Forcing Your Opponent to Stay in Control of a Loony Dots-and-Boxes Endgame

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Introduction to Loony Endgames and Controlled Value

The reader is assumed to be familiar with the game of Dots and Boxes. An excellent introduction can be found in [WW], [Nowakowski] or [D&B].

Some results about this game are more easily described in terms of the dual game, called Strings-and-Coins [D&B, Chapter 2]. This game is played on a graph G, whose nodes are called coins and whose branches are called strings. In this graph, the ends of each string are attached to two different coins or to a coin and the *ground*, which is a special uncapturable node. The *valence* of any coin is the number of strings attached to it. It has long been known that typical endgames reach a *loony* stage in which no coin has valence 1, and every coin of valence 2 is part of a chain of at least 3 such coins or of a loop of at least 4 such coins. When such a position occurs, the player who has just completed his turn is said to be *in control.* Let's call him Right. All moves now available to his opponent, called Left, are of the type called *loony.* Wherever Left plays, Right will then (possibly after capturing a few coins) have a choice between two options: (1) he can complete his turn by declining the last two or four coins of the chain or loop (respectively) which Left just began, thereby forcing her to play the first move elsewhere, or (2) he can capture all currently available coins and play the next move elsewhere himself. If Right chooses the former option, he is said to *retain control.*

When a loony endgame is played well, Right will score at least as many points as Left. Right strives to maximize the difference between their scores, while Left strives to minimize it. The *value* of the loony endgame G, denoted by $\mathrm{v}(G)$, is Right's net margin of victory if both players play optimally. If Left has already acquired a lead of t points before the position G is reached, then Left can win the game if and only if $\mathrm{v}(G) < t$.

If Right chooses to stay in control at least until the last or second-last turn of the game, and if Left chooses her best strategy knowing that this is how Right intends to play, then Right's resulting net score is called the *controlled value* of G, denoted by $\mathrm{cv}(G)$. It is known [D&B, pp. 82–84] that $\mathrm{v}(G) \geq \mathrm{cv}(G)$. The controlled value can often be computed much more quickly than the actual value. It also turns out that under certain conditions the two are known to be equal. This paper weakens those conditions substantially.

The Formula for Controlled Value

A *joint* is a coin with valence three or four. A coin which is immediately capturable has valence one. Other coins that are not joints have valence two. Let c be the number of coins in G, and let j be the number of joints. No matter what the valence of the ground is, we treat it as a special joint, even though it is counted in neither c nor j. Let v be the total valence of all joints (including the ground). The *excess valence* is $v - 2j$.

Recall from [D&B, pp. 82–84] that the controlled value is given by the following formula:

$$\text{If } p > \tfrac{1}{4}, \text{ then } \mathrm{cv}(G) = 8 + c + 4j - 2v - 8p,$$

where p is the maximum adjusted number of node-disjoint loops in G, obtained by counting each loop as

1 if it excludes the ground,

$\frac{1}{2}$ if it includes the ground and at least 4 other nodes,

$\frac{1}{4}$ if it includes the ground and 3 other nodes.

Since the ground is a single uncapturable node, and independent chains are viewed as loops through the ground, at most one chain can be included in p. So $p = \frac{1}{4}$ only in the degenerate case in which G consists entirely of independent chains of length 3.

Overview of Conditions for $\mathrm{v}(G) = \mathrm{cv}(G)$

Let K be the subgraph of G consisting of the p node-disjoint loops. In [WW, pp. 543–544] it was shown that that Left minimizes Right's score by playing in such a way that all of the joints in each of these loops is eliminated before the loop is played. One easy way for Left to accomplish this is to play all other chains before playing K. This works if $\mathrm{cv}(K)$ is sufficiently large. In particular, it is easy to see that

$$\text{if } \mathrm{cv}(K) \geq c/2, \text{ then } \mathrm{v}(G) = \mathrm{cv}(G).$$

Here K is the subgraph of G which remains as the final stage of the loony endgame. The main result of the present paper concerns another latestage endgame subgraph, K, as well as another subgraph, H, which represents the position after a relatively short opening phase. We show that $\mathrm{v}(G) = \mathrm{cv}(G)$

