Math 113, **First Midterm Exam**
SOLUTIONS

(1) Let H be a proper subgroup of G. By Lagrange's Theorem (10.10), the order of H is a proper factor of pq. If $|H| = 1$ then $H = \{e\}$ which is cyclic. Otherwise, the order of H is a prime number, and then H is cyclic because the cyclic subgroup generated by any non-identity element must be equal to H, again by Lagrange's Theorem.

The ambient group G need not be abelian. For an example, consider the symmetric group $G = S_3$. It is not abelian but its order equals pq for the primes $p = 2$ and $q = 3$.

(2) Besides the whole group A_4 and the trivial subgroup $\{e\}$, there are precisely **8** other subgroups in the alternating group A_4. There are three subgroups of order 2, namely $\langle(12)(34)\rangle$, $\langle(13)(24)\rangle$, and $\langle(14)(23)\rangle$. The union of these three subgroups is the unique subgroup of order 4, which is isomorphic to the Klein group $\mathbf{Z}_2 \times \mathbf{Z}_2$. Finally, there are four subgroups of order 3, namely $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, and $\langle(234)\rangle$. Note that there is no subgroup of order 6, as we know from the failure of the converse of Lagrange's Theorem. The subgroup diagram of A_4, and plenty of other useful information, can be found at

`http://shell.cas.usf.edu/~wclark/algctlg/small_groups.html`

(3) There are **20** distinct homomorphisms from $\mathbf{Z}$ to $\mathbf{Z}_{20}$. To specify such a homomorphism ψ, it is necessary and sufficient to specify the image $\psi(1)$ of the generator 1 of $\mathbf{Z}$, and each element of $\mathbf{Z}_{20}$ is allowed as the image of 1. The image of ψ will then be the cyclic subgroup generated by $\psi(1)$. None of these homomorphisms can be injective because $\mathbf{Z}$ is infinite while $\mathbf{Z}_{20}$ is finite. The number of surjective homomorphisms equals the number of generators of $\mathbf{Z}_{20}$, which is $\phi(20) = \mathbf{8}$, the value of the Euler-phi function.

(4) Since D_5 is not abelian, its commutator subgroup C is a proper subgroup. Moreover, C is a normal subgroup (by Theorem 15.20). Since D_5 has order 10, we conclude that the order of C equals 1, 2 or 5. But it cannot be 2 because any two reflections of the regular pentagon are conjugate in D_5, so D_5 has no normal subgroups of order 2. The order of C cannot be 1 because the following rotation can be written as a commutator:

$$(12345) \;=\; \sigma\tau\sigma^{-1}\tau^{-1} \qquad \text{for } \sigma = (13)(45) \text{ and } \sigma = (25)(34).$$

Thus C has order 5, and it coincides with the group of all rotations of the pentagon. The quotient group D_5/C has order 2, so it is isomorphic to $\mathbf{Z}_2$, the unique group of order 2.

(5) The identity element (e,e) is in H, and H is closed under products $(g,g)(h,h) = (gh,gh)$ and inverses $(g,g)^{-1} = (g^{-1},g^{-1})$. So, it is a subgroup, and we get (a). For parts (b) and (c) see the solutions to # 3 in the following midterm exam from Spring 2010:

`tbp.berkeley.edu/examfiles/math/math113-sp10-mt1-Denis%20Auroux-soln.pdf`

Please check out some of the other old exams posted by the Math Dept.