeed true it, then the face that $n - m = 2$ must do something eliminate our possible choices in directions to move when executing the simplex method. One possible explanation could be that when $n - m = 2$, then either the polyhedron is unbounded, in which case simplex terminates at optimal cost being $-\infty$ or the polyhedron is bounded and it happens that in the polytope all the basic feasible solutions are nondegenerate, and then the simplex method on this polytope will have to terminate.