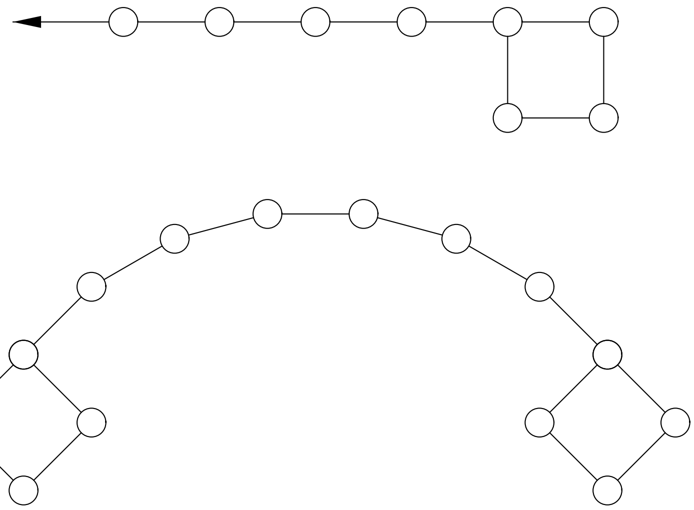


Figure 1. A dipper and earmuffs.

under appropriate restrictions on H and K. The major improvement is that the former condition

$$\mathrm{cv}(K) \geq c/2$$

is now replaced by the much weaker constraint

$$\mathrm{cv}(K) \geq 10.$$

"Very Long" Defined

A *very long loop* is a loop containing at least 8 nodes. A *very long chain* is a chain containing at least 5 nodes. Loops of length at least 4 but less than 8, and chains of length 3 or 4 are called *moderate*. This is a minor but deliberate change from [D&B, p. 84], where chains of length 4 were considered *very long*.

We will see that Left has a rather powerful strategy which postpones playing all very long loops and very long chains until very near the end of the game, after she has extracted all of the profits she can get from playing 3-chains and moderate loops. But moderate loops can be interconnected with the remainder of the graph in ways which require Left to play with considerable care. A few of the simpler such configurations, including those shown in Figure 1, appeared in [D&B].

We now undertake a more thorough investigation of those subgraphs of a loony endgame which contain the moderate loops.

Theorem. *In a loony endgame position, a moderate loop can include at most one joint.*

Proof. If it had two or more joints A and B, these joints would partition the loop into two sets, one running clockwise from A to B, and another running clockwise from B to A. Since the entire moderate loop has at most 7 nodes, including A and B, then there are only 5 other nodes, and so at least one of these proper chains has at most 2 nodes. That means it is short and so the position is not loony. □

Arbitrary strings-and-coins graphs might have joints with valence 5 or more, but such graphs lie beyond the scope of our present study. Since graphs arising from dots-and-boxes positions cannot have valence more than 4,

we henceforth assume that every joint has valence 3 or 4.

Definition. A pair of loops which each has only one joint, which they share, is called a *twin*.

A twin is always best-played as if it were a dipper. The loop of the dipper is the longer loop of the twin, and the handle of the dipper is the shorter loop of the twin.

Definition. A moderate loop with a joint, that is not part of a twin, is called a *trinket*. A trinket and the chain or chains connected to it form part of a *bracelet*. A bracelet contains only a single path of chains, each internal joint of which connects to a trinket. The bracelet may also include a chain at either of its ends which adjoins only one trinket; such a chain is called a *tag*.

In a loony endgame graph whose joints have valence at most 4, there are only four possible kinds of environments of moderate loops:

(1) isolated loops
(2) twins
(3) bracelets

It is possible for a bracelet to be cyclic, in the sense that the path between its chains forms a cycle. A cyclic bracelet may be isolated, or it can have one joint connecting it to other parts of the graph. In the latter case, the chains of the bracelet which adjoin that joint are regarded as the bracelet's tags. If a cyclic bracelet had two or more joints without trinkets, it would be regarded as two separate bracelets, each of whose tags happened to share a joint with a tag of another bracelet.

