UC Berkeley Math 254A
Problem Set 3

November 12, 2014

Last Updated: November 12, 2014. Please let me know if you find any typos.

We will discuss these questions in class on **Mon. Nov. 24th.**
Written solutions are due on **Mon. Nov. 24nd.**

1 Cyclotomic Fields

1. Prove Dirichlet's theorem: for n a positive integer, there are infinitely many primes p which are congruent to $1 \mod n$.
2. Suppose that A is an arbitrary finite abelian group. Show that there exists a Galois extension L of $\mathbb{Q}$ such that $\mathrm{Gal}(L|\mathbb{Q}) \cong A$.
3. Let ζ be a primitive p-th root of unity, p an odd prime. Prove that $\mathbb{Z}[\zeta]^\times = \langle \zeta \rangle \cdot \mathbb{Z}[\zeta + \zeta^{-1}]^\times$.
4. Prove that every quadratic extension of $\mathbb{Q}$ is contained in some cyclotomic field (without simply quoting the Kronecher-Weber theorem).

2 Valued fields and their completions

1. Prove that the map $\mathbb{Z}[[x]] \to \mathbb{Z}_p$ defined by $x \mapsto p$ induces an isomorphism $\mathbb{Z}[[x]]/(x-p) \cong \mathbb{Z}_p$.
2. Suppose that K is a field which is complete with respect to an absolute value $|\bullet|_K$. Let L be a finite exntension of K and put $n := [L : K]$. Prove that the unique extension of $|\bullet|_K$ to L, denoted $|\bullet|_L$, is given by
$$|x|_L := \sqrt[n]{\left|N_{L|K}(x)\right|_K}.$$
3. Suppose that K is a discretely valued field, with valuation ring $(\mathcal{O}, \mathfrak{m})$, and let $\hat{K}$ be the completion of K with respect to the induced aboslute value. Let $(\hat{\mathcal{O}}, \hat{\mathfrak{m}})$ be the valuation ring of $\hat{K}$. Prove that there is a canonical isomorphism $\hat{\mathcal{O}}^\times \xrightarrow{\cong} \varprojlim_n \mathcal{O}^\times / U^{(n)}$.
4. Prove the second form of Hensel's Lemma: Suppose that K is a complete discretly valued field with valuation ring $(\mathcal{O}, \mathfrak{m})$ and residue field κ. Suppose that $f \in \mathcal{O}[x]$ is a polynomial, with at least one coefficient being a unit. Suppose that $a_0 \in \mathcal{O}$ is such that
$$f(a_0) = 0 \mod f'(a_0)^2 \cdot \mathfrak{m}.$$
Prove that there exists an element $a \in \mathcal{O}$ such that $f(a) = 0$ and such that $a = a_0 \mod \mathfrak{m}$.

3 Multiplicative groups in $\mathbb{Q}_p^\times$

1. Prove that there is a unique multiplicative section $\mathbb{F}_p \to \mathbb{Z}_p$ of the canonical map $\mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$, whose image is $\mu_{p-1} \cup \{0\} \subset \mathbb{Z}_p$.
2. Prove that $\mathbb{Q}_p^\times$ is isomorphic to the internal direct product $\langle p \rangle \times \mu_{p-1} \times (1 + p \cdot \mathbb{Z}_p)$.
3. Suppose that ℓ is a prime which is different from p. What is the size of $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times \ell}$?
4. What is the size of $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times p}$? Find an explicit basis for $\mathbb{Q}_2^\times / \mathbb{Q}_2^{\times 2}$ as a (multiplicative) vector space over $\mathbb{F}_2$.