Math 113 (Bernd Sturmfels), **Final Exam**
Thursday, August 17, 8:10 a.m.–10:00 a.m.

Please start by writing your name and your student ID on the cover of your blue book. **This exam is closed book**. Do not use any notes, calculators, cell phones etc. You must show all your work to get credit. There are ten problems, each worth 10 points, for a total of 100 points.

(1) Let u be the class of 11 in $\mathbf{Z}/23\mathbf{Z}$. Determine u^{-1} and $u+u^{-1}$.

(2) What is the smallest order of a non-abelian group ?

(3) How many units does the ring $\mathbf{Z}/60\mathbf{Z}$ have ?

(4) Determine the 11^{th} cyclotomic polynomial $\Phi_{11}(x)$.

(5) For which values of a in $\mathbf{F}_5$ is the ring $\mathbf{F}_5[x]/\langle x^3+2x^2+a\rangle$ a field ?

(6) Consider the ideal $I = \langle x^2+y^2-1, xy+2\rangle$ in $\mathbf{Q}[x,y]$. Find a polynomial $f(x)$ which generates the principal ideal $I\cap\mathbf{Q}[x]$ in $\mathbf{Q}[x]$.

(7) Prove or disprove: If H_1 and H_2 are subgroups of a finite group G then their product $H_1\cdot H_2$ is a subgroup of G.

(8) Prove or disprove: If I is an ideal in a principal ideal domain R then every ideal in the quotient ring R/I is principal.

(9) Prove or disprove: There exists a term ordering such that the subset $\{x^3+y^2, x^2+y\}$ of $\mathbf{Q}[x,y]$ is a Gröbner basis.

(10) Prove or disprove: If R is the quotient ring $\mathbf{Q}[x,y,z]/\langle x\cdot z-y^2\rangle$ then every irreducible element in R is prime.