The loony endgame position of Figure 2 includes a bracelet, a 12-loop, and a 6-chain. The bracelet includes 8 joints, labeled A, B, C, D, E, F, G, and H. This bracelet has no tags. The lengths of its chains and trinkets are labeled in the diagram of Figure 3. The trinkets at A and H are 6-loops; all of the other trinkets are 4-loops. The negative controlled value of each chain and trinket is shown in this table:

$AB: -5$	$BC: -6$	$CD: -1$	$DE: -4$	$EF: -1$	$FG: -6$	$GH: -5$	
$A: +2$	$B: +4$	$C: +4$	$D: +4$	$E: +4$	$F: +4$	$G: +4$	$H: +2$

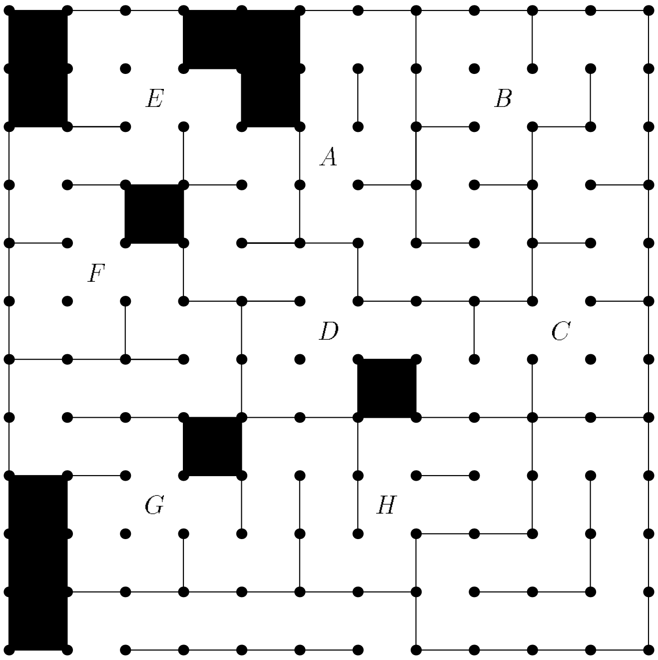
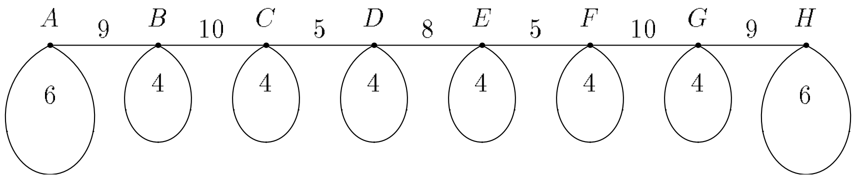


Figure 2. A loony endgame position. All 11 shaded boxes have been taken by L.



36	boxes in trinkets
+ 56	boxes in chains within the bracelet
92	boxes in bracelet
11	already taken
12	loop in southeast corner
+ 6	chain at bottom
121	total boxes on the board

Figure 3. The bracelet corresponding to the position of Figure 2.

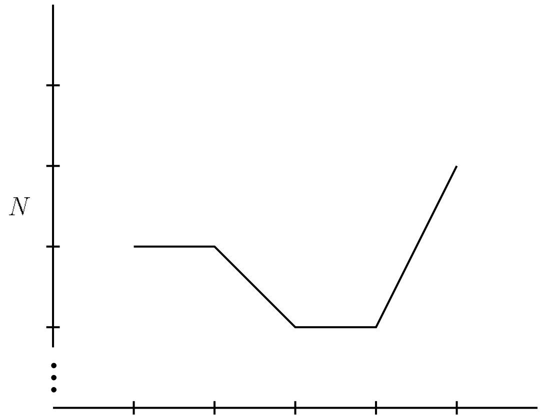


Figure 4. Move-by-move graph of 3-chain.

In total, Left should enjoy a net gain of 28 points on the trinkets, but at a net cost of 28 points on the bracelet's chains. So, altogether, the controlled value of the entire bracelet is 0. The controlled value of the 12-loop in the southeast corner of Figure 2 is $8 - 4 = 4$, and when the 6-chain along the bottom of the figure is played on the very last turn, Right will take it all. This yields a controlled value of 10 points for the entire position shown in Figure 2. Since Left is 11 points ahead, she can win unless at some point Right can afford to relinquish control.

A naive strategy for Left is to offer all of the chains within the bracelet before offering any of the trinkets. It has long been known that this naive Left strategy is optimal when the rest of the game includes such a large number of nodes on very long loops and very long chains that the controlled value is very big. However, in the present problem, there are 92 points within the bracelet, and the controlled value of the independent chains and loops elsewhere is only 10, which is far too small. The naive strategy won't work for the problem of Figure 2, because at some point along the way, Right will gain a lead that will enable him to win by relinquishing control.

Let N denote Left's net score, that is, the number of boxes Left has taken minus the number of boxes Right has taken. Assume (for now) that Right must remain in control. Suppose Left offers a 3-chain. We know that Right takes one box and declines two, resulting in a net profit of one box for Left. Let's take a closer look, though, at the changes in N move-by-move. Left offers the 3-chain and there is no change in N. Right takes one box and N decreases by one. Right declines the last two boxes with his next move and there is no change in N. Finally, Left takes these two boxes at once and N increases by two.

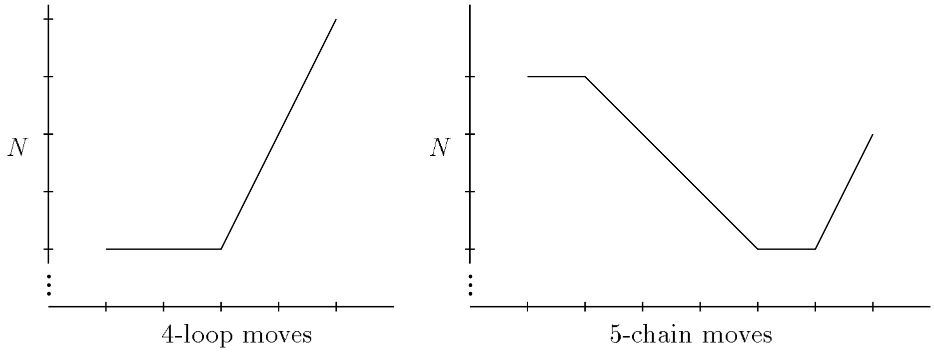


Figure 5. Move-by-move graphs of 4-loop and 5-chain.

We know that a 3-chain results in a net increase in N of one, but it is not an immediate gain for Left. As can be seen in Figure 4, when Left offers a 3-chain, N first dips by one and then climbs by two.

More examples of how N changes move-by-move are given in Figure 5. When Left offers a 4-loop, N increases by four and never dips. When she offers a 5-chain, N eventually decreases by one but only after partially rebounding from an earlier dip of three.

The following flight analogy is useful when discussing Left's net score: let N be the altitude of a plane and let the move-by-move graph represent the plane's path over time. The plane gains altitude with little or no dipping when Left offers moderate loops and 3-chains. It loses altitude when she offers long loops and chains. More complex pieces can result in small gains in altitude with big dips. Other pieces, such as a 4-chain, might cause dips but not change the cruising altitude.

Recall that the controlled value, $\mathrm{cv}(G)$, of a loony endgame G is the value of the game to Right if he remains in control (until the last or perhaps penultimate move). Since it is Right's choice whether or not to stay in control, the actual value of the game, $\mathrm{v}(G)$, is always at least $\mathrm{cv}(G)$. Right would have to concede control at some point in order to do better than $\mathrm{cv}(G)$. Therefore,

$$\text{if Left can force Right to stay in control, then } \mathrm{cv}(G) = \mathrm{v}(G).$$

Returning to our flight analogy, let the ground be at $N = -\mathrm{cv}(G)$. If Right concedes control when $N = n > -\mathrm{cv}(G)$, then if Left stays in control for the rest of the game, the plane ends at $n + (n + \mathrm{cv}(G))$. Since $2n + \mathrm{cv}(G) > -\mathrm{cv}(G)$, Right cannot afford to lose control before $N = -\mathrm{cv}(G)$. If ever $N < -\mathrm{cv}(G)$, then Right will do better by giving up control. In order for Left to force Right to stay in control, the plane must not crash. By that we mean the flight path must maintain an altitude at least as high as the ground at $-\mathrm{cv}(G)$.

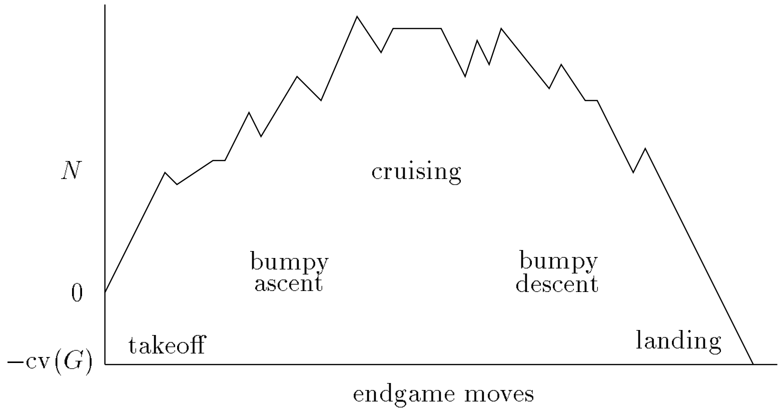


Figure 6. Successful flight for Left, with N measured relative to takeoff, $\mathrm{cv}(G)$ above ground level.

To summarize:

> If Left's plane can take off, get to a sufficiently high cruising altitude and land, all without crashing, Left can force Right to stay in control.

Figure 6 illustrates a successful flight for Left with labeled flight segments.

We now return to the problem of Figure 2. Another Left strategy is to play through the bracelet in the following order: AB, A, BC, B, CD, C, DE, D, EF, E, FG, F, GH, G, H. (Here each pair of letters refers to the chain between the respective joints, and each single letter refers to the trinket at that joint, which becomes a loop before it is played.) This is a special case of the *direct* strategy that Left can use to play any bracelet: play the chains in order from one end to the other, but intersperse these chain plays with plays on the trinkets (= loops), playing each loop as soon as it becomes detached.

Suppose Left plays the direct strategy on the bracelet of Figure 3, and also suppose that Right elects to stay in control by declining each piece of the bracelet. Then Left's net score, move by move as the bracelet is played, is shown in Figure 7. During the first move of any chain, the score remains constant as Left concludes her turn by offering that chain. For the next several moves, Left's net score then declines by one point per move as Right takes all but the last two boxes of the chain. Then the score stays constant as Right makes the double-dealing move which concludes his turn. On the last move of the chain, Left's net

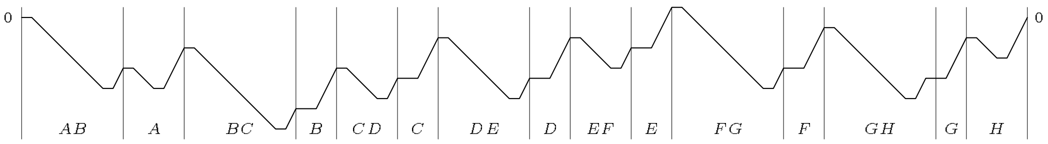


Figure 7. Bumpty Direct Flight from AB to H, for the problem of Figure 2, shown move by move (refer to table on page 4).

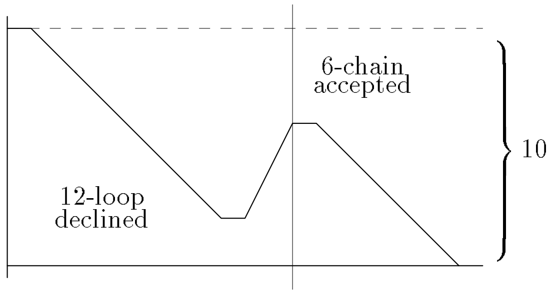


Figure 8. A smooth landing.

score increases by two points as he makes the double-cross. Left then concludes her turn by starting another piece of the bracelet with a move that leaves the net score unchanged. If Left has offered a loop, then after taking all but four of its boxes, Right concludes his turn with a move that offers four boxes at the end of the loop with a double double-deal. Left then finishes the play on the loop with two moves *each* of which increases Left's score by two points, after which Left begins play on another piece of the bracelet.

Portions of the flight path of Figure 7 which correspond to separate pieces of the bracelet are delineated by vertical lines. Notice that the relative score at each of these transitions between pieces is the same as one gets from the partial sums of the turn-by-turn analysis of the table on page 4:

$$-5 + 2 - 6 + 4 - 1 + 4 - 4 + 4 - 1 + 4 - 6 + 4 - 5 + 4 + 2 = 0.$$

Altogether, if Right stays in control while Left plays the bracelet in this way, then Left's relative net score at the end of the bracelet is the same as it was at the beginning. In Figure 2, this would mean that when the play of the bracelet was concluded, Left would again be ahead by 11 points, just as she was when the play of the bracelet began. Left would then go on to win the game by playing out the *landing* plan shown in Figure 8.

However, an astute Right will not play this way. Measured relative to the ground level, the altitude at the start and end of the bracelet's flight path is 10, but the altitude at the lowest point within BC is -1. **At this point the plane has crashed!** Right can thwart Left's "direct flight" strategy by accepting BC rather than declining it. In the present example, that enables Right to win the game.

So Left's "direct flight" plan for playing the bracelet as indicated in Figure 7 is too bumpy. Let's be more specific:

DEFINITION. A *k-point bump* in a flight is a descent of k followed by an ascent of k. A *k-point blip* is an ascent of k followed by a descent of k.

The flight plan of Figure 3 has an 11-point bump, beginning at the start of AB and ending one move prior to the end of E.

For some bracelets, the direct flight plan in the reverse direction can be more or less bumpy than the direct flight plan in the forward direction. However, as seen in Figure 3, this particular bracelet has palindrome symmetry, so the direct flight plan has the same bumpiness in either direction.

Extracting a Profitable Subbracelet

In the big bump of the flight plan of Figure 7, from the lowest point near the end of the segment BC, to the highest point at the beginning of FG, there is a net 12-point rise. This 12-point rise includes all of the segments B, CD, C, DE, D, EF, and E. Omitting the first two loops (B and C), we find the sequence CD, DE, D, EF, E with a net gain of 2. *These segments form a subbracelet with tags at both end.* Evidently, instead of playing in the *direct* order shown in Figure 7, Left could instead play this subbracelet first, in this order: CD, DE, D, EF, E. This subbracelet yields Left a gain of 2 and a maximum bump of 7.

So to win the game shown in Figure 7, Left should begin by playing the profitable subbracelet CD, DE, D, EF, and E. Since this flight plan stays above ground, Right does better to decline all five of those pieces than he can do by accepting any of them.

The play of the profitable subbracelet increases Left's lead from 11 points to 13. Left can then play each of the two residual bracelets, and then land safely as in Figure 8.

This example is readily generalized to the following theorem:

Theorem. *If the direct flight plan of a bracelet includes a bump of* 11 *or more, than the bracelet includes a playable profitable proper sub-bracelet.*

Definition. To be *playable*, a subbracelet must either have tags at both ends, or end with a trinket that is also an end of the original bracelet.

Note: The place to find this profitable subbracelet is evidently within the ascending part of the bump.

Proof. Except at the beginning (which might be either one or two consecutive chains) and the end (which might be either one or two consecutive loops), any bracelet's direct flight plan alternates between chains and loops. Each chain is a net descent, and each loop is a net ascent. Since we have assumed (without loss of generality) that every chain within the bracelet is very long, we know that every chain ends at a lower altitude than it began. So there is no loss of generality in assuming that the bump ends within a loop, which we keep as the last loop of the subbracelet. The bump's ascent must have begun from an elevation at least 11 points lower. So, starting from the beginning of this ascent, let us purge the next two loops. (If the ascent starts within a loop, we include that loop as one of the two which we purge.) The maximum ascent of the purged moves can be at most ten: 4 within each of the two purged loops, and possibly another 2 within

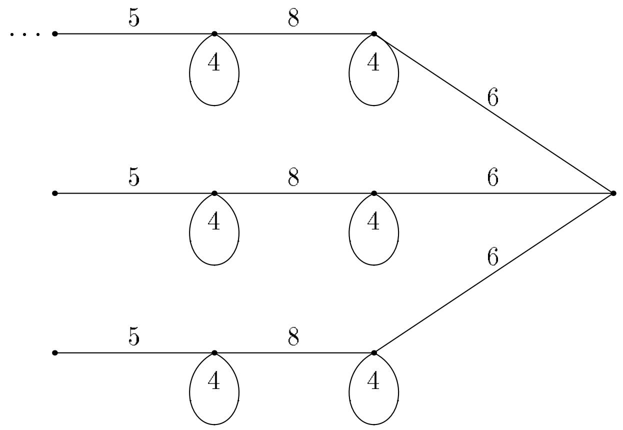


Figure 9. Interconnected bracelets.

the preceding chain within which the ascent might have begun. So the unpurged portion of the ascent still includes an ascent of at least one point. It remains to verify that these segments constitute a subbracelet with tags and both ends. In the direct flight plan, the initial tag of the subbracelet appeared between the first two loops, and the final tag appeared just before the final loop. The purged direct flight plan can be used as the direct flight plan for the subbracelet. □

COROLLARY 1. *If a bracelet includes no playable profitable subbracelets, its direct flight plan includes no bumps greater than* 10.

COROLLARY 2. *A minimal profitable sub-bracelet includes no bumps greater than* 10.

These corollaries will ensure the success of the following algorithm:

ALGORITHM (FINDING A SMOOTH FLIGHT PLAN THROUGH ANY CONFIGURATION OF BRACELETS).

1. If any bracelet contains a profitable sub-bracelet, play a minimal profitable sub-bracelet.
2. If no bracelet contains any profitable sub-bracelet(s), proceed with a direct flight through any one of the bracelets.

Figure 9 shows a situation in which three identical bracelets have tags which share a common joint. Initially, each of these three bracelets has a profitable subbracelet. However, after such a profitable subbracelet is played, the profitable subbracelets of the other two disappear into an unprofitable merger.

Nevertheless, it is easily seen that

THEOREM. *By using the preceding algorithm, Left can play any set of bracelets in such a way that the overall flight plan incurs no bump greater than* 10.

This 10 is also easily seen to be the best possible: One of the simplest nondegenerate bracelets is a pair of earmuffs with muffs that are 4-loops connected by a headband that is a chain of length 12 or more. The only feasible flight plan for this bracelet has a bump of 10.

So we are now ready to state our main result, which includes one more (final) term of technical jargon.

DEFINITION. A subgraph H of a graph G is said to be *compatible* if there is a maximal set of node-disjoint loops in H which is a subset of a maximal set of node-disjoint loops in G.

THEOREM. *Let G be a loony endgame, which contains a compatible subgraph H, which contains a compatible subgraph K, such that*

$$\mathrm{cv}(G) \geq 0,$$
$$\mathrm{cv}(H) \geq 10,$$
$$\mathrm{cv}(K) \geq 10,$$

and in which K contains no moderate loops or moderate chains, and where Left has a flight plan that ascends from G to H without crashing.

Then $\mathrm{v}(G) = \mathrm{cv}(G)$.

DISCUSSION: In the example of Figure 2, $G = H =$ initial position, and K consists only of the 12-loop in the southeast and the 6-chain at the bottom.

In a typical general case, isolated loops, profitable twins, and some moderate chains can be played on the ascent en route from G to H. The route from H to K plays all of the bracelets. K contains all of the very long loops counted in the set which defined the controlled value, plus all very long chains not within bracelets.

Comments

COMMENT 1. For some graphs G Left can succeed by leaving some portions of bracelets within K. All that is really required of K is that $\mathrm{cv}(K)$ be large enough to remain above the bumpiness from H to K, and that a safe landing from K be possible. If the graph contains unprofitable bracelets which aren't very bumpy (as surely happens, for example, if all of their trinkets are 6-loops rather than 4-loops), then some such bracelets might be included in the early portion of the landing from K.

COMMENT 2. Ascending from G to H is usually a simple matter of playing isolated loops and moderate chains. However, if there aren't enough such profitable opportunities available, then some profitable but not-too-bumpy subbracelets might also be included in the latter portions of this ascent.

COMMENT 3. In general, deciding which 3-chains to play in order to ascend from G to H can be challenging. The difficulty is that playing such a chain

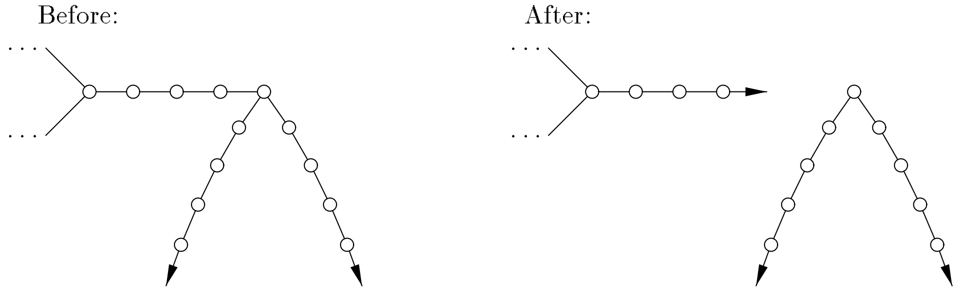


Figure 10. Detaching.

can eliminate a joint, fusing other chains to possibly eliminate other 3-chains, or possibly extend the tag of some bracelet in a way which makes a subbracelet unprofitable.

COMMENT 4. In general, Left begins by finding a maximal set of node-disjoint loops in G. Although a slight (nonplanar) generalization of this problem is NP-hard, in practice it is easily solved by hand for any loony dots-and-boxes positions that can be constructed on boards of sizes up to at least 11×11. The grounded loop and the other very long loops in such a position will be played last, so all other chains connected to them can effectively be detached and viewed as terminating at the ground instead of at joints which are parts of these very long loops.

If there were any loop whose nodes were disjoint from these very long loops and the bracelets, then the supposedly maximal set of loops could have been increased, so it is apparent that after the very long loops and bracelets have been detached, the remaining graph forms a disjoint set of trees. If there are few or no isolated moderate loops or twins, Left may be eager to play as many moderate chains as possible. Within each tree, she can find a maximal set of moderate chains as follows:

Start: If there is a moderate chain which has one end grounded, play it.

If some joint has two or more chains of length at least 4 connecting it directly to the ground, detach all other chains from this joint, and return to start.

(The detachment is illustrated in Figure 10.)

End: When this point is reached, the tree contains no joints adjacent to the ground, so it must be empty.

However, this approach may be ill advised when bracelets are connected to these trees. The problem is that if a tag of a bracelet terminates at the same joint as a moderate chain of the tree, then playing the moderate chain might fuse the tag of the bracelet into a longer chain, thereby eliminating a profitable subbracelet that might have contributed more to the ascent then the moderate chain Left wanted to play.

Comment 5. When Dots-and-Boxes games are played on the 25-square 5×5 board, only rather short bracelets such as the dipper and earmuffs of Figure 1 appear, and even these are relatively rare. The most common loony endgames are merely sums of chains and loops of assorted lengths. But even a sum of chains and loops can be fairly complicated. An extensive study of such cases appears in Scott [2000].

References

[WW] Berlekamp, E. R.; Conway, J. H.; Guy, R. K. *Winning Ways for Your Mathematical Plays* (two volumes). Academic Press, London, 1982.

[Nowakowski] Nowakowski, R. "...Welter's game, sylver coinage, dots-and-boxes, ...", pp. 155–182 in *Combinatorial Games*, Proc. Symp. Appl. Math **43**, edited by Richard K. Guy. AMS, Providence, 1991

[D&B] Berlekamp, E. R. *The-dots-and boxes games: sophisticated child's play,* A K Peters, Natick (MA), 2000.

[Scott] Scott, K. *Loony endgames in dots & boxes*, Master's thesis, University of California, Berkeley, 2000.